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# RESULTS ON SUMS OF CONTINUED FRACTIONS

BY

JAMES L. HLAVKA

ABSTRACT. Let $F(m)$ be the (Cantor) set of infinite continued fractions with partial quotients no greater than $m$ and let $F(m) + F(n) = \{\alpha + \beta: \alpha \in F(m), \beta \in F(n)\}$. We show that $F(3) + F(4)$ is an interval of length 1.14 ... so every real number is the sum of an integer, an element of $F(3)$ and an element of $F(4)$. Similar results are given for $F(2) + F(7)$, $F(2) + F(2) + F(4)$, $F(2) + F(3) + F(3)$ and $F(2) + F(2) + F(2) + F(2)$. The techniques used are applicable to any Cantor sets in R for which certain parameters can be evaluated.

Marshall Hall, Jr. [3] proved that $F(4) + F(4) \equiv \mathbf{R}$ (mod 1) (all notation is defined in the next paragraph) and posed the question: is $F(3) + F(4) \equiv \mathbf{R}$ (mod 1)? In this paper we prove $F(3) + F(4) \equiv \mathbf{R}$ (mod 1) and several other results, summarized in Table 1. Only two questions concerning when a sum of $F(m_i) \equiv \mathbf{R}$ remain open: $F(2) + F(5) \equiv \mathbf{R}$? and $F(2) + F(6) \equiv \mathbf{R}$? We conjecture that they are both false.

| | | |
|---|---|---|
| $F(2) + F(4) \not\equiv \mathbf{R}$ | $F(3) + F(3) \not\equiv \mathbf{R}$ | $F(2) + F(2) + F(3) \not\equiv \mathbf{R}$ |
| $F(2) + F(5)$ ? | $F(3) + F(4) \equiv \mathbf{R}$ | $F(2) + F(2) + F(4) \equiv \mathbf{R}$ |
| $F(2) + F(6)$ ? | | $F(2) + F(3) + F(3) \equiv \mathbf{R}$ |
| $F(2) + F(7) \equiv \mathbf{R}$ | | $F(2) + F(2) + F(2) + F(2) \equiv \mathbf{R}$ |

Table 1. All congruences are modulo 1

We let $\mathbf{N}$ be the natural numbers and $\mathbf{R}$ the real numbers. Lower case Roman letters except $g$ and $h$ will be elements of $\mathbf{N}$; Greek letters elements of $\mathbf{R}$. Let

$$\langle a_1, a_2, \ldots \rangle = \frac{1}{a_1} + \frac{1}{a_2} + \ldots,$$

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$$\langle a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_s}\rangle$$
$$= \langle a_1, \ldots, a_r, a_{r+1}, \ldots, a_s, a_{r+1}, \ldots, a_s, \ldots\rangle,$$

and

$$F(m) = \{\langle a_1, a_2, \ldots\rangle:\ 1 \le a_i \le m \text{ for all } i \in \mathbf{N}\}.$$

When working with continued fractions it is convenient to write intervals without ordering their endpoints, so we define

$$(\alpha, \beta) = \{\xi \in \mathbf{R}:\ \min(\alpha, \beta) < \xi < \max(\alpha, \beta)\}$$

and

$$[\alpha, \beta] = \{\xi \in \mathbf{R}:\ \min(\alpha, \beta) \le \xi \le \max(\alpha, \beta)\}.$$

If $A$ and $B$ are subsets of $\mathbf{R}$, let $\overline{A}$ = the span of $A = \sup(\alpha - \beta)$ over all $\alpha, \beta \in A$ and $A + B = \{\alpha + \beta: \alpha \in A, \beta \in B\}$. Write "$A + B \equiv \mathbf{R} \pmod 1$" to mean "$\xi \in \mathbf{R}$ implies $\xi \equiv \alpha + \beta \pmod 1$ for some $\alpha \in A$, $\beta \in B$." Let $P(m)$ be the special closed interval $[\langle\overline{m, 1}\rangle, \langle\overline{1, m}\rangle]$.

Note that $\langle\overline{m, 1}\rangle$ and $\langle\overline{1, m}\rangle$ are the least and greatest elements of $F(m)$, respectively, so $F(m) \subset P(m)$. Moreover $F(m) \subset F(m + 1)$. The latter inclusion immediately shows that every question of the form $\sum_i F(m_i) \equiv \mathbf{R}$? is covered in Table 1.

$F(m)$ is a Cantor set so the natural approach to computing $\sum F(m_i)$ is by deleting intervals from the $P(m_i)$. Our objective is to devise an algorithm, called a construction of $F(m)$, which controls the order of these deletions sufficiently to establish that $\sum F(m_i) = \sum P(m_i)$ whenever this is true. (This is the same approach Hall used to investigate $F(4) + F(4)$.)

**Lemma 1.** *$F(m) = P(m) \setminus O(m)$ = the set-theoretic difference of $P(m)$ and $O(m)$, where $O(m)$ is defined by*

$$O(m) = \bigcup \{(\langle a_1, \ldots, a_s, \overline{1, m}\rangle, \langle a_1, \ldots, a_s + 1, \overline{m, 1}\rangle):\ s \in \mathbf{N},$$
(1)
$$1 \le a_i \le m \text{ for } i \le s \text{ and } a_s \ne m\}.$$

**Proof.** We show that $O(m)$ is composed of precisely those intervals which are deleted from $P(m)$ to form $F(m)$.

First assume $\alpha \in P(m)\setminus F(m)$. Then $\alpha$ has a first partial quotient $a_{r+1}$ which is greater than $m$, with $r > 0$ and $a_r \ne 1$ when $r = 1$. Now if $a_r = 1$ then

$$\alpha = \langle a_1, \ldots, a_{r-1}, a_r, \ldots\rangle$$
$$\in (\langle a_1, \ldots, a_{r-1} + 1, \overline{m, 1}\rangle, \langle a_1, \ldots, a_{r-1}, \overline{1, m}\rangle)$$

and if $a_r > 1$ then

$$\alpha \in (\langle a_1, \dots, a_r - 1, \overline{1, m}\rangle, \langle a_1, \dots, a_r, \overline{m, 1}\rangle),$$

so $\alpha \in O(m)$. Conversely if $\alpha \in (\langle a_1, \dots, a_s, \overline{1, m}\rangle, \langle a_1, \dots, a_s + 1, \overline{m, 1}\rangle) \subset O(m)$, then

(2) $\alpha \in (\langle a_1, \dots, a_s, \overline{1, m}\rangle, \langle a_1, \dots, a_s + 1\rangle)$,

(3) $\alpha = \langle a_1, \dots, a_s + 1\rangle$, or

(4) $\alpha \in (\langle a_1, \dots, a_s + 1\rangle, \langle a_1, \dots, a_s + 1, \overline{m, 1}\rangle)$.

If (2) then $\alpha = \langle a_1, \dots, a_s, \alpha'\rangle$ where $\langle\overline{1, m}\rangle < \alpha' < 1$ so $\alpha' \notin F(m)$. Hence $\alpha \notin F(m)$. If (4) then $\alpha = \langle a_1, \dots, a_{s+1}, \alpha'\rangle$ where $\alpha' < \langle\overline{m, 1}\rangle$ and again $\alpha \notin F(m)$. Lastly, (3) implies $\alpha \notin F(m)$ since $F(m)$ contains only infinite continued fractions. □

We now define a construction of $F(m)$ as follows. Let $I_m^2 = P(m)$. Choose an interval $O_m^2$ of $O(m)$ and delete it from $I_m^2$, leaving two new intervals $I_m^3$ and $I_m^4$. From each delete an interval of $O(m)$, say $O_m^3$ and $O_m^4$ respectively. $I_m^3$ will be split into two intervals $I_m^5$ and $I_m^6$; $I_m^4$ will be split into $I_m^7$ and $I_m^8$. From each of $I_m^5, \dots, I_m^8$ delete the interval $O_m^5, \dots, O_m^8$ respectively. Continue in this way. This procedure is demonstrated in Figures 1a and 1b. If $O(m) = \bigcup_{i=2}^{\infty} O_m^i$ we call this procedure a construction $C$ of $F(m)$. We call $F_m^k = \bigcup_{j=2^{k-1}+1}^{2^k} I_m^j$ the $k$th step in the construction of $F(m)$.

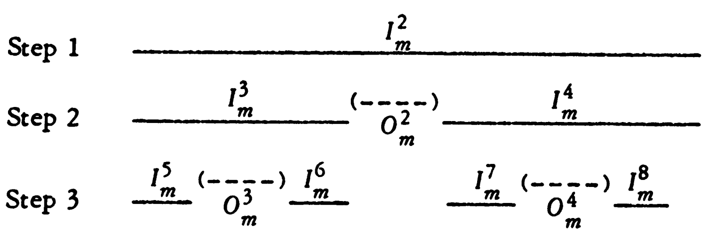


Figure 1a

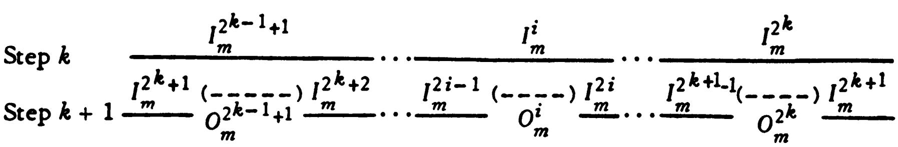


Figure 1b

**Lemma 2.** *If $C$ is any construction of $F(m)$, $m > 1$, and $F_m^k$ is the kth step in this construction, then $F(m) = \bigcap_{k=1}^{\infty} F_m^k$.*

**Proof.** Obvious from Figure 1b. □

**Definition.** If $C$ is a construction of $F(m)$, then

$$g_m = g_m(C) = \sup_i (\overline{O_m^i}/\overline{I_m^i}),$$

$$h_m = h_m(C) = \inf\left(\inf_i(\overline{I_m^{2i-1}}/\overline{I_m^i}),\ \inf_i(\overline{I_m^{2i}}/\overline{I_m^i})\right),$$

and

$$h'_m = h'_m(C) = \sup\left(\sup_i(\overline{I_m^{2i-1}}/\overline{I_m^i}),\ \sup_i(\overline{I_m^{2i}}/\overline{I_m^i})\right).$$

**Theorem 3.** *If there exist constructions $C$ and $C'$ of $F(m)$ and $F(n)$ respectively such that*

*(5) $g_m(C) \cdot g_n(C') \le h_m(C) \cdot h_n(C')$ and*

*(6) $g_m(C) \cdot \overline{P(m)} \le \overline{P(n)}$ and $g_n(C') \cdot \overline{P(n)} \le \overline{P(m)}$*

*then $F(m) + F(n) = P(m) + P(n)$.*

**Proof.** Let $\{I_m^i\}_{i=2}^{\infty}$ and $\{I_n^j\}_{j=2}^{\infty}$ be the intervals appearing in the constructions $C$ and $C'$ respectively. Call the intervals $I_m^i$ and $I_n^j$ compatible (with respect to $C$ and $C'$), written $I_m^i \sim I_n^j$, iff

$$g_m \cdot \overline{I_m^i} \le \overline{I_n^j} \quad \text{and} \quad g_n \cdot \overline{I_n^j} \le \overline{I_m^i}. \tag{7}$$

Call the intervals $I_m^i$ and $I_n^j$ $M$-dividable, written $I_m^i \overset{M}{\approx} I_n^j$, iff (8) or (9) is true, where (8) and (9) are the following (symmetric) conditions.

(8)

$$\text{(8.1)} \qquad I_m^{2i-1} \sim I_n^j \quad \text{and} \quad I_m^{2i} \sim I_n^j,$$

$$\text{(8.2)} \qquad (I_m^{2i-1} + I_n^j) \cup (I_m^{2i} + I_n^j) = I_m^i + I_n^j, \quad \text{and}$$

$$\text{(8.3)} \qquad M \cdot \overline{I_m^i} \ge \overline{I_n^j}.$$

(9)

$$\text{(9.1)} \qquad I_m^i \sim I_n^{2j-1} \quad \text{and} \quad I_m^i \sim I_n^{2j},$$

$$\text{(9.2)} \qquad (I_m^i + I_n^{2j-1}) \cup (I_m^i + I_n^{2j}) = I_m^i + I_n^j, \quad \text{and}$$

$$\text{(9.3)} \qquad M \cdot \overline{I_n^j} \ge \overline{I_m^i}.$$

The four pairs of intervals appearing in (8.1) and (9.1) are said to be derived from the pair $(I_m^i, I_n^j)$.

It suffices to show that for some $M \in \mathbf{R}^+$, $I_m^i \sim I_n^j$ implies $I_m^i \overset{M}{\approx} I_n^j$ for all $i, j \geq 2$. To prove this, set $S_0 = \{(I_m^2, I_n^2)\}$ and

$$S_{r+1} = \{(I, J)\colon\ I \sim J \text{ and } (I, J) \text{ is derived from a pair } (I_0, J_0) \in S_r\}.$$

Clearly

$$\bigcup\{I + J\colon (I, J) \in S_{r+1}\} = \bigcup\{I + J\colon (I, J) \in S_r\} = \cdots = I_m^2 + I_n^2 = P(m) + P(n).$$

If $(I, J) \in S_r$ then $\overline{I} \cdot \overline{J} \leq \lambda^r \cdot \overline{I_m^2} \cdot \overline{I_n^2} \to 0$ as $r \to \infty$, where $\lambda =$ $\max(1 - h_m, 1 - h_n) < 1$ (if $h_m$ or $h_n = 0$ then $g_m$ or $g_n = 0$ by (5) so $F(m)$ or $F(n)$ is not a Cantor set—contradiction). Since $I \sim J$, the ratio $\overline{I}/\overline{J}$ is bounded so $\overline{I} \to 0$ and $\overline{J} \to 0$. Therefore for each $i$ there is an $r_0$ such that $O_m^i$ has been deleted from every $I$ appearing in a pair $(I, J) \in S_r$, $r > r_0$. Since $O(m) = \bigcup_{i=2}^\infty O_m^i$,

$$O(m) \cap \left(\bigcap_{r=0}^{\infty} \left(\bigcup\{I\colon (I, J) \in S_r\}\right)\right) = \varnothing. \tag{10}$$

But for all $r$, $F(m) \subset \bigcup\{I\colon (I, J) \in S_r\} \subset P(m)$ so (10) and Lemma 1 yield

$$F(m) = \bigcap_{r=0}^{\infty} \left(\bigcup\{I\colon (I, J) \in S_r\}\right).$$

Similarly for $F(n)$. Since the sequence $\{\bigcup\{I\colon (I, J) \in S_r\}\}_{r=0}^\infty$ is a nested sequence of compact sets, we obtain directly the result

$$F(m) + F(n) = \bigcap_{r=0}^{\infty} \left(\bigcup\{I + J\colon (I, J) \in S_r\}\right) = \bigcap_{r=0}^{\infty} (P(m) + P(n)) = P(m) + P(n).$$

Now fix $M \geq \max(h_m/g_n, h_n/g_m)$ and assume $I_m^i \sim I_n^j$. Since $g_m g_n \leq h_m h_n$, we must have

$$\overline{I_m^i}/\overline{I_n^j} \geq g_n/h_m \tag{11}$$

or

$$\overline{I_m^i}/\overline{I_n^j} \leq h_n/g_m. \tag{12}$$

Assuming (11) we will verify (8). Similarly (9) will follow from (12), so this will show $I_m^i \overset{M}{\approx} I_n^j$. So assume (11) and set $k = 2i - 1$ or $2i$. Then recalling the definition of $h_m$,

$$\overline{I^k_m} \geq h_m \cdot \overline{I^i_m} \geq g_n \cdot \overline{I^j_n} \quad \text{and} \quad \overline{I^j_n} \geq g_m \cdot \overline{I^i_m} \geq g_m \cdot \overline{I^k_m},$$

so $I^k_m \sim I^j_n$. To check (8.2), let $I^{2i-1}_m = [\alpha, \beta]$, $I^{2i}_m = [\gamma, \delta]$ and $I^j_n = [\alpha_0, \delta_0]$, with $\alpha < \beta < \gamma < \delta$ and $\alpha_0 < \delta_0$. Since

$$\overline{I^j_n} \geq g_m \cdot \overline{I^i_m} \geq \overline{O^i_m}, \tag{13}$$

we have $\delta_0 - \alpha_0 \geq \gamma - \beta$ or $\beta + \delta_0 \geq \gamma + \alpha_0$. Then

$$(I^{2i-1}_m + I^j_n) \cup (I^{2i}_m + I^j_n) = [\alpha + \alpha_0, \beta + \delta_0] \cup [\gamma + \alpha_0, \delta + \delta_0]$$
$$= [\alpha + \alpha_0, \delta + \delta_0] = I^i_m + I^j_n.$$

Lastly, (8.3) is satisfied by our choice of $M$. □

**Lemma 4.** *If* $\alpha = \langle a_1, \ldots, a_s, \alpha' \rangle$, $\beta = \langle a_1, \ldots, a_s, \beta' \rangle$, $\alpha' > 0$, $\beta' > 0$, $\langle a_1, \ldots, a_s \rangle = p_s/q_s$, $\langle a_1, \ldots, a_{s-1} \rangle = p_{s-1}/q_{s-1}$ *and* $Q = q_{s-1}/q_s$, *then*

$$(\alpha' - \beta')/(\alpha - \beta) = (Q + \alpha')(Q + \beta')(-1)^{s+1} q_s^2.$$

**Proof.** We have

$$\alpha - \beta = \frac{p_s\alpha' + p_{s-1}}{q_s\alpha' + q_{s-1}} - \frac{p_s\beta' + p_{s-1}}{q_s\beta' + q_{s-1}}$$
$$= \frac{p_s}{q_s}\left\{\frac{\alpha' + Q + p_{s-1}/p_s - Q}{\alpha' + q_{s-1}/q_s} - \frac{\beta' + Q + p_{s-1}/p_s - Q}{\beta' + q_{s-1}/q_s}\right\}$$
$$= \frac{p_s}{q_s}\left(\frac{p_{s-1}}{p_s} - Q\right)\left\{\frac{1}{\alpha' + Q} - \frac{1}{\beta' + Q}\right\}$$
$$= \frac{(-1)^{s+1}(\alpha' - \beta')}{q_s^2(\alpha' + Q)(\beta' + Q)},$$

since $p_{s-1}q_s - p_sq_{s-1} = (-1)^s$. The result follows immediately. □

**Lemma 5.** *If* $\gamma = \langle a_1, \ldots, a_s, \gamma' \rangle$, $\delta = \langle a_1, \ldots, a_s, \delta' \rangle$, $\gamma' > 0$, $\delta' > 0$, *and* $\alpha, \beta, \alpha', \beta', Q$ *are as in Lemma* 4, *then*

$$\frac{\alpha - \beta}{\gamma - \delta} = \frac{\alpha' - \beta'}{\gamma' - \delta'} \cdot \frac{(Q + \gamma')(Q + \delta')}{(Q + \alpha')(Q + \beta')} \tag{14}$$

*and* $Q \in [1/(a_s + 1), 1]$.

**Proof.** Statement (14) is an immediate corollary of Lemma 4. The restriction on $Q$ follows from the well-known result that $Q = \langle a_s, \ldots, a_1 \rangle$. □

**Theorem 6.** $F(3) + F(4) = P(3) + P(4) = [\langle\overline{3, 1}\rangle + \langle\overline{4, 1}\rangle, \langle\overline{1, 3}\rangle + \langle\overline{1, 4}\rangle] = [.4709\ldots, 1.6197\ldots]$.

**Proof.** We produce constructions of $F(3)$ and $F(4)$ satisfying the hypotheses of Theorem 3. Let us begin by defining a canonical construction, $C_m$, of $F(m)$ for any $m$, as follows. $I_m^2$ must be $[\overline{m, 1}, \overline{1, m}]$. If

$$I_m^i = [\langle a_1, \ldots, a_s, j, \overline{m, 1}\rangle, \langle a_1, \ldots, a_s, m, \overline{1, m}\rangle]$$

with $s \geq 0$ and $j \neq m$ then

$$O_m^i = (\langle a_1, \ldots, a_s, j, \overline{1, m}\rangle, \langle a_1, \ldots, a_s, j+1, \overline{m, 1}\rangle)$$

so that

$$I_m^{2i-1} = [\langle a_1, \ldots, a_s, j, \overline{m, 1}\rangle, \langle a_1, \ldots, a_s, j, \overline{1, m}\rangle]$$

and

$$I_m^{2i} = [\langle a_1, \ldots, a_s, j+1, \overline{m, 1}\rangle, \langle a_1, \ldots, a_s, m, \overline{1, m}\rangle].$$

It is relatively easy to show that $\bigcup_{i=2}^{\infty} O_m^i = O(m)$ so this does define a construction of $F(m)$. The value of the constructions $C_m$ is that we can readily calculate $g_m(C_m)$ and $h_m(C_m)$ (and $h_m'(C_m)$, which will be needed later). We have

$$\begin{aligned} g_m(C_m) &= \max_{i \geq 2} (\overline{O_m^i}/\overline{I_m^i}) \\ &= \max \frac{\langle a_1, \ldots, a_s, j, \overline{1, m}\rangle - \langle a_1, \ldots, a_s, j+1, \overline{m, 1}\rangle}{\langle a_1, \ldots, a_s, j, \overline{m, 1}\rangle - \langle a_1, \ldots, a_s, m, \overline{1, m}\rangle} \end{aligned}$$

over $1 \leq j \leq m$, $s \geq 0$ and $1 \leq a_i \leq m$ for $i \leq s$. Using Lemma 5, we obtain

$$g_m(C_m) \leq \max\left\{\frac{\langle j, \overline{1, m}\rangle - \langle j+1, \overline{m, 1}\rangle}{\langle j, \overline{m, 1}\rangle - \langle m, \overline{1, m}\rangle} \cdot \frac{(Q + \langle j, \overline{m, 1}\rangle)(Q + \langle m, \overline{1, m}\rangle)}{(Q + \langle j, \overline{1, m}\rangle)(Q + \langle j+1, \overline{m, 1}\rangle)}\right\}$$

over $1 \leq j \leq m$ and $Q \in [1/(m+1), 1]$. For each allowable value of $j$ this expression is a rational function in $Q$ whose maximum on the interval $[1/(m+1), 1]$ can be readily calculated. A Univac 1108 was used to perform these calculations and then maximize over $j$ (and for similar calculations arising later). We thus obtain the bounds

$$g_3(C_3) \leq .2992\ldots \quad \text{and} \quad g_4(C_4) \leq .2278\ldots.$$

Similarly

$$h_m(C_m) = \min_{i\geq 2}\left(\min\left(\overline{I_m^{2i-1}}/\overline{I_m^i},\ \overline{I_m^{2i}}/\overline{I_m^i}\right)\right)$$

$$= \min_{j,a_i}\left(\min\left(\frac{\langle a_1, \ldots, a_s, j, \overline{1, m}\rangle - \langle a_1, \ldots, a_s, j, \overline{m, 1}\rangle}{\langle a_1, \ldots, a_s, m, \overline{1, m}\rangle - \langle a_1, \ldots, a_s, j, \overline{m, 1}\rangle}\right.\right.$$

$$\left.\left.\frac{\langle a_1, \ldots, a_s, m, \overline{1, m}\rangle - \langle a_1, \ldots, a_s, j+1, \overline{m, 1}\rangle}{\langle a_1, \ldots, a_s, m, \overline{1, m}\rangle - \langle a_1, \ldots, a_s, j, \overline{m, 1}\rangle}\right)\right)$$

$$\geq \min_{j, Q}\left(\min\left(\frac{\langle j, \overline{1, m}\rangle - \langle j, \overline{m, 1}\rangle}{\langle m, \overline{1, m}\rangle - \langle j, \overline{m, 1}\rangle}\cdot\frac{(Q + \langle m, \overline{1, m}\rangle)}{(Q + \langle j, \overline{1, m}\rangle)},\right.\right.$$

$$\left.\left.\frac{\langle m, \overline{1, m}\rangle - \langle j+1, \overline{m, 1}\rangle}{\langle m, \overline{1, m}\rangle - \langle j, \overline{m, 1}\rangle}\cdot\frac{(Q + \langle j, \overline{m, 1}\rangle)}{(Q + \langle j+1, \overline{m, 1}\rangle)}\right)\right),$$

from which we obtain

$$h_3(C_3) \geq .2471\ldots \quad \text{and} \quad h_4(C_4) \geq .2963\ldots.$$

A simple multiplication shows that

$$g_3(C_3)\cdot g_4(C_4) \leq .0667 < .0731 \leq h_3(C_3)\cdot h_4(C_4).$$

Also

$$g_3 \cdot \overline{P(3)} \leq .2992 \times .5276 < P(4) = .6212\ldots < \frac{.5276}{.2278} \leq \frac{\overline{P(3)}}{g_4}$$

so by Theorem 3 we have the result $F(3) + F(4) = P(3) + P(4)$. □

**Corollary 7.** $F(3) + F(4) \equiv \mathbf{R} \pmod 1$.

**Proof.** This is obvious since $F(3) + F(4)$ contains an interval of length greater than one. □

The values of $g_3$, $g_4$, $h_3$ and $h_4$ are in fact equal to the bounds given because these bounds arise from $Q = 1/(m+1)$ or $Q = 1$, which are possible values of $Q$. For $g_2$ and $h_2$, below, this does not happen.

Applying Theorem 3 to the canonical constructions of $F(2)$ and $F(12)$ as above we can establish $F(2) + F(12) = P(2) + P(12)$, but now we find that the canonical construction of $F(12)$ is not an optimal construction in terms of minimizing the ratio $g_{12}/h_{12}$. This is because the maximal value of $\overline{O_m^i}/\overline{I_m^i}$ always occurs at $j = m - 1$ but the minimal value of $\min(\overline{I_m^{2i-1}}, \overline{I_m^{2i}})/\overline{I_m^i}$

only occurs at $j = m - 1$ if $m \le 4$. A noncanonical construction can allow us to lower the number 12, but the best result is obtained by extending Theorem 3.

**Theorem 8.** *If there exist constructions C and C′ of F(m) and F(n) respectively, such that*

(15) $$\overline{O^i_m} \cdot \overline{O^j_n} \le \min(\overline{I^{2i-1}_m}, \overline{I^{2i}_m}) \cdot \min(\overline{I^{2j-1}_n}, \overline{I^{2j}_n}) \quad \text{for all } i, j \ge 2,$$

(16) $$g_m \cdot \overline{P(m)} \le \overline{P(n)} \text{ and } g_n \cdot \overline{P(n)} \le \overline{P(m)},$$

*and*

(17) $$\max(h'_m, h'_n) \cdot g_m g_n \le h_m h_n,$$

*then* $F(m) + F(n) = P(m) + P(n)$.

**Proof.** Call the intervals $I^i_m$ and $I^j_n$ M-dividable* iff (8), (9) or

(18)

(18.1) $$(I^{2i-1}_m + I^{2j-1}_n) \cup (I^{2i-1}_m + I^{2j}_n) \cup (I^{2i}_m + I^{2j-1}_n) \cup (I^{2i}_m + I^{2j}_n) = I^i_m + I^j_n \text{ and}$$

(18.2) $$I^{2i-1}_m \sim I^{2j-1}_n,\ I^{2i-1}_m \sim I^{2j}_n,\ I^{2i}_m \sim I^{2j-1}_n, \text{ and } I^{2i}_m \sim I^{2j}_n$$

holds. The proof now parallels the proof of Theorem 3; the only significant difference being to show that a compatible pair is M-dividable* when neither (11) nor (12) holds. So assume

(19) $$h_n/g_m \le \overline{I^i_m}/\overline{I^j_n} \le g_n/h_m.$$

Let $I^{2i-1}_m = [\alpha, \beta]$, $I^{2i}_m = [\gamma, \delta]$, $I^{2j-1}_n = [\alpha_0, \beta_0]$, and $I^{2j}_n = [\gamma_0, \delta_0]$. From (15) we obtain

$$\overline{O^i_m} \le \min(\overline{I^{2j-1}_n}, \overline{I^{2j}_n}) \quad \text{or} \quad \overline{O^j_n} \le \min(\overline{I^{2i-1}_m}, \overline{I^{2i}_m});$$

assume for simplicity the latter. Then $\gamma_0 - \beta_0 \le \min(\beta - \alpha, \delta - \gamma)$ so $\alpha + \gamma_0 \le \beta + \beta_0$ and $\gamma + \gamma_0 \le \delta + \beta_0$. Then the LHS of (18.1) is

$$[\alpha + \alpha_0, \beta + \beta_0] \cup [\alpha + \gamma_0, \beta + \delta_0] \cup [\gamma + \alpha_0, \delta + \beta_0] \cup [\gamma + \gamma_0, \delta + \delta_0]$$
$$= [\alpha + \alpha_0, \beta + \delta_0] \cup [\gamma + \alpha_0, \delta + \delta_0] = [\alpha + \alpha_0, \delta + \delta_0] = I^i_m + I^j_n.$$

For $k = 2i - 1$ or $2i$, $l = 2j - 1$ or $2j$, we have

$$\overline{I^k_m} \ge h_m \cdot \overline{I^i_m} \ge \frac{h_m h_n}{g_m} \cdot \overline{I^j_n} \ge \frac{h_m h_n}{g_m h'_n} \cdot \overline{I^l_n} \ge g_n \cdot \overline{I^l_n}$$

and similarly $\overline{I_n^l} \geq g_m \cdot \overline{I_m^k}$ so $\overline{I_m^k} \sim \overline{I_n^l}$. □

Theorem 9. $F(2) + F(7) = P(2) + P(7) = [.4928\ldots, 1.6195\ldots] \equiv \mathbf{R}$ (mod 1).

**Proof.** The canonical constructions can be used. We again apply Lemma 5 to obtain the bounds

$$g_2 \leq .4456\ldots, \quad h_2 \geq .1686\ldots, \quad h_2' \leq .4589\ldots, \quad g_7 \leq .1343\ldots,$$

$$h_7 \geq .2906\ldots, \quad h_7' \leq .6359\ldots, \quad \text{and} \quad \frac{\overline{O_7^j}}{\min(\overline{I_7^{2j-1}}, \overline{I_2^{2j}})} \leq .3594\ldots.$$

Since $\min(\overline{I_2^{2i-1}}, \overline{I_2^{2i}})/\overline{O_2^i} \geq h_2/g_2$, it is now easy to verify the hypotheses of Theorem 8. □

For any constructions of $F(2)$ and $F(6)$, equation (15) fails whenever an interval of the type $(\langle a_1, \ldots, a_s, 5, \overline{1, 6}\rangle, \langle a_1, \ldots, a_s, 6, \overline{6, 1}\rangle)$ is deleted from $F(6)$. Thus, as (15) is intuitively a "best possible" condition in the sense that no weakening approximations were made, it is probable that $F(2) + F(6) \neq P(2) + P(6)$. Moreover, since (15) fails infinitely often, $F(2) + F(6)$ may possibly be a Cantor set.

The negative results $F(3) + F(3) \not\equiv \mathbf{R}$ (mod 1) and $F(2) + F(4) \not\equiv \mathbf{R}$ (mod 1) can be verified directly. $P(2) + P(4)$ has length less than one and $F(3) \subset [\langle\overline{3, 1}\rangle, \langle 3, \overline{3, 1}\rangle] \cup [\langle 2, \overline{1, 3}\rangle, \langle\overline{1, 3}\rangle]$ yields

$$F(3) + F(3) \subset [.5274\ldots, .62178\ldots] \cup [.62200\ldots, 1.5826\ldots].$$

We now look at sums of $F(m_1) + \ldots + F(m_s)$ with $s > 2$.

**Theorem 10.** *Let $C_1, \ldots, C_s$ be constructions of $F(m_1), \ldots, F(m_s)$ respectively (not necessarily canonical constructions). If*

(20) $h_m \leq \overline{I_m^2}/\overline{I_n^2}$ *for* $m, n \in \{m_1, \ldots, m_s\}$, *and*

(21) $g_m + h_m \leq \sum_{i=1}^{s} h_{m_i}$ *for* $m \in \{m_1, \ldots, m_s\}$

*then* $F(m_1) + \ldots + F(m_s) = P(m_1) + \ldots + P(m_s)$.

**Proof.** Again we mimic the proof of Theorem 3. Let $(I_{m_1}^{j_1}, \ldots, I_{m_s}^{j_s})$ be compatible iff

(22) $$h_{m_i} \cdot \overline{I_{m_k}^{j_k}} \leq \overline{I_{m_i}^{j_i}} \quad \text{for } 1 \leq i, k \leq s.$$

Call $(I_{m_1}^{j_1}, \dots, I_{m_s}^{j_s})$ dividable iff

$$(I_n^{2j-1} + (I_{m_1}^{j_1} + \cdots + \widehat{I_n^j} + \cdots + I_{m_s}^{j_s})) \cup (I_n^{2j} + (I_{m_1}^{j_1} + \cdots + \widehat{I_n^j} + \cdots + I_{m_s}^{j_s}))$$

$$= I_{m_1}^{j_1} + \cdots + I_{m_s}^{j_s}, \tag{23}$$

and $(I_{m_1}^{j_1}, \dots, I_n^{2j-1}, \dots, I_{m_s}^{j_s})$ and $(I_{m_1}^{j_1}, \dots, I_n^{2j}, \dots, I_{m_s}^{j_s})$ are compatible, where $\widehat{\ }$ means omission and $n$ is such that

$$\overline{I_n^j} \geq \overline{I_{m_i}^{j_i}} \quad \text{for } 1 \leq i \leq s. \tag{24}$$

Hypothesis (20) says the beginning $s$-tuple $(I_{m_1}^2, \dots, I_{m_s}^2)$ is compatible, so the proof reduces to showing that every compatible $s$-tuple is dividable. Let $k = 2j - 1$ or $2j$. Then $\overline{I_n^k} \geq h_n \cdot \overline{I_n^j} \geq h_n \cdot \overline{I_{m_i}^{j_i}}$ for $1 \leq i \leq s$, and the other combinations of subscripts occurring in (22) trivially produce correct inequalities, so $(I_{m_1}^{j_1}, \dots, I_n^k, \dots, I_{m_s}^{j_s})$ is compatible. From (21), we have

$$g_n \leq h_{m_1} + \cdots + \widehat{h_n} + \cdots + h_{m_s}$$

so that

$$\overline{I_{m_1}^{j_1}} + \cdots + \widehat{\overline{I_n^j}} + \cdots + \overline{I_{m_s}^{j_s}} \geq h_{m_1} \cdot \overline{I_n^j} + \cdots + \widehat{h_n \cdot \overline{I_n^j}} + \cdots + h_{m_s} \cdot \overline{I_n^j}$$

$$= (h_{m_1} + \cdots + \widehat{h_n} + \cdots + h_{m_s}) \cdot \overline{I_n^j} \geq g_n \cdot \overline{I_n^j} \geq \overline{O_n^j}.$$

This is the analog of equation (13) and is precisely the inequality needed to establish (23). □

**Theorem 11.** *The following are all true.*

$$F(2) + F(2) + F(4) = P(2) + P(2) + P(4) \equiv \mathbf{R} \pmod 1,$$
$$F(2) + F(3) + F(3) = P(2) + P(3) + P(3) \equiv \mathbf{R} \pmod 1,$$

*and*

$$F(2) + F(2) + F(2) + F(2) = P(2) + P(2) + P(2) + P(2) \equiv \mathbf{R} \pmod 1.$$

**Proof.** Apply Theorem 10 to the canonical constructions of $F(2)$, $F(3)$ and $F(4)$. □

The final result listed in Table 1, $F(2) + F(2) + F(3) \not\equiv \mathbf{R}$, results from inspecting the first few subdivisions of $F(2)$ and $F(3)$.

T. W. Cusick and R. A. Lee [1], [2] have investigated $\Sigma_i S(m_i)$, where

$$S(m) = \{\langle a_1, a_2, \ldots\rangle : a_i \geq m \text{ for all } i\}$$
$$\cup\{\langle a_1, \ldots, a_s\rangle : a_i \geq m \text{ for } 1 \leq i \leq s, \ s \geq 1\} \cup \{0\}.$$

They have shown [2] that

$$\sum_{i=1}^{m} S(m) = [0, 1]. \tag{25}$$

Our Theorem 10 can be applied to $\Sigma_i S(m_i)$ in place of $\Sigma_i F(m_i)$, whereupon (25) follows as a relatively easy special case.

More generally, Theorems 3, 8 and 10 are applicable to any Cantor sets for which $g_m$, $h_m$ and $h'_m$ can be evaluated.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53705