

SUBMERSIVE AND UNIPOTENT GROUP QUOTIENTS  
AMONG SCHEMES OF A COUNTABLE TYPE OVER  
A FIELD  $k$

BY

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**ABSTRACT.** An algebraic group  $G$  is called *submersive* if every quotient in affine schemes  $c^G: \text{Spec } A \rightarrow \text{Spec } A^G$  which is surjective is also submersive. We prove that every unipotent group is submersive. Suppose  $G$  is submersive. We show that if  $c^G(\text{Spec } A)$  is open in  $\text{Spec } A^G$  or if some restrictions on the action of  $G$  on  $A$  are made,  $c^G$  is a topological quotient. A criterion for semisimplicity of points is extended to the case where  $G$  is unipotent. Finally, applications of the theory are provided.

**0. Introduction.** Let  $X$  be an affine scheme of finite type over a field  $k$  and suppose that  $G$  is an irreducible algebraic group over  $k$  which has a closed action on  $X$  via a  $k$  morphism  $\sigma: G \times X \rightarrow X$ . If  $X = \text{Spec } A$ ,  $A^G$  consists of the functions in  $A$  invariant under  $G$  and  $X^G = \text{Spec } A^G$ , we call the map  $c^G: X \rightarrow X^G$  induced by the inclusion  $A^G \rightarrow A$  the *algebraic group quotient of  $X$  by  $G$* . It is easily seen that if  $G$  is affine, the mapping  $c^G$  is the coequalizer of  $\sigma$  and  $P_2$ , the projection of  $G \times X$  onto  $X$ , in the category of affine schemes of a countable type over  $k$  (see [2]).

**Definition 1.** Let  $X$  be a free variable. If  $c^G: X \rightarrow X^G$  is surjective implies that  $c^G: X \rightarrow X^G$  is submersive, we say that  $G$  is *submersive*.

Every reductive group is submersive. See Mumford [6, p. 27, Theorem 1.1]. In §4, we show that every unipotent group is submersive.

The basic notation to be used here is that of Mumford [6]. However, we present now some of the conceptions to be found in Mumford [6] in a more compact form using some category theory.

**Definition 2.** Let  $D_1$  be a subcategory of  $D_2$  and  $D_2$  be a subcategory of  $D_3$  (thus,  $D_1$ ,  $D_2$  and  $D_3$  are categories). Suppose that  $f, g: Y_1 \rightrightarrows Y_2$

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are two maps in  $D_1$ . A  $D_3$  coequalizer in  $D_2$  relative to  $D_1$  is a map  $c: Y_2 \rightarrow Y_3$  in  $D_2$  such that, for each map  $b: Y_2 \rightarrow Y_4$  in  $D_3$  satisfying  $b \circ f = b \circ g$ , there is a unique map  $i: Y_3 \rightarrow Y_4$  in  $D_3$  with  $i \circ c = b$ .

The most important theorem of this paper is the theorem that follows. In the paper  $A^G \neq k$ .

**Theorem 1.** *Let  $X$  and  $X^G$  be those affine schemes and  $G$  be that algebraic group considered above. Assume that  $G$  is submersive and affine. Then, under each of the following conditions:*

- (i)  $c^G(X)$  is open in  $X^G$ ,
- (ii) for each  $h \in A$  and each  $\sigma \in G$ , there is a sequence of functions  $h_1, \dots, h_n$  in  $A$  such that

$$h^\sigma = h + h_1, \quad h_1^\sigma = h_1 + h_2, \quad \dots, \quad h_{n-1}^\sigma = h_{n-1} + h_n$$

with  $h_n \in A^G$ ,  $c^G: X \rightarrow c^G(X)$  is a Top and Sch coequalizer of the maps

$$\sigma, P_2: G \times X \rightrightarrows X$$

in Sch relative to Aff. Here, Top is the category of topological spaces, Sch is the category of schemes and Aff is the category of affine schemes.

In §2 we show that condition (ii) of Theorem 1 implies condition (i). Then, in §6, we provide some examples where condition (ii) is satisfied.

**Corollary 1.** *Let  $X$  and  $X^G$  be those affine schemes and  $G$  be that algebraic group of Theorem 1. Suppose that  $G$  is a unipotent group. Then,  $c^G: X \rightarrow c^G(X)$  is a Top and Sch coequalizer of  $\sigma$  and  $P_2$  in Sch relative to Aff.*

**Proof.** In §4, we show that every unipotent group is submersive. Apply Theorem 1.

**Corollary 2.** *Let  $X$  and  $X^G$  be those affine schemes and  $G$  be that algebraic group of Theorem 1. Suppose that  $c^G(X)$  is of finite type over  $k$  and that  $k$  is an algebraically closed field. If  $X$  is irreducible and normal and if the residue field of the generic point of  $c^G(X)$  has characteristic 0, then  $c^G: X \rightarrow c^G(X)$  is a geometric quotient of  $X$  by  $G$ , i.e., the mapping  $c^G$  is a Top and Sch coequalizer of  $\sigma$  and  $P_2$  in Sch relative to Aff and, in addition,  $c^G$  is universally submersive.*

**Proof.** One applies Mumford [6, Proposition 0.2, p. 7].

**Remark 1.** Mumford [6, p. 6] indicates that if  $X$  is normal over  $k$  and  $c^G: X \rightarrow c^G(X)$  is a geometric quotient, then  $c^G(X)$  must be of finite type

over  $k$ . Nagata's example (see Dieudonné [4, p. 45]) shows us that  $X$  can be of finite type and normal but that  $X^G$  may not be of finite type. A simple example illustrates, however, that  $c^G(X)$  could be of finite type over  $k$  even though it is an open subscheme of an affine scheme of countable type over  $k$ . Let  $i, *: k^1 \rightrightarrows k^2$  be two maps where  $i$  is a closed immersion identifying  $k^1$  with  $X = 0$ ,  $(0, 0) \in i(k^1)$  and  $*(k^1) = (0, 0)$ . Then,  $i$  and  $*$  have a coequalizer  $k^2/k^1 = \text{Spec}(k + (X))$  in  $\underline{\text{Sch}}$  where  $(X)$  is the ideal generated by  $X$  in  $k[X, Y]$ . Furthermore,  $k + (X)$  is not of finite type over  $k$ ; see [2]. Localize at  $X \neq 0$ . Then,  $(k + (X))_X \simeq k[X, Y]_X$ . Let  $q: k^2 \rightarrow k^2/k^1$  be the quotient map. It follows that  $k^2/k^1 - q(k^1) \simeq k^2 - k^1$  is of finite type over  $k$ . This suggests the following refinement of

**Hilbert's Fourteenth Problem.** *Is  $c^G(X)$  a scheme of finite type over  $k$ ?*

In order to provide some applications of Theorem 1, we will prove the next result in §5.

**Theorem 2.** *Let  $G$  be a unipotent affine algebraic group which acts on projective space  $\mathbb{P}^n$  by a closed action over  $k$ . Consider the cone  $k^{n+1}$  lying above  $\mathbb{P}^n$ . If  $x$  is a point of  $\mathbb{P}^n$ , suppose that  $x'$  denotes a point lying above  $x$ . Note that the action of  $G$  on  $\mathbb{P}^n$  extends to an action of  $G$  on  $k^{n+1}$ . Then,  $x$  is semistable if and only if the closure of the orbit of  $x'$  does not contain 0.*

For further information on the notions contained in this theorem, the reader is referred to Mumford [6, p. 50].

Examples of unipotent actions on projective spaces, which appear as linear systems, will be provided in §6.

**1. A partial generalization of Chevalley's theorem.**

**Lemma 1.** *Let  $f: X \rightarrow Y$  be a dominant morphism of affine schemes defined over  $k$  and suppose that  $X$  is of finite type over  $k$ . Then,  $f(X)$  contains an open subset  $U$  of  $Y$ .*

**Proof.** Clearly, we can assume that  $X$  and  $Y$  are reduced. Suppose that  $X_1, X_2, \dots, X_m$  are the components of  $X$  (finite in number as  $X$  is of finite type). Consider the mapping  $f_i: X_i \rightarrow \overline{f(X_i)}$ , the restriction of  $f$  to  $X_i$  mapped to the closure of  $f(X_i)$ ,  $i = 1, 2, \dots, m$ . If  $f(X_i)$  contains an open subset  $U_i$  of  $\overline{f(X_i)}$ , then  $f(X_i)$  also contains an affine open subset  $U(g_i)$  of  $\overline{f(X_i)}$ , the complement of the hyperplane  $g_i$ . In that case,  $f(X) =$

$\bigcup_{i=1}^m f(X_i)$  contains an affine open  $U(g_1 g_2 \cdots g_m)$  of  $\overline{f(X)} = \bigcup_{i=1}^m \overline{f(X_i)}$ . Therefore, we can assume in Lemma 1 that  $X$  and  $Y$  are irreducible and reduced.

The remainder of the proof, modulo some minor modifications, can be found in Mumford [7, pp. 94 and 95].

**Remark.** When  $Y$  is of finite type, Lemma 1 follows from Chevalley's theorem which asserts (see Grothendieck [5]) that  $f(X)$  is constructible.

2.  $c^G(X)$  is open in  $X^G$ . We employ the notation of §0. In order to show that  $c^G: X \rightarrow c^G(X)$  is a Top coequalizer in Sch, we must show that  $c^G(X)$  is a scheme. This result is a consequence of the next lemma.

**Lemma 2.** *If condition (ii) of Theorem 1 holds,  $c^G(X)$  is open in  $X^G$ .*

**Proof.** Lemma 1 implies that there is an affine open subset  $U(g_1)$  of  $X^G$  such that  $U(g_1) \subset c^G(X) \subset X^G$ . If  $c^G(X) = X^G$  or  $c^G(X) = U(g_1)$ , we are finished. Suppose, then, that  $c^G(X) \neq X^G$  and  $c^G(X) \neq U(g_1)$  and consider the map

$$c_1^G: \text{Spec } A/(g_1) \rightarrow \text{Spec } A^G/(g_1)$$

where  $(g_1)$  denotes the ideal generated by  $g_1$  in  $A$  and  $A^G$ , respectively.

Suppose that  $c_1^G$  is dominant. Lemma 1 implies that there is an open subset  $U(g_2)$  of  $\text{Spec } A^G/(g_1)$  such that

$$U(g_2) \subset c_1^G(\text{Spec } A/(g_1)) \subset \text{Spec } A^G/(g_1).$$

If (a)<sub>1</sub>  $U(g_2) = c_1^G(\text{Spec } A/(g_1))$  or (b)<sub>1</sub>  $c_1^G(\text{Spec } A/(g_1)) = \text{Spec } A^G/(g_1)$ , we are finished. In case (a)<sub>1</sub>,  $c^G(X) = X^G - (V(g_1) \cap V(g_2))$  (where  $V(g)$  denotes the hypersurface defined by  $g$  in  $X^G$ ) and hence  $c^G(X)$  is open in  $X^G$ . In case (b)<sub>1</sub>,  $c^G(X) = X^G$ . This contradicts a previous assumption. Assume therefore that (a)<sub>1</sub> and (b)<sub>1</sub> do not hold.

If  $A$  is an integral domain,  $g_1 \in A^G$  and  $x = ag_1 \in A^G$ , then  $a \in A^G$ .

In this case, one sees easily that  $A^G/(g_1) \rightarrow A/(g_1)$  is injective and hence  $c_1^G$  is dominant.

*Claim (n).* If  $g_1, \dots, g_n \in A^G$ , then

$$c^G: \text{Spec } A^G/(g_1, \dots, g_n) \rightarrow \text{Spec } A/(g_1, \dots, g_n)$$

is injective.

**Proof of Claim (n).** Suppose that  $g_1, g_2, \dots, g_n \in A^G$  and

$$(\gamma) \quad a_1g_1 + a_2g_2 + \dots + a_n g_n = 0$$

with  $a_1, a_2, \dots, a_n \in A$ . We show that there are  $a'_1, a'_2, \dots, a'_n \in A^G$ , not all zero, such that  $a'_1g_1 + a'_2g_2 + \dots + a'_n g_n = 0$ .

If  $f = a_1g_1 + a_2g_2 + \dots + a_{n-1}g_{n-1} \in A^G$ , taking  $a_n = -1$  and  $g_n = f$  above,  $f = a'_1g_1 + \dots + a'_{n-1}g_{n-1}$  for elements  $a'_1, \dots, a'_{n-1} \in A^G$ . This implies that the map  $A^G/(g_1, \dots, g_{n-1}) \rightarrow A/(g_1, \dots, g_{n-1})$  is injective and proves the claim.

Suppose that  $a_1 \notin A^G$ . There is then a  $\sigma \in G$  such that  $a_1^\sigma \neq a_1$ . Condition (ii), Theorem 1, can be applied to determine infinite sequences  $a_j^i$ ,  $i = 0, 1, \dots$ ;  $j = 1, \dots, n$ , of functions such that, for  $j = 1, \dots, n$ ,  $(a_j^i)^\sigma = a_j^i + a_j^{i+1}$  where  $a_j = a_j^0$  and, for some integer  $m(j)$ ,  $a_j^{m(j)}$  is in  $A^G$ . Note that  $a_j^{m(j)+1} = 0$ . We can assume that  $a_j^i$ ,  $i < m(j)$ , is not invariant under  $\sigma$  and that  $m = \max(m(j)) = m(1)$ .

Applying  $\sigma$  to  $\gamma$ , we find that

$$(a_1 + a_1^1)g_1 + (a_2 + a_2^1)g_2 + \dots + (a_n + a_n^1)g_n = 0$$

or using  $\gamma$  that

$$a_1^1g_1 + a_2^1g_2 + \dots + a_n^1g_n = 0.$$

This process can be used  $m$  times to show that

$$a_1^m g_1 + a_2^m g_2 + \dots + a_n^m g_n = 0.$$

Since  $a_1^{m-1}$  is not invariant under  $\sigma$ ,  $a_1^m \neq 0$ . Clearly,  $a_1^m, a_2^m, \dots, a_n^m \in A^G$ . The claim is proven.

Continuing as we have, we obtain at stage  $n$  an affine open subset  $U(g_n)$  of  $\text{Spec } A^G/(g_1, g_2, \dots, g_{n-1})$  from a dominant map  $c_{n-1}^G$  such that

$$U(g_n) \subset c_{n-1}^G(\text{Spec } A/(g_1, g_2, \dots, g_{n-1})) \subset \text{Spec } A^G/(g_1, \dots, g_{n-1}).$$

If

$$(a)_n \quad U(g_n) = c_{n-1}^G(\text{Spec } A/(g_1, \dots, g_{n-1})), \text{ or}$$

$$(b)_n \quad c_{n-1}^G(\text{Spec } A/(g_1, \dots, g_{n-1})) = \text{Spec } A^G/(g_1, \dots, g_{n-1}),$$

we are done. In case  $(a)_n$ ,  $c^G(X) = X^G - (\bigcap_{i=1}^n V(g_i))$ . In case  $(b)_n$  is true, we find that  $(b)_{n-1}$  is true. Assume that  $(a)_n$  and  $(b)_n$  are not true.

Let

$$X_n = (c^G)^{-1} \left( X^G - \left( \bigcap_{i=1}^n V(g_i) \right) \right).$$

Then,  $X_n$  is an open subset of  $X$ ; and because of the nature of our construction, i.e. we assume  $(b)_n$  is not true, if  $X \neq X_n$ , then  $X_n \subsetneq X_{n+1}$ . But, as  $X$  is of finite type over  $k$  and hence satisfies the ascending chain condition on open sets, there is an integer  $N$  such that  $X_N = X$ . It follows that

$$c^G(X) = c^G(X_N) = X^G - \left( \bigcap_{i=1}^N V(g_i) \right). \quad \text{Q.E.D.}$$

**3. Proof of Theorem 1.** According to Mumford [6, p. 8], a categorical quotient  $(c^G(X), c^G)$  to  $\sigma$  and  $P_2$  exists when the following conditions hold:

- (i)  $c^G \circ \sigma = c^G \circ P_2$ .
- (ii)  $\underline{c}_{c^G(X)}$  is the sheaf of invariants of  $c_*^G(\underline{c}_X) |_{c^G(X)}$ .
- (iii) If  $W$  is an invariant closed subset of  $X$ ,  $c^G(W)$  is closed in  $X^G$ ; if  $\{W_i\}$ ,  $i \in I$ , form a set of invariant closed subsets of  $X$ , then

$$c^G\left(\bigcap_{i \in I} W_i\right) = \bigcap_{i \in I} c^G(W_i).$$

Let  $\{U_j\}_{j \in J}$  be a covering of  $c^G(X)$  by affine opens. We employ the proposition below demonstrated in [3].

**Proposition 1.** *In the category of countably generated  $k$  algebras, localization preserves equalizer.*

As a consequence of this proposition, the maps

$$c_j^G: (c^G)^{-1}(U_j) \rightarrow U_j \quad (j \in J)$$

are algebraic quotients of  $(c^G)^{-1}(U_j)$  by  $G$  and surjective. Furthermore, it is not difficult to see that if (i), (ii) and (iii) hold for  $c_j^G$  and  $c_j^G$  is a topological group quotient,  $j \in J$ , (i), (ii) and (iii) hold for  $c^G$  and  $c^G$  is a topological group quotient. Therefore, we can assume that  $c^G(X) = X^G$ .

Having made the last assumption, (i) is obvious. (ii) follows from Proposition 1 and the fact that the defining property of a sheaf is an equalizing diagram. As  $G$  is submersive, (iii) will follow from the next proposition. See Mumford [6, p. 28].

**Proposition 2.** *If the mapping  $c^G: X \rightarrow X^G$  is surjective and submersive and if  $A_i$ ,  $i \in I$ , are the invariant (reduced) ideals in  $A$  corresponding to the  $W_i$  in (iii), then there is an equality*

$$\left(\sum_{i \in I} A_i\right)^{1/2} \cap A^G = \left(\sum_{i \in I} (A_i \cap A^G)\right)^{1/2}.$$

**Proof of Proposition 2.** We have

$$\left(\sum_{i \in I} A_i\right)^{1/2} \cap A^G = \left(\sum_{i \in I} \left(\bigcap_{p \in W_i} p\right)\right)^{1/2} \cap A^G = \left(\bigcap_{p \in \bigcup W_i} p\right) \cap A^G.$$

As  $c^G$  is submersive and surjective,

$$(*) \quad A_i \cap A^G = \bigcap_{p \in W_i} (p \cap A^G).$$

Therefore,

$$\left(\sum_{i \in I} (A_i \cap A^G)\right)^{1/2} = \left(\sum_{i \in I} \left(\bigcap_{p \in W_i} (p \cap A^G)\right)\right)^{1/2} = \bigcap_{p \in \bigcup W_i} (p \cap A^G).$$

The first and last equalities imply the proposition. Note that the intersection in (\*) contains all the prime ideals of  $A^G$  containing  $A_i \cap A^G$ . Otherwise, we cannot conclude the last equality.

To demonstrate that  $c^G$  is a topological quotient and, thus, to complete the proof of Theorem 1, we must show that under the action of  $G$ , distinct orbits are mapped by  $c^G$  into distinct points. The action of  $G$  on  $X$  is closed. Let  $A_1$  and  $A_2$  be the defining ideals of two distinct orbits  $X_1$  and  $X_2$  of  $X$  under  $G$ . Then

$$A^G = (A_1 + A_2)^{1/2} \cap A^G = ((A_1 \cap A^G) + (A_2 \cap A^G))^{1/2}.$$

Hence,  $1 = f + g$ , where  $f, g \in A^G$ ,  $f$  vanishes on  $c^G(X_1)$  and  $g$  vanishes on  $c^G(X_2)$ . It must be the case, then, that  $c^G(X_1) \neq c^G(X_2)$ . Q.E.D.

**4. Unipotent groups are submersive.** Note that every representation of a unipotent group is through a unipotent action. See Borel [1, p. 151]. The fact that unipotent groups are submersive follows, then, immediately from the next proposition.

**Proposition 3.** *Let  $G$  be an algebraic group such that every finite dimensional representation of  $G$  has at least one fixed point. Then  $G$  is submersive.*

**Proof.** Suppose that  $c^G: X \rightarrow X^G$  is surjective where  $X = \text{Spec } A$  and  $X^G = \text{Spec } A^G$ . Consider an invariant closed set  $W$  of  $X$  and let  $J$  be an

invariant ideal whose zeroes equal  $W$ .  $J$  is the union of finite dimensional  $G$  invariant subspaces. See Mumford [6, p. 25]. As every finite dimensional representation of  $G$  has at least one fixed point, clearly  $J$  contains an element  $g_1$  of  $A^G$ .

*Claim.*  $c_1^G: \text{Spec } A/(g_1) \rightarrow \text{Spec } A^G/(g_1)$  is surjective.

*Proof of claim.* If  $q \in \text{Spec } A^G/(g_1)$ ,  $q$  has the form  $p \cap A^G$  as  $c^G$  is surjective. As  $q$  contains  $g_1$ , so does  $p$ . Therefore,  $p \in \text{Spec } A/(g_1)$ .

Repeating this argument with  $c_1^G$  instead of  $c^G$  and  $J/(g_1)$  instead of  $J$ , we obtain a surjective map  $c_2^G: \text{Spec } A/(g_1, g_2) \rightarrow \text{Spec } A^G/(g_1, g_2)$  where  $g_2 \in J$  but  $g_2 \notin (g_1)$ .

Continuing with this process, at stage  $n$ , we obtain a surjective map

$$c_n^G: \text{Spec } A/(g_1, g_2, \dots, g_n) \rightarrow \text{Spec } A^G/(g_1, g_2, \dots, g_n)$$

where  $(g_1, g_2, \dots, g_n) \subset J$  but  $g_i \notin (g_1, g_2, \dots, g_{i-1})$ . As  $X$  is of finite type over  $k$ , there is an integer  $N$  such that  $J = (g_1, g_2, \dots, g_N)$ .

$$c_N^G: \text{Spec } A/(g_1, g_2, \dots, g_N) \rightarrow \text{Spec } A^G/(g_1, g_2, \dots, g_N)$$

is surjective. As  $W = \text{Spec } A/J$  and  $\text{Spec } A^G/(g_1, g_2, \dots, g_N)$  is closed in  $X^G$ ,  $c^G(W)$  is a closed subset of  $X^G$ . Q.E.D.

**5. Proof of Theorem 2.** The direction:  $x$  is semistable implies that  $0$  is not in the closure of the orbit  $O(x')$  of  $x'$  is clear. See Mumford [6, p. 50, Proposition 2.2].

Suppose that  $0$  is not in the closure of the orbit  $O(x')$  of  $x'$ . Let  $W_1$  be the closure of  $O(x')$  and  $W_2$  be  $0$ .  $W_1$  and  $W_2$  are disjoint invariant closed subsets of  $K^{n+1}$ . To prove that  $x$  is semistable and hence to prove Theorem 2, we need to show that for some positive integer  $m$ , there is an  $F \in H^0(\mathbb{P}^n, \mathfrak{O}_{\mathbb{P}^n}(m))$  such that  $F(x) \neq 0$ . First, we consider the affine situation where the following lemma is needed.

**Lemma 3.** *Suppose that the map  $c^G: X \rightarrow X^G$  is submersive. If  $X_1$  and  $X_2$  are two disjoint invariant closed subschemes of  $X$ , there is an element  $f \in \Gamma(X^G, \mathfrak{O}_{X^G})$  which is 1 on  $X_1$  and 0 on  $X_2$ .*

**Proof.** As in the proof of Theorem 1, as  $c^G$  is submersive, applying Proposition 2, one arrives at the equation

$$A^G = (A_1 + A_2)^{1/2} \cap A^G = ((A_1 \cap A^G) + (A_2 \cap A^G))^{1/2}$$



where  $A_1$  and  $A_2$  are the defining ideals of  $X_1$  and  $X_2$ , respectively. Then  $1 = f + g$  where  $f$  vanishes on  $X_2$ ,  $g$  vanishes on  $X_1$  and  $f, g \in A^G$ .

Q.E.D.

Now, as we have seen in the proof of Theorem 1, the map  $c^G$  restricts to mappings of affine opens  $c_j^G: (c^G)^{-1}(U_j) \rightarrow U_j$  where  $\{U_j\}_{j \in J}$  is an open affine covering of  $c^G(X)$ . We examine such maps in our present situation, i.e.,  $X = k^{n+1}$  and  $G$  is a unipotent group acting on  $X$ . Also, we place  $V_j = (c^G)^{-1}(U_j)$ .

Let  $V_j$  be the complement of the hypersurface  $g_j$ . As  $k^{n+1}$  is of affine type over  $k$ , only a finite number of  $V_j, j = 1, \dots, m$ , are needed to cover  $k^{n+1}$ . Pick a point  $Q \in W_1$ . By taking an appropriate linear combination of the  $g_j, j = 1, \dots, m, g_0 = d_1g_1 + d_2g_2 + \dots + d_mg_m$ , one can guarantee that  $0, Q \in V_0$  where  $V_0$  is the complement of the hypersurface  $g_0$ . Thus, also,  $W_1 \cap V_0 \neq \emptyset$ . Proposition 3 implies, as  $G$  is unipotent, that  $c_0^G: V_0 = (c^G)^{-1}(U_0) \rightarrow U_0$  is submersive. By Lemma 3, there is a function  $f$  on  $V_0$  which is one on  $W_1 \cap V_0$  and such that  $f(0) = 0$ .  $f$  has the form  $h/(g_0)^m$  where  $h \in k[X_0, X_1, \dots, X_n], h$  does not vanish on  $W_1 \cap V_0$  and  $h(0) = 0$ .  $h$  is an invariant function as  $f$  and  $g_0$  are invariant functions.  $h$  is, thus, constant on  $O(x')$ . But, then,  $h$  must be constant nonzero on  $W_1$ . Write  $h = h_l + \dots + h_{l+r}$  where  $h_i$  is homogeneous of degree  $i$ . As  $h(0) = 0, l \geq 1$ . For some  $i, l \leq i \leq l+r, h_i$  does not vanish at  $x'$ . Let  $F$  be that  $h_i$ . Then,  $F$  is an invariant homogeneous polynomial such that  $F(x) \neq 0$ .

Q.E.D.

6. Examples of quotients by unipotent actions. In this section  $k$  is the field of complex numbers.

Example 1. Let  $U$  be the unipotent group which acts on  $k^2$  and consists of matrices of the form  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ . The action of  $U$  on  $k^2$  induces an action of  $U$  on lines of the form  $a_0 + a_1X + a_2Y = 0$  in  $k^2$  where  $a_2 \neq 0$ . Under this action, lines which pass through a fixed point on the  $Y$  axis (other than  $X = 0$ ) are identified. Thus, as it is easy to verify  $X^U = c^U(X) = k^1$  where  $X = k^2$  represents the collection of lines where  $a_2 \neq 0$ . Extending the action of  $U$  to  $P^2$ , one sees that  $X^U = P^1$ .

Example 2. Let  $H$  be a hyperplane in  $P^2$  and consider the operation of  $PGL(2)$  on  $|mH|$ . See Mumford [6, p. 79]. The action of  $PGL(2)$  induces an action of the unipotent subgroup  $U$  of  $SL(3)$  whose elements are of the form

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

on  $|mH|$ . Suppose that  $m = 4$ . Then  $|4H|$  consists of polynomials of the form

$$\begin{aligned} & a_{40}X_1^4 + a_{31}X_1^3 + a_{22}X_1^2X_2^2 + a_{13}X_1X_2^3 + a_{04}X_2^4 \\ & + a_{30}X_0X_1^3 + a_{21}X_0X_1^2X_2 + a_{12}X_0X_1X_2^2 + a_{03}X_0X_2^3 \\ & + a_{20}X_0^2X_1^2 + a_{11}X_0^2X_1X_2 + a_{02}X_0^2X_2^2 \\ & + a_{10}X_0^3X_1 + a_{00}X_0^4 + a_{01}X_0^3X_2. \end{aligned}$$

The action of  $U$  on  $|4H|$  sends  $(a_{ij})$  onto

$$\begin{aligned} & (a_{40}, a_{31}, a_{22}, a_{13}, a_{04}, a_{30} + ta_{31}, a_{21} + 2ta_{22}, a_{12} + 3ta_{13}, a_{03} + 4ta_{04}, \\ & a_{20} + t^2a_{22} + ta_{21}, a_{11} + 3t^2a_{13} + 2ta_{12}, a_{02} + 3ta_{03} + 6t^2a_{04}, \\ & a_{10} + t^3a_{13} + t^2a_{12} + ta_{11}, a_{01} + 4t^3a_{04} + 3t^2a_{03} + 2ta_{02}, \\ & a_{01} + t^4a_{04} + t^3a_{03} + t^2a_{02} + ta_{01}). \end{aligned}$$

We apply Theorem 2. It follows from the above representation of the action of  $U$  that the orbit of  $(a_{ij})$  as a point of  $k^{15}$  does not have 0 as a limit point and that condition (ii) of Theorem 1 is satisfied. Therefore, employing Theorem 1, there is a quotient of  $|4H|$  by  $U$  and this quotient is topological. See [6, p. 38].

Note that the quotient of  $|4H|$  by  $SL(3)$  may not exist. See Mumford [6, p. 80]. The chief problem in taking a unipotent quotient is that little is known about the interpretation of unipotent quotients.

**Example 3.** Suppose that we are in the same situation as in Example 2 except let  $U$  now be the full unipotent subgroup of  $SL(3)$ , i.e.,  $U$  consists of all matrices of the form

$$\begin{pmatrix} 1 & w & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of  $U$ , in this case, on  $|4H|$  sends  $(a_{ij})$  into  $(a'_{ij})$  where

$$a'_{40} = a_{40} + ua_{31} + u^2a_{22} + u^3a_{13} + u^4a_{04},$$

$$a'_{31} = a_{31} + 2ua_{22} + 3u^2a_{13} + 4u^3a_{04},$$

$$a'_{22} = a_{22} + 3ua_{13} + 6u^2a_{04},$$

$$a'_{13} = a_{13} + 4ua_{04},$$

$$a'_{04} = a_{04},$$

$$a'_{30} = a_{30} + 4wa_{40} + (t + 3wu)a_{31} + (2ut + 2uw^2)a_{22} \\ + (3u^2t + u^3w)a_{13} + 4u^3ta_{04} + ua_{21} + u^2a_{12} + u^3a_{03},$$

$$a'_{21} = a_{21} + 3wa_{31} + (2t + 4uw)a_{22} + (6ut + 3u^2w)a_{13} \\ + 12u^2ta_{04} + 2ua_{12} + 3u^2a_{03},$$

$$a'_{12} = a_{12} + 2wa_{22} + (3t + 3uw)a_{13} + 12uta_{04} + 3ua_{03},$$

$$a'_{03} = a_{03} + wa_{13} + 4ta_{04},$$

$$a'_{20} = a_{20} + 6w^2a_{40} + (3tw + 3w^2u)a_{31} + (4utw + t^2 + w^2u^2)a_{22} \\ + (3ut^2 + 3u^2t)a_{13} + 6u^2t^2a_{04} + 3wa_{30} + (t + 2wu)a_{21} \\ + (2ut + u^2w)a_{12} + 3u^2ta_{03} + ua_{11} + u^2a_{02},$$

$$a'_{11} = a_{11} + 3w^2a_{31} + (4wt + 2uw^2)a_{22} + (3t^2 + 6utw)a_{13} \\ + 12ut^2a_{04} + 2wa_{21} + (2t + 2uw)a_{12} + 6uta_{03} + 2ua_{02},$$

$$a'_{02} = a_{02} + w^2a_{22} + 3twa_{13} + 6t^2a_{04} + wa_{12} + 3ta_{03},$$

$$a'_{10} = a_{10} + 4w^3a_{40} + (3w^2t + w^3u)a_{31} + (2wt^2 + 2utw^2)a_{22} \\ + (t^3 + 3uwt^2)a_{13} + 4ut^3a_{04} + 3w^2a_{30} + (2wt + uw^2)a_{21} \\ + (t^2 + 2ut)a_{12} + 3t^2ua_{03} + 2wa_{20} + (t + uw)a_{11} + 2uta_{02} + ua_{01},$$

$$a'_{01} = a_{01} + w^3a_{31} + 2w^2a_{22} + 3t^2wa_{13} + 4t^3a_{04} + w^2a_{21} \\ + 2twa_{12} + 3t^2a_{03} + wa_{11} + 2ta_{02},$$

$$\begin{aligned}
 a'_{00} = & a_{00} + w^4 a_{40} + w^3 t a_{31} + w^2 t^2 a_{22} + w t^3 a_{13} + t^4 a_{04} + w^3 a_{30} \\
 & + w^2 t a_{21} + t^2 w a_{12} + t^3 a_{03} + w^2 a_{20} + w t a_{11} + t^2 a_{02} + w a_{10} + t a_{01}.
 \end{aligned}$$

If the orbit of  $(a_{ij})$  has 0 as a limit point, this implies in turn that  $a_{04}$ ,  $a_{13}$ ,  $a_{22}$ ,  $a_{13}$ ,  $a_{40}$ ,  $a_{03}$ ,  $a_{12}$ ,  $a_{21}$ ,  $a_{30}$ ,  $a_{02}$ ,  $a_{11}$ ,  $a_{20}$ ,  $a_{01}$ ,  $a_{10}$ , and, finally,  $a_{00}$  are zero. Thus, again, because of Theorems 1 and 2, the quotient of  $|4H|$  by  $U$  exists in schemes and this quotient is topological.

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