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FIXED POINTS IN REPRESENTATIONS OF CATEGORIES

BY

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ABSTRACT. Fixed points of endomorphisms of representations, i.e. functors into the category of sets, are investigated. A necessary and sufficient condition on a category K is given for each of its indecomposable representations to have the fixed point property. The condition appears to be the same as that found by Isbell and Mitchell for Colim: $\mathrm{Ab}^K \to \mathrm{Ab}$ to be exact. A well-known theorem on mappings of Katětov and Kenyon is extended to transformations of functors.

Introduction. A representation of a category K, i.e. a covariant functor F from K to the category of sets, is said to have the *fixed point property* if each transformation $\tau: F \to F$ has a fixed point, i.e. $x \in FM$ with $\tau^M(x) = x$. The aim of the current paper is to express the fixed point property by algebraic means. Clearly, all representations of a category never have the fixed point property: if F is any representation then the transformation of the sum $F \vee F$ which interchanges copies of F has no fixed points. Therefore it seems natural to try to characterize those categories whose all indecomposable representations have the fixed point property; a representation F is *indecomposable* if it is nontrivial, i.e. distinct from the constant functor to $\emptyset$, and cannot be expressed as $F = F_1 \vee F_2$ with F_i nontrivial. We call these categories *Brouwerian.*

It is our pleasant duty to express our gratitude to Věra Trnková for the attention paid to our work, which she has also initiated by putting the problem of generalization of the theorem on mappings by Katětov and Kenyon (see below).

I. Properties of Brouwerian categories.

Theorem 1.1. *Each indecomposable representation of a Brouwerian category is hereditarily indecomposable, i.e. each of its subfunctors is indecomposable. Equivalently, each two nontrivial subfunctors have a nontrivial intersection.*

Proof. Assume the contrary. Then we have indecomposable representations H, H_1, H_2 with $H_1 \subset H$, $H_2 \subset H$ and $H_1 \cap H_2 = \emptyset$. The colimit of

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the diagram

$$(*) \qquad \begin{tikzcd} H_1 \arrow[r] \arrow[d] & H & H_2 \arrow[l] \arrow[d] \\ H & & H \\ H_2 \arrow[r] \arrow[u] & H & H_1 \arrow[l] \arrow[u] \end{tikzcd}$$

is a functor F

$$\begin{tikzcd} H_1 \arrow[r] \arrow[d] \arrow[dr, "i_7"] & H \arrow[d, "i_0"] & H_1 \arrow[l] \arrow[d] \arrow[dl, "i_1"'] \\ H \arrow[r, "i_6"] & F & H \arrow[l, "i_2"'] \\ H_2 \arrow[r] \arrow[u] \arrow[ur, "i_5"] & H \arrow[u, "i_4"'] & H_2 \arrow[l] \arrow[u] \arrow[ul, "i_3"'] \end{tikzcd}$$

which is indecomposable. Indeed, a representation is indecomposable iff its colimit is a singleton set 1. Now, denote by $D: \mathfrak{D} \longrightarrow \mathrm{Set}^K$ the diagram (*) and define $J: \mathfrak{D} \times K \longrightarrow \mathrm{Set}$ by $J(d, X) = (Dd)X$. Then $F = \mathrm{Colim}_d J(d, -)$ and so $\mathrm{Colim}\, F = \mathrm{Colim}_X \mathrm{Colim}_d J(d, X) = \mathrm{Colim}_d \mathrm{Colim}_X J(d, X)$ and since H, H_1, H_2 are indecomposable, we have $\mathrm{Colim}_d \mathrm{Colim}_X J(d, X) = \mathrm{Colim}_d 1 = 1$.

Because of the symmetry of (*), $\{i_{n+4}\}_{n=0}^{7}$ is also its direct bound (+ is the addition mod 8). Thus, there is a unique transformation $\tau: F \longrightarrow F$ with $\tau \cdot i_n = i_{n+4}$. The transformation has no fixed points. Indeed, F consists of four copies of H glued as follows: the first one with the second one in H_1, the second one with the third one in H_2, the third one with the fourth one in H_1 and the fourth one with the first one in H_2. In particular, the nth copy of H ($n = 1, 2, 3, 4$) is disjoint with the $(n+2)$nd (+ is the addition mod 4). But τ sends the nth copy isomorphically just onto the $(n+2)$nd so that τ cannot have fixed points. This concludes the proof.

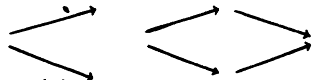
The condition of the above theorem is not sufficient for a category to be Brouwerian, as will be seen from the characterization of Brouwerian categories and

Proposition 1.2. *Given a category K, the following conditions are equivalent.*

(i) *Each indecomposable representation of K is hereditarily indecomposable.*

(ii) *For each indecomposable representation F of K and each* $x \in FM$, $y \in FN$ *there exist morphisms* $f: M \longrightarrow X$, $g: N \longrightarrow X$ *with* $fx = gy$.

(iii) *Each diagram* (a) *in* K *can be embedded into a commutative one* (b):



(a) (b)

Proof. (i) $\to$ (iii). Let f_1: $M \to N_1$, f_2: $M \to N_2$ be given. Let $H_1 \subset \operatorname{Hom}(M, -)$, $H_1X = \{hf_1;\ h\colon N_1 \to X\}$, analogously H_2. As $\operatorname{Hom}(M, -)$ is indecomposable and H_1, H_2 are nontrivial, they are not disjoint and (iii) follows.

(iii) $\to$ (ii). Fix $x \in FM$ and for each object A denote $GA = \{t \in FA$; there exist $f\colon M \to X$, $g\colon A \to X$ with $fx = gt\}$ and $G_2A = FA - G_1A$. (iii) implies that G_1 is a subfunctor of F; in fact, let $h\colon A \to B$, $t \in G_1A$. Then $fx = gt$ for some g, f; due to (iii), there exist g', h' with $g'g = h'h$. Then $(fg')x = (h'h)t$. Thus $ht \in G_1B$. Therefore G_1 is a subfunctor of F. Clearly, G_2 is a subfunctor of F, too: let $h\colon A \to B$, $t \in G_2A$. If $ht \in G_1B$ then $fx = ght$ for some f, g and so $t \in G_1A$ which is a contradiction. Therefore, $ht \in G_2B$.

As F is indecomposable and G_1 is nontrivial ($x \in G_1M$), $F = G_1$.

(ii) $\to$ (i). Let F be an indecomposable representation of K, G_1 and G_2 its disjoint subfunctors. Then G_1 or G_2 is trivial because otherwise there are $x \in G_1M$, $y \in G_2N$ and, given $f\colon M \to X$, $g\colon N \to X$ with $fx = gy = z$ we have $z \in G_1X \cap G_2X$. This concludes the proof. □

The following theorem generalizes a theorem on mappings proved by Katětov [5] and Kenyon [6]: For each set X and each mapping $f\colon X \to X$ without fixed points there exists a decomposition of X into disjoint subsets X_1, X_2, X_3 such that $f(X_i) \cap X_i = \emptyset$, $i = 1, 2, 3$. Our paper was, in fact, initiated by Věra Trnková, who suggested that the above theorem should be generalized for functor-categories.

Theorem 1.3. *A category is Brouwerian iff for each of its representations* F *and each transformation* $\tau\colon F \to F$ *without fixed points there exists a decomposition* $F = F_1 \vee F_2 \vee F_3$ *such that* $\tau(F_i) \cap F_i$ *is trivial,* $i = 1, 2, 3$.

Proof. Let K be Brouwerian, $F\colon K \to \mathrm{Set}$. Each element $x \in FM$ generates an indecomposable subfunctor $H_x \subset F$, $H_xN = \{y;\ y = fx$ for some $f\colon M \to N\}$. Therefore, F is a disjoint union of its maximal indecomposable subfunctors, $F = \bigvee_{j \in J} F_j$. If $\tau\colon F \to F$ then clearly for each $j \in J$ there is $t_j \in J$ with $\tau(F_j) \subset F_{t_j}$. Apply the above theorem on the mapping $i \to t_i$: As K is Brouwerian, if τ has no fixed points, clearly $i \neq t_i$ for all i, and so the set J can be decomposed into subsets J_1, J_2, J_3 as above. Put $F_1 = \bigvee_{j \in J_1} F_j$, analogously F_2, F_3.

Let K not be Brouwerian. Let F be an indecomposable representation of K and let τ: $F \to F$ have no fixed points. F cannot be decomposed into F_1, F_2, F_3 with $\tau(F_i) \cap F_i$ trivial simply because two of the three functors would be trivial, as F is indecomposable.

Theorem 1.4. *Let F be an indecomposable representation of a Brouwerian category. Then every collection $\tau_1, \cdots, \tau_n$ of endomorphisms of F has a common fixed point, i.e. there exists M and $x \in FM$ with $\tau_i^M(x) = x$, $i = 1, \cdots, n$.*

Proof. For each $i = 1, \cdots, n$, let F_i be the subfunctor of F such that for every object M, F_iM is just the set of all fixed points of τ_i^M. By Theorem 1.1, two, and hence finitely many, nontrivial subfunctors of F have always a nontrivial intersection. Thus $\bigcap_{i=1}^n F_i$ is nontrivial so that $\tau_1, \cdots, \tau_n$ have a common fixed point.

The preceding theorem can be regarded as a fixed point theorem for a multiple of transformations. The following theorem is a formulation of the fixed point point property for a multivalued transformation.

Given a representation F of a category K, by a *multitransformation* τ: $F \to F$ we shall mean a partial nonvoid multivalued transformation, that is, a family $\{\tau^M\}$ such that

(1) M runs over all K-objects,

(2) each τ^M is a partial multivalued mapping of FM to FM, i.e., simply $\tau^M \subset FM \times FM$,

(3) if $(x, y) \in \tau^M$ and f: $M \to N$ then $(fx, fy) \in \tau^N$,

(4) some τ^M is nonvoid.

Note that F, equipped with a multitransformation, can be regarded as a functor into the category of graphs and compatible maps. In fact, τ^M is a graph on FM, and condition (3) ensures the compatibility of maps Ff: $FM \to FN$.

We shall need some graph-theoretical notions. Let (X, R) be a graph. A sequence $a_0, a_1, \cdots, a_n$ of its vertices is called a *chain* from a_0 to a_n with length n if for each $i = 1, \cdots, n$ either $(a_{i-1}, a_i) \in R$ or $(a_i, a_{i-1}) \in R$. In the former case put $m_i = 1$, in the latter put $m_i = -1$; then $m = |\Sigma_{i=1}^n m_i|$ is called the *characteristic* of the chain. If $a_0 = a_n$, the chain is called a *cycle*. A chain with pairwise distinct vertices is *regular* if $m = n$, i.e., roughly speaking, if the direction of arrows (a_{i-1}, a_i) is either always the same as in R or the opposite one. Analogously *regular cycle*. Given an infinite sequence a_i of vertices, it is called a chain (a regular chain) if each subsequence $a_0, \cdots, a_n$ is a chain (regular chain).

Theorem 1.5. *A category is Brouwerian iff every multitransformation of each of its indecomposable representations has a cycle with characteristic* 1.

Proof. If a category fulfills the above condition then it is Brouwerian: if τ is a transformation then all its cycles with pairwise distinct vertices are regular and thus a cycle with characteristic 1 is just x, x with x a fixed point of τ.

Let K be a Brouwerian category, $\tau\colon F \longrightarrow F$ a multitransformation of an indecomposable representation F of K. We shall show that τ has a cycle with characteristic 1. Define a congruence $\sim$ on F as follows: if $x, y \in FM$ then $x \sim_M y$ iff there is a chain with characteristic 0 from x to y in τ^M. Clearly for $f\colon M \longrightarrow N$ in K, $x \sim_M y$ implies $fx \sim_N fy$ and so we may define a factor functor of F under $\sim$, $G = F/\sim$. As a factor functor of an idecomposable functor, G is also indecomposable. For each M, let σ^M be the quotient graph of τ^M with respect to $\sim_M$: $\sigma^M = \{([x], [y]);\ (x, y) \in \tau^M\}$ where [ ] denotes the congruence classes. Then $\sigma = \{\sigma^M\}$ is a multitransformation of G. Moreover, σ is single-valued and one-to-one, that is:

(a) for each $t \in GM$, $(t, u) \in \sigma^M$ for at most one $u \in GM$,

(b) for each $u \in GM$, $(t, u) \in \sigma^M$ for at most one $t \in GM$.

To prove (a), assume $(t, u_i) \in \sigma^M$ for $i = 1, 2$. Then we have some $(x_i, y_i) \in \tau^M$ where $[x_i] = t$, $[y_i] = u_i$, $i = 1, 2$. As $[x_1] = [x_2]$, there is a chain $x_1 = a_0, a_1, \cdots, a_n = x_2$ with characteristic 0 in τ^M. Then $y_1, a_0, \cdots, a_n, y_2$ is a chain with characteristic $-1 + 0 + 1 = 0$ from y_1 to y_2, hence $[y_1] = [y_2]$, i.e. $u_1 = u_2$. (b) is analogous.

So we have proved that each component of (GM, σ^M) is either a regular cycle or a regular chain. There are two possibilities:

I. *Some* (GM, σ^M) *contains a regular cycle*, $a_0, \cdots, a_n, a_0$. Let H be the subfunctor of G generated by the set $\{a_0, \cdots, a_n\}$, let ρ be the restriction of σ to H. Then ρ is a transformation with $\rho^{n+1} = 1$. By Theorem 1.1, H is indecomposable, so that ρ has a fixed point, i.e. there is M and $t \in GM$ with $(t, t) \in \rho^M$. We have a chain $b_0, \cdots, b_m$ with $[b_0] = [b_m] = t$ in τ^M, $(b_m, b_0) \in \tau^M$. Then $b_0, \cdots, b_m, b_0$ is a cycle with characteristic 1. That concludes the proof.

II. *No* (GM, σ^M) *contains a cycle*. Then components of any (GM, σ^M) are regular chains. Moreover, there is an object Z such that (GZ, σ^Z) contains a regular chain of length ≥ 2 (e.g., consider a chain f_1x, f_1y, f_2y where $(x, y) \in \sigma^M$ is arbitrary, $f_1\colon M \longrightarrow Z$, $f_2\colon M \longrightarrow Z$ are chosen so that $f_1 x = f_2 y$; see 1.2). Let H be a subfunctor of G generated by a regular chain of length ≥ 2. Let $\approx$ be a congruence on H, $x \approx_M y$ iff there is a chain of an even length from x to y in σ^M. Put $H' = H/\approx$. For each M, let σ_1^M be the quotient graph of σ^M with respect to $\approx_M$. Then components of σ_1^M are regular cycles of length 2 so that $\{\sigma_1^M\}$ is a transformation without fixed points. This is a contradiction because H' is clearly indecomposable; thus case II cannot occur. □

II. Characterization. Recall that a category K is *filtered* [7] if

(1) for every pair M, N of objects there is an object Z with $\mathrm{Hom}(M, Z) \neq \emptyset \neq \mathrm{Hom}(N, Z)$,

(2) for every pair $f_1, f_2 \colon M \longrightarrow N$ of morphisms there is $h \colon N \longrightarrow Z$ with $hf_1 = hf_2$.

We shall say that a category K is *quasifiltered* if it satisfies (1) above and

(2′) for every pair $f_1, f_2 \colon M \longrightarrow N$ of morphisms there are morphisms $h_0, h_1, \cdots, h_n \colon N \longrightarrow Z$ such that

$$\begin{aligned} h_0 f_{i_1} &= h_1 f_{j_1}, \\ h_1 f_{i_2} &= h_2 f_{j_2}, \\ &\cdots \\ h_{n-1} f_{i_n} &= h_n f_{j_n}, \\ h_n f_{i_{n+1}} &= h_0 f_{j_{n+1}}, \end{aligned}$$

$$\begin{tikzcd} M \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & N \arrow[r, bend left=50, "h_0"] \arrow[r, "h_1"] \arrow[r, bend right=20, "h_2"] \arrow[r, bend right=60, "h_n"'] & Z \end{tikzcd}$$

where i_t, j_t are 1 or 2 and $|\Sigma_{t=1}^{n+1} (i_t - j_t)| = 1$.

Let us observe that a category has filtered (quasifiltered) components iff it fulfills (2) ((2′)) and (iii) of 1.2. A filtered category is quasifiltered, of course. The converse fails to be true.

Example (Isbell and Mitchell [4]). The category of finite ordinals and order preserving injections is quasifiltered but not filtered.

Theorem 2.1 (The Characterization Theorem). *A category is Brouwerian iff it has quasifiltered components.*

Proof. *Sufficiency.* Let K have quasifiltered components. Consider a transformation $\tau \colon F \longrightarrow F$ where F is an indecomposable representation of K. We shall prove that τ has a fixed point.

Choose an arbitrary object M with $FM \neq \emptyset$ and choose $a \in FM$. Applying Proposition 1.2 (ii) on a, $\tau^M(a)$ we get K-morphisms $f_1, f_2 \colon M \longrightarrow N$

with $f_1 a = f_2 r^M(a)$. There exist K-morphisms $h_0, h_1, \cdots, h_n, h_{n+1} = h_0$: $N \to Z$ with $h_{t-1} f_{i_t} = h_t f_{j_t}$ where $|\Sigma(i_t - j_t)| = 1$. Denote

$$x_k = (r^M)^k(a); \qquad y_k = (h_0 f_1) x_k \quad \text{and} \quad p_k = \sum_{t=1}^{k} (i_t - j_t)$$

(for $k \le n+1$). The proof will be concluded when we show that for all $k \le n+1$,

$$(+) \qquad (h_k f_1) x_n = y_{n-p_k}.$$

Indeed, then $h_0 = h_{n+1}$ yields $y_n = y_{n+p}$ and since $p_{n+1} = \pm 1$, clearly y_n is a fixed point of r (recall that $r^Z(y_{n-1}^{n+1}) = y_n$ and $r^Z(y_n) = y_{n+1}$).

The proposition (+) holds for $k = 0$; let us prove it for $k+1$ assuming that it holds for $k \le n$. We have $h_k f_{i_{k+1}} = h_{k+1} f_{j_{k+1}}$ which we apply to x_n if $j_{k+1} = 1$, or to x_{n+1} if $j_{k+1} = 2$. Then the proof is very easy when we take into consideration that $p_{k+1} = p_k + i_{k+1} - j_{k+1}$ and that $r^Z(y_{n-1-p_k}) = y_{n-p_k}$ while $r^Z(y_{n-p_k}) = y_{n+1-p_k}$.

Necessity. If K is a Brouwerian category then it satisfies (iii) of 1.2, so that to show that K has quasifiltered components it suffices to prove (2′). Let $f_1, f_2 \colon M \to N$ be given. Define a multitransformation r: Hom$(M, -) \to$ Hom$(M, -)$ by $r^X = \{(hf_1, hf_2); h \colon N \to X\}$. By 1.5 there is a cycle $a_0, a_1, \cdots, a_n, a_0$ with characteristic 1 in some Hom(M, Z). For every $t = 1, \cdots, n$ either $(a_{t-1}, a_t) \in r^Z$ or $(a_t, a_{t-1}) \in r^Z$. Thus there are morphisms $h_0, \cdots, h_n \colon N \to Z$ and numbers $i_1, j_1, i_2, j_2, \cdots, i_{n+1}, j_{n+1} = 1, 2$ such that

$$(a_0, a_1) = (h_0 f_{j_{n+1}}, h_0 f_{i_1}),$$
$$(a_1, a_2) = (h_1 f_{j_1}, h_1 f_{i_2}),$$
$$\cdots$$
$$(a_{n-1}, a_n) = (h_{n-1} f_{j_{n-1}}, h_{n-1} f_{i_n}),$$
$$(a_n, a_0) = (h_n f_{j_n}, h_n f_{i_{n+1}}).$$

Now (2′) follows immediately. The characteristic of the cycle $a_0, \cdots a_n, a_0$ is

$$1 = |(i_1 - j_{n+1}) + (i_2 - j_1) + (i_3 - j_2) + \cdots + (i_n - j_{n-1}) + (i_{n+1} - j_n)|$$
$$= \left| \sum_{t=1}^{n+1} (i_t - j_t) \right|.$$

The necessary and sufficient condition on a small category K to be

Brouwerian turns out to be the same as that for Colim: $\mathrm{Ab}^K \to \mathrm{Ab}$ to be exact (Isbell and Mitchell [4]); Ab is the category of Abelian groups. In fact, Colim: $\mathrm{AB}^K \to \mathrm{Ab}$ is exact iff the category aff K has filtered components [4]. Here aff K denotes the category with the same objects as K such that morphisms from M to N in aff K are those elements $\Sigma\alpha_i f_i$ of the free Abelian group over $\mathrm{Hom}_K(M, N)$ with $\Sigma\alpha_i = 1$. The composition in aff K is defined by $(\Sigma\alpha_i f_i)(\Sigma\beta_j g_j) = \Sigma(\alpha_i\beta_j)f_i g_j$. It follows easily by [4, Lemma 1] that aff K has filtered components iff K has quasifiltered ones. This enables us to formulate a proposition of [4] in terms of Brouwerian categories.

Theorem 2.2. *Let each component C of a category K possess a weak terminal object, i.e. an object T such that* $\mathrm{Hom}(M, T) \neq \emptyset$ *for $M \in C$. Then K is Brouwerian iff it has filtered components.*

Corollary 2.3 [3]. *A monoid is Brouwerian iff it is filtered.*

Let us note that 2.3 follows from [1] too.

It follows immediately from the characterization theorem that a preordered class is a Brouwerian category iff it has directed components. More generally,

Proposition 2.4. *Let K be a category such that for each object M there is a natural number $n(M)$ which is bigger or equal to the number of morphisms from M to any given object. Then K is Brouwerian iff it has filtered components.*

Proof. Let K be a category which is Brouwerian and fulfills the above condition. Let $f, g: M \to N$. We have to find k with $kf = kg$. As K fulfills 1.2 (iii) the following defines a congruence $\sim$ on $\mathrm{Hom}(M, -)$: if $p, q \in \mathrm{Hom}(M, X)$ then $p \sim_X q$ iff $kp = kq$ for some k. Denote $F = \mathrm{Hom}(M, -)/\sim$. Clearly, for any morphism f, the mapping Ff is one-to-one. Moreover, for any object X, $|FX| \leq |\mathrm{Hom}(M, X)| \leq n(M)$. Thus, we can choose among all X with $\mathrm{Hom}(M, X) \neq \emptyset$ such an object C that $|FC|$ is maximal. Let h: $N \to C$. Notice that for each morphism r with domain C, Fr is a bijection. Due to 1.2 (iii), there are $f', g': C \to D$ with $f'hf = g'hg$. We shall show that $Ff' = Fg'$. Then we get $Ff'([hf]) = Fg'([hf])$, i.e. $[f'hf] = [g'hf]$ and so there is k' with $k'g'hf = k'f'hf = k'g'hg$; put $k = k'g'h$; then $kf = kg$.

To prove $Ff' = Fg'$, put $f_1 = f'$ and $f_2 = g'$ and let $h_0, \cdots, h_n$, $h_{n+1} = h_0$ be from condition (2'), i.e. $F(h_t f_{i_{t+1}}) = F(h_{t+1} f_{j_{t+1}})$ where $|\Sigma_{t=1}^{n+1}(i_t - j_t)| = 1$. Then for the morphism $m = Ff_2(Ff_1)^{-1}$ we get:

$$Fh_0 = Fh_{n+1} = Fh_n \cdot m^{(i_{n+1}-j_{n+1})} = Fh_{n-1} \cdot m^{(i_{n+1}-j_{n+1})} \cdot m^{(i_n-j_m)}$$
$$= \cdots = Fh_0 \cdot m^{\Sigma(i_t-j_t)} = Fh_0 m^{\pm 1}.$$

Therefore $Fh_0 = Fh_0 \cdot m$ and since Fh_0 is one-to-one, clearly $m = \mathrm{id}$, i.e. $Ff_1 = Ff_2$.

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