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CHARACTERISTIC PRINCIPAL BUNDLES

BY

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ABSTRACT. Characteristic principal bundles are the duals of commutative twisted group algebras. A principal bundle with locally compact second countable (Abelian) group and base space is characteristic iff it supports a continuous eigenfunction for almost every character measurably in the characters, also iff it is the quotient by Z of a principal E-bundle for every E in $\mathrm{Ext}(G, Z)$ and a measurability condition holds. If a bundle is locally trivial, n.a.s.c. for it to be such a quotient are given in terms of the local transformations and Čech cohomology of the base space. Although characteristic G-bundles need not be locally trivial, the class of characteristic G-bundles is a homotopy invariant of the base space. The isomorphism classes of commutative twisted group algebras over G with values in a given commutative C^*-algebra A are classified by the extensions of G by the integer first Čech cohomology group of the maximal ideal space of A.

1. Introduction. Characteristic principal bundles were introduced in [15] and shown to be precisely the maximal ideal spaces of commutative twisted group algebras, as defined in [3]. In accordance with [1], they can thus also be thought of as the duals of the commutative and separable homogeneous Banach *-algebraic bundles discussed by J. M. G. Fell [5].

We use the definition of principal bundle given in [10], so bundles need not be locally trivial. Let G be a locally compact second countable (lcsc) Abelian group, with identity e and let $G^\wedge$ denote its Pontryagin dual, with identity 1. All groups are presumed Hausdorff. A principal G-bundle X over a lcsc Hausdorff space B is said to be *characteristic* if there is a function F from $G^\wedge \times X$ to the unit circle, measurable on $G^\wedge$ and continuous on X, such that

(1) $F(1, t) = 1$ for all t in X and

(2) $F(\chi, gt) = \chi(g)F(\chi, t)$

for all g in G, t in X and almost all χ in $G^\wedge$. In the terminology of [11], X has eigenfunctions with almost every character as eigenvalue—measurably.

Bundles which are not locally trivial have apparently received little attention from geometers and topologists, but they appear to be of interest for

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functional analysis. Recently, J. M. G. Fell [6] has generalized the results of [15] to obtain a broader class of commutative Banach *-algebraic bundles which have as duals all principal bundles rather than just the characteristic ones. The characteristic bundles remain of particular interest, however, because of the central importance of homogeneous Banach *-algebraic bundles and also because of their connection with twisted group algebras. In this paper we study some functorial aspects of characteristic bundles.

An example of a characteristic bundle which is not locally trivial is given in [15]. Conversely, Fell has pointed out [6] that locally trivial principal bundles need not be characteristic. He observed, for instance, that S^2, regarded as a principal Z_2-bundle over RP^2, cannot be characteristic since, by the Borsuk-Ulam theorem, there can be no antipodes preserving map of S^2 to S^1 and hence (2) cannot be satisfied with χ the nontrivial character of Z_2. In §2 we explore, in more traditional topological terms, precisely which principal bundles are characteristic—particularly in the locally trivial case.

The set of locally trivial principal G-bundles over a given lcsc base space B, denoted $k_G(B)$ in [10], is known to be a homotopy invariant of B and thus to induce a functor on the homotopy classes of such spaces. Although characteristic G-bundles need not be locally trivial we show that the set of such bundles over B depends only on the first Čech cohomology group of B and is thus *a fortiori* a homotopy invariant. This yields some general functorial observations on commutative twisted group algebras and separable homogeneous commutative Banach *-algebraic bundles.

We conclude this section with two general remarks which are sometimes useful in considering special examples of characteristic bundles. The author [15] gave a rather elaborate proof, using Fourier transforms, that characteristic bundles with finite group G are locally trivial. This is an immediate consequence of the following basic remark, which is probably well known to experts, but which does not appear in the standard reference [10].

Remark 1. A principal bundle with discrete group is locally trivial.

Proof. Let X be a principal G-bundle over B with projection π. Suppose X is not locally trivial at $b = \pi(t)$. Then there can be no local cross-section over any neighborhood of b. Let $\{U_\alpha\}$ be the directed family of open neighborhoods of t. Since π is open, $\{\pi(U_\alpha)\}$ is a directed family of open neighborhoods of b. The set U_α will be a local cross-section over $\pi(U_\alpha)$ iff π is 1-1 on U_α. Thus for each U_α we must have points t_α and $\gamma_\alpha t_\alpha$ both in U_α, with $\gamma_\alpha \neq e$. The nets t_α and $\gamma_\alpha t_\alpha$ both converge to t so, by continuity of the translation function r, $r(t_\alpha, \gamma_\alpha t_\alpha) = \gamma_\alpha \to r(t, t) = e$. This is not possible, however, since $\gamma_\alpha \neq e$ and G is discrete.

Remark 2. Any G-space X supporting a function F satisfying conditions (1) and (2) is a principal G-bundle.

Proof. If $gt = t$ for some g and t then, by (2), $\chi(g) = 1$ for almost all χ in $G^{\wedge}$. Thus $g = e$ and X is an effective (i.e. free) G-space. If $G^{\wedge}$ is compact, consider the continuous function

$$U(g, t, t') = \int_{G^{\wedge}} \overline{\chi}(g) \frac{F(\chi, t')}{F(\chi, t)}\, d\chi.$$

By (2),

$$\begin{aligned} U(g, t, \gamma t) &= 0 \quad \text{if } g \neq \gamma, \\ &= 1 \quad \text{if } g = \gamma. \end{aligned}$$

The required translation function (see [10]) on $X^* = \{(t, t') : t = \gamma t \text{ for some } \gamma \text{ in } G\}$ to G can thus be taken to be the support in G of the restriction of U to X^*. If $G^{\wedge}$ is not compact, the same end can be achieved by introducing a sequence of "Féjer kernels" in the integral and taking intersections of supports.

2. **Lifting of principal bundles.** Let E be a lcsc Abelian extension of G by the lcsc Abelian group K, so K is closed in E and G is E/K. Every principal E-bundle X over a base space B is mapped via a bundle morphism over B to the principal G-bundle X/K (see [10]). In this section we will be concerned with the inverse question: When is a principal G-bundle the image of an E-bundle in this fashion? When a principal G-bundle X is such an image we say it can be *lifted* to E and we denote the lifted E-bundle by X_E, so $X = X_E/K$. Although some of the results can be generalized, we are only concerned here with extensions of G by the integers, so $K = Z$.

The set of characters in $G^{\wedge}$ of the form $\chi(g) = \exp(2\pi i\phi(g))$, where ϕ is a homomorphism of G to R, the real numbers, is a closed subgroup of $G^{\wedge}$. We denote the quotient of $G^{\wedge}$ by this subgroup as $\mathrm{Ext}(G, Z)$. This terminology is justified by

Remark 3. The group $\mathrm{Ext}(G, Z)$ can be identified with the isomorphism classes of lcsc Abelian extensions of G by Z.

Proof. Let $\phi(g) = (2\pi i)^{-1} \operatorname{Log} \chi(g)$, where Log denotes the principal branch of the logarithm. Then

$$\alpha(g, \gamma) = \phi(g) + \phi(\gamma) - \phi(g\gamma)$$

is an integer valued symmetric 2-cocycle on G and thus defines an Abelian extension of G by Z (see [7]). Since ϕ is measurable, the extension can be given a locally compact topology using Weil's converse to Haar's theorem as described, for instance, in [3]. If two characters differ by a member of the identity coset of $\mathrm{Ext}(G, Z)$, however, then the resulting ϕ's differ only by an integer-valued function on G and thus determine isomorphic extensions, since the corresponding cocycles are cohomologous.

Conversely, given an extension of G by Z, there is always a measurable cross-section (see [4]) $\phi^\sim$ of G to E such that $\alpha(g, \gamma) = \phi^\sim(g) + \phi^\sim(\gamma) - \phi^\sim(g\gamma)$ is a measurable integer-valued (and hence also real-valued) cocycle. From the structure theory of locally compact Abelian groups, however, any extension of G by R splits and α produces such an extension as in the preceding paragraph. Thus there is a real-valued measurable function ϕ on G such that $\phi(e) = 0$ and $\alpha(g, \gamma) = \phi(g) + \phi(\gamma) - \phi(g\gamma)$. Since α is integer-valued, $\exp(2\pi i\phi)$ is a character of G and ϕ can be chosen to make the character continuous. Obviously, however, ϕ is determined by the above equation only to within a homomorphism of G to R, so the given extension only determines a coset in $\mathrm{Ext}(G, Z)$ rather that a particular character.

Lemma 1. *A characteristic G-bundle can be lifted to any extension of G by Z.*

Proof. Since any extension E of G by Z splits over R, we can write the extension as the subgroup $\{(n + \phi(g), g)\}$ in $R \times G$ where ϕ is as in the converse of the previous remark. Define $\theta(t) = (2\pi i)^{-1} \operatorname{Log} F(\chi, t)$ where χ is $\exp(2\pi i\phi(t))$. Consider the subset $\{(n + \theta(t), t)\}$ in $R \times X$ where X is the given characteristic G-bundle. Let the action of E on this subset be

$$(n + \phi(g), g) \cdot (m + \theta(t), t) = (n + m + \phi(g) + \theta(t), gt)$$
$$= (n + m + \beta(g, t) + \theta(gt), gt)$$

where β is the integer-valued function $\beta(g, t) = \phi(g) + \theta(t) - \theta(gt)$. It is easily verified that this subset of $R \times X$ is a principal E-bundle under this action and since Z is imbedded in E as the set $\{(n, e)\}$, the identification of t with the Z-orbit of $(\theta(t), t)$ shows that it is a lifting of X to E.

Lemma 2. *Let X_E be a lifting of an lcsc Hausdorff principal G-bundle X to an extension E of G by Z and let χ be a member of the coset in* $\mathrm{Ext}(G, Z)$ *corresponding to E. Then there is a continuous eigenfunction $F_\chi(t)$ on X such that $F_\chi(gt) = \chi(g)F_\chi(t)$.*

Proof. We note that X_E is a principal Z-bundle over X and by Remark 1 it is locally trivial. It thus has a locally trivial prolongation to a principal R-bundle over X (see [10, 2.2]). However, every locally trivial principal R-bundle over an lcsc space is trivial (by [16, 12.3]), so we can identify X_E with a subset $\{(n + \theta(t), t)\}$ of $R \times X$, where $\theta(t)$ is continuous modulo Z

(see [16, 9.5] or [10, 2.31]). Let $\phi^\sim$ be a measurable cross-section of E over G. Then the action of $m + \phi^\sim(g)$ on this realization of X_E can be written as

$$(m + \phi^\sim(g)) \cdot (n + \theta(t), t) = (n + m + \beta(g, t) + \theta(gt), gt)$$

where β is integer-valued. Associativity of this group action yields

$$\phi^\sim(g) + \phi^\sim(\gamma) - \phi^\sim(\gamma g) = \beta(g, t) + \beta(\gamma, gt) - \beta(\gamma g, t)$$

for all t. However, the left-hand side of this equation must be $\alpha(g, \gamma)$, a cocycle determining the extension and, as in the proof of Remark 3, we can find a real-valued ϕ satisfying the same relationship such that $\exp(2\pi i\phi) = \chi$. Then $\phi(g) + \theta(t) - \theta(gt)$ is an integer, and defining $F_\chi = \exp(2\pi i\theta)$ we have the desired eigenfunction.

We will say a family of liftings, as in Lemma 2, is measurable if the functions θ can be chosen to depend measurably on $\operatorname{Ext}(G, Z)$. We thus have

Theorem 1. *A principal G-bundle is characteristic iff it has a measurable family of liftings over* $\operatorname{Ext}(G, Z)$.

This theorem makes it easy to decide, for instance, that S^2 is not a characteristic Z_2-bundle over RP^2 since it is evident geometrically that S^2 cannot be lifted to the nontrivial extension of Z_2 by Z.

The proofs of the lemmas make it apparent that the result can be cast in more general terms and extended to a cohomology theory of lifting of transformation groups, less dependent on particular realizations. Adequate development of this theory would lead far afield from our present purpose, however, and will be presented in a separate paper. The result which is relevant here can be read off from the proofs of the lemmas: We must consider two cohomology theories on G. In the first, the action of G on a Z-valued function on G is trivial. In the second, the left action of G on a Z-valued function on X is translation and the right action is trivial. The relationship $\alpha(g, \gamma) = \beta(g, t) + \beta(\gamma, gt) - \beta(\gamma g, t)$ says that every 2-cocycle in the first theory is a 2-coboundary in the second and thus cohomologous to zero. Our proofs above use the triviality of the prolongations from Z to R to bypass these considerations.

When the bundle X is locally trivial the problem is considerably simplified. Lemma 2 admits a more direct proof and the techniques of the proof reduce determination of whether X can be lifted to E to a computation involving the structure of G, the local transformations, and the Čech cohomology of the base space, B.

Alternative proof of Lemma 2 (X locally trivial over B). We note that X_E and X have the common base space B, a local cross-section on X_E

determines one on X, and X_E is locally trivial if X is. Let U_j be a locally-finite open covering of B such that $X_E|U_j$, and hence $X|U_j$, is trivial and let $\{u_j\}$ be a partition of unity subordinated to $\{U_j\}$. Let $\{\psi_j\}$ be a collection of local cross-sections of X_E over $\{U_j\}$. Define $f_j(t) = n + \phi(g)$ for t in $(n + \phi(g), g) \cdot \psi_j$, where E is imbedded in $R \times G$ as before. For all t in X_E, let $f(t) = \Sigma f_j(t)u_j(p(t))$ where p is the projection of X_E on B. Then

$$f((n + \phi(g), g) \cdot t) = \sum (n + \phi(g) + f_j(t))u_j(p(t)) = n + \phi(g) + f(t)$$

so $F = \exp(2\pi i f)$ is well defined on X and has the desired properties (1) and (2). Moreover F clearly depends measurably on the character $\chi = \exp(2\pi i\phi)$.

Suppose we attempted to carry out the construction of F along the preceding lines, but without directly invoking the existence of the lifted bundle X_E. Let $\psi_j^{\sim}$, in this case, be a cross-section of $X|U_j$. Suppose first that $\chi = \exp(2\pi i\phi)$ where ϕ is a real-valued homomorphism of G. Define f_j on $X|U_j$ by

$$f_j(t) = \phi(\tau(t, \psi_j^{\sim}(p(t)))) = \phi(\tau_j(t))$$

where p is projection, τ is the translation function of X, and $\tau_j(t) = \tau(t, \psi_j^{\sim}(p(t)))$. Let $\{u_j\}$ be a partition of unity subordinated to $\{U_j\}$. Then

$$F(t) = \exp\left(2\pi i \sum f_j(t)u_j(p(t))\right)$$

has the desired properties, since

$$F(gt) = \exp\left(2\pi i \sum(\phi(g) + f_j(t))u_j(p(t))\right) = \chi(g)F(t).$$

If ϕ is not a homomorphism, on the other hand, we have

$$\chi(g)F(t) = F(gt)\exp\left[2\pi i \sum \alpha(g, \tau_j(t))u_j(p(t))\right]$$

where $\alpha(g, \gamma) = \phi(g) + \phi(\gamma) - \phi(g\gamma)$ and the desired relation is satisfied iff $\Sigma\, \alpha(g, \tau_j(t))u_j(p(t))$ is always an integer. However, $\phi(g)$ is only determined by $\chi(g)$ to within an integer, so we can add an integer-valued function $n_j(t)$ to $f_j(t)$. Doing so, calculation shows that the relation $F(gt) = \chi(g)F(t)$ will hold provided the functions $n_j(t)$ can be chosen to satisfy

$$\alpha(g, \tau_j(t)) - \alpha(g, \tau_k(t)) = [n_j(gt) - n_j(t)] - [n_k(gt) - n_k(t)]$$

whenever U_j and U_k intersect. (Note that the alternative proof of Lemma 2 furnishes us with functions n_j when we are given a function F satisfying (1) and (2).)

We can identify $X|U_j$ with $G \times U_j$ by the mapping $t \to (\tau_j(t), p(t))$. Local transformations are defined by $g_{jk}\tau_k(t) = \tau_j(t)$ for t in $U_j \cap U_k$. Thus the local triviality of X allows us to focus on the action of G on itself. We are concerned, then, with two actions of G on the measurable Z-valued functions on G. The first is the usual trivial action and the second is left translation on the left and trivial on the right. Each action defines a Z-valued cohomology theory on G. For instance, in the first theory the coboundary of a Z-valued function is given by $(\delta f)(\gamma, g) = f(\gamma) - f(\gamma g) + f(g)$ while in the second theory $(\partial f)(\gamma, g) = f(\gamma g) - f(g)$. (Note that the first expression is a 2-coboundary and the second is a 1-coboundary.) Let $Z^2(G)$ denote the 2-cocycles on $G \times G$ under the first theory, i.e. measurable integer-valued functions f on $G \times G$ such that $\delta f = 0$. Let $\mathfrak{B}^1(G)$ be the coboundaries in the second theory i.e. functions on $G \times G$ of the form ∂f. The structure of these groups $Z^2(G)$ and $\mathfrak{B}^1(G)$ depends only on G. The structure of the bundle X, on the other hand, is specified by the G-valued Čech 1-cocycle $\{g_{jk}\}$. The function $\alpha(g, r_j(t))$ defines a Čech 0-cochain taking values in the measurable Z-valued functions on $G \times G$. The structure of this Čech 0-cochain is determined by the choice of α in $Z^2(G)$ and by the structure of the bundle through relations of the form $\alpha(g, r_j(t)) = \alpha(g, g_{jk} r_k(t))$. On the other hand, the functions $n_j(g r_j(t)) - n_j(r_j(t))$ also define arbitrary Čech cochains with values in $\mathfrak{B}^1(G)$. These observations yield

Theorem 2. *A locally trivial principal G-bundle X can be lifted to an extension E of G by Z iff the Čech coboundary of the cochain defined by $\alpha(g, r_j(t))$, where α is an integer-valued cocycle determining the extension E, is equal to the Čech coboundary of a $\mathfrak{B}^1(G)$-valued Cech 0-cochain.*

This theorem reduces the determination of whether a bundle can be lifted to an extension to a direct computation, especially when the group G is finite and the base space is triangulable. It is also a convenient tool for constructing examples of noncharacteristic principal bundles.

3. Classification of principal bundles. In the preceding section the lifting of locally trivial bundles was found to depend specifically on the Čech cohomology of the base space. In this section we will show that this dependence persists for characteristic bundles which are not locally trivial. It is well known that the class of locally-trivial principal G-bundles over a lcsc Hausdorff base space B is a homotopy invariant of B, denoted $k_G(B)$ in [10]. We show here that the class of characteristic G-bundles—whether locally trivial or not—depends only on the first Čech cohomology group of B. We will be concerned with continuous functions on the Stone-Čech compactification of B, so we will assume throughout that B is compact with the understanding that it is to be replaced by its compactification when this is not the case.

It was shown in [15] that a characteristic principal G-bundle is the dual of a commutative twisted group algebra $L^1(A, G^\wedge; \alpha)$ where B is the maximal ideal space of the separable commutative Banach*-algebra A and α is a measurable symmetric cocycle on $G^\wedge$ taking values in the unitary double centralizers of A. Thus α has a Gel'fand representation as a cocycle on $G^\wedge$ with values defined a.e. in the circle-valued functions on (the Stone-Čech compactification of) B. Two cocycles α and α' define isomorphic twisted group algebras iff they differ by a coboundary

$$\alpha'(\chi, \eta)/\alpha(\chi, \eta) = \psi(\chi)\psi(\eta)/\psi(\chi\eta),$$

where ψ is a measurable function from $G^\wedge$ to the unitary double centralizers of A (see [3] or [15]). (Since we are assuming B compact, we can take the double centralizers to be identical to A rather than adjoining continuous functions on the compactification.)

Let $C(B)$ denote the continuous complex-valued functions on B, $R(B)$ the real-valued, $R^+(B)$ the positive real-valued functions, and $U(B)$ the unit circle-valued functions in $C(B)$. For any commutative Banach algebra A, let A^{-1} denote the group of units of A and $\exp(A)$ the subgroup of A^{-1} obtained by applying the exponential function to A. By the Arens-Royden theorem, we know that there are isomorphisms

$$C^{-1}(B)/\exp(C(B)) = A^{-1}/\exp(A) = \mathcal{H}^1(B)$$

where $\mathcal{H}^1(B)$ is the first Čech cohomology group of B with integer coefficients (see [8]). For simplicity we will assume below that $A = C(B)$, since any characteristic principal G-bundle arises as the maximal ideal space of $L'(A, G^\wedge; \alpha)$ for such an A.

We note that $U(B) = C^{-1}(B)/R^+(B)$ so we also have an isomorphism

$$\mathcal{H}^1(B) = U(B)/[\exp(C(B))/\exp(C(B)) \cap R^+(B)].$$

A cocycle $\alpha: G^\wedge \times G^\wedge \to U(B)$ thus induces a cocycle $\alpha^-: G^\wedge \times G^\wedge \to \mathcal{H}^1(B)$. (We can give $\mathcal{H}^1(B)$ the discrete quotient topology induced by the above isomorphisms, so that α^- will also be a Borel function.)

Lemma 3. *If two symmetric Borel cocycles α_1 and α_2 induce the same $\mathcal{H}^1(B)$-valued cocycle, $\alpha_1^- = \alpha_2^-$, then the twisted group algebras $L^1(A, G^\wedge; \alpha_1)$ and $L^1(A, G^\wedge; \alpha_2)$ are isomorphic.*

Proof. We note that $\alpha = \alpha_1/\alpha_2$ is also a cocycle and α^- is the identity in $\mathcal{H}^1(B)$. Thus α has its values (a.e.) in $\exp(C(B))/[R^+(B) \cap \exp(C(B))]$, i.e. the circle-valued functions on B with continuous logarithms. Let $\beta = (2\pi i)^{-1}\operatorname{Log}(\alpha)$ where Log denotes a logarithm continuous on B. Let f_b denote the evaluation of the function f at b in B. Then β is an additive cocycle modulo Z, i.e.

$$\beta_b(\chi, \eta) + \beta_b(\theta, \chi\eta) - \beta_b(\theta, \chi) - \beta_b(\theta\chi, \eta) = K_b(\theta, \chi, \eta)$$

is an integer.

For each b, K_b is a measurable integer-valued 3-cocycle representing a member of $\operatorname{Ext}^2(G, Z)$. Every sclc Abelian group extension, however, can be regarded as a characteristic principal bundle (see [3], [15]) and Lemma 1, with a minor modification to provide the multiplication in the lifted bundle, shows that $\operatorname{Ext}^2(G, Z)$ is trivial. (For a more general result, cf. [18].) Thus K_b must be the coboundary of some integer-valued measurable function $\beta_b^\sim$ on $G^\wedge \times G^\wedge$, so that $\beta_b - \beta_b^\sim$ is a real-valued 2-cocycle. Since β_b is a continuous function of b, K_b is a continuous integer-valued function on B. But $\beta_b^\sim$ can be taken to be constant on the subsets of B where K is constant and hence will also be continuous. Thus (by substituting $\beta - \beta^\sim$ if necessary) we can assume β_b to be a real-valued cocycle.

Let $C^1(G^\wedge, R)$ denote the group of real-valued measurable functions on $G^\wedge$ with the topology of local convergence in measure, $\operatorname{Hom}(G^\wedge, R)$ the subgroup of continuous real-valued homomorphisms of $G^\wedge$, and $B^2(G^\wedge, R)$ the group of real coboundaries—real-valued measurable functions on $G^\wedge \times G^\wedge$ of the form

$$\beta(\chi, \eta) = \delta\phi(\chi, \eta) = \phi(\chi) + \phi(\eta) - \phi(\chi\eta),$$

for ϕ in $C^1(G^\wedge, R)$—again with the topology of local convergence in measure. These are metrizable topological groups and a minor modification of the arguments of [15] shows that $B^2(G^\wedge, R)$ is topologically isomorphic to the quotient of $C^1(G^\wedge, R)$ by $\operatorname{Hom}(G, R)$. By the structure theorem for sclc Abelian groups (cf. [13]) and the triviality of real-valued homomorphisms on compact and torsion groups, $\operatorname{Hom}(G, R)$ is isomorphic to a countable product of R. By [19, Corollary 7.3], therefore, there is a continuous-cross-section from $B^2(G^\wedge, R)$ to $C^1(G^\wedge, R)$ and by a slight modification of the argument of [15, Lemma 8] the cross-section can be chosen to preserve a.e. convergence of sequences. Now β is a mapping of the separable space B into $B^2(G^\wedge, R)$ which carries convergent sequences into sequences which converge a.e. and by composing β with the cross-section we obtain a map of B to $C^1(G^\wedge, R)$ which carries convergent sequences to a.e. convergent sequences. Let ϕ_b be the image of b under this mapping. Then $\psi_b(\chi) = \exp(2\pi i\phi_b(\chi))$ defines a.e. a continuous function with coboundary α. Thus α_1 and α_2 are cohomologous and define isomorphic twisted group algebras.

Let A be any separable commutative Banach*-algebra and let B be the Stone-Čech compactification of its maximal ideal space. For any locally compact Abelian topological groups K, let $\mathrm{Ext}(G, K)$ be the group of symmetric measurable K-valued 2-cocycles on the lcsc Abelian group G modulo the coboundaries. Let $\mathcal{H}^1(B)$ be the first integer-valued Čech cohomology group of B, with the discrete topology. We summarize our results:

Theorem 3. *There is a* 1-1 *correspondence between*

(a) $\mathrm{Ext}(G^\wedge, \mathcal{H}^1(B))$, *modulo the trivial automorphisms induced by homeomorphisms of* B *and automorphisms of* $G^\wedge$;

(b) *the isomorphism classes of commutative twisted group algebras on* $G^\wedge$ *with values in* $C(B)$;

(c) *the isomorphism classes of separable homogeneous commutative Banach**-*algebraic bundles over* $G^\wedge$ *with* $C(B)$ *as fibre over the identity*;

(d) *the characteristic principal* G-*bundles over* B;

(e) *the principal* G-*bundles over* B *which can be lifted measurably to every integer extension of* G *by* Z.

Proof. The correspondence of (a) and (b) is the preceding lemma, that of (b) and (c) follows from results of [1], that of (b) and (d) from [15] and [2] and that of (e) and (d) from Theorem 1.

The above theorem provides a controvariant functor from the homotopy classes of sclc Hausdorff spaces to the principal G-bundles which does not depend on an assumption of local triviality and also a covariant functor on commutative Banach*-algebras to twisted group algebras. In general, it should be noted, twisted group algebras may have isomorphisms which are not special—i.e. which do not maintain the algebra and the group invariant. These correspond to homeomorphisms of the related characteristic bundles which are not bundle morphisms. For the case $A = C(B)$, however, it is shown in [2] that all isomorphisms are special.

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