

MUTUAL EXISTENCE OF PRODUCT INTEGRALS
 IN NORMED RINGS

BY

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ABSTRACT. Definitions and integrals are of the subdivision-refinement type, and functions are from $R \times R$ to N , where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. If G is a function from $R \times R$ to N , then $G \in OM^*$ on $[a, b]$ only if (i) $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$ and (ii) if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leq p < q \leq n$, then

$$\left| \prod_{x_p}^{x_q} (1 + G) - \prod_{i=p+1}^q (1 + G_i) \right| < \epsilon;$$

and $G \in OM^\circ$ on $[a, b]$ only if (i) $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$ and (ii) the integral $\int_a^b |1 + G - \prod(1 + G)|$ exists and is zero. Further, $G \in OP^\circ$ on $[a, b]$ only if there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 < p \leq q \leq n$, then $|\prod_{i=p}^q (1 + G_i)| < B$.

If F and G are functions from $R \times R$ to N , $F \in OP^\circ$ on $[a, b]$, each of $\lim_{x, y \rightarrow p^+} F(x, y)$ and $\lim_{x, y \rightarrow p^-} F(x, y)$ exists and is zero for $p \in [a, b]$, each of $\lim_{x \rightarrow p^+} F(p, x)$, $\lim_{x \rightarrow p^-} F(x, p)$, $\lim_{x \rightarrow p^+} G(p, x)$ and $\lim_{x \rightarrow p^-} G(x, p)$ exists for $p \in [a, b]$, and G has bounded variation on $[a, b]$, then any two of the following statements imply the other:

(1) $F + G \in OM^*$ on $[a, b]$, (2) $F \in OM^*$ on $[a, b]$, and (3) $G \in OM^*$ on $[a, b]$.

In addition, with the same restrictions on F and G , any two of the following statements imply the other:

(1) $F + G \in OM^\circ$ on $[a, b]$, (2) $F \in OM^\circ$ on $[a, b]$, and (3) $G \in OM^\circ$ on $[a, b]$.

The results in this paper generalize a theorem contained in a previous paper by the author [Proc. Amer. Math. Soc. 42 (1974), 96–103]. Additional background on product integration can be obtained from a paper by B. W. Helton [Pacific J. Math. 16 (1966), 297–322].

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All definitions are of the subdivision-refinement type, and functions are from $R \times R$ to N , where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. Functions are assumed to be defined only for elements $\{x, y\}$ of $R \times R$ such that $x < y$.

If G is a function from $R \times R$ to N , then $\int_a^b G$ exists only if there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D , then $|L - \sum_{i=1}^n G_i| < \epsilon$, where $G_i = G(x_{i-1}, x_i)$. Similarly, ${}_a\Pi^b(1 + G)$ exists only if there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D , then $|L - \Pi_{i=1}^n(1 + G_i)| < \epsilon$.

The statements that G is bounded on $[a, b]$, $G \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D , then

- (1) $|G_i| < B$ for $1 \leq i \leq n$,
- (2) $|\Pi_{i=p}^q(1 + G_i)| < B$ for $1 \leq p \leq q \leq n$, and
- (3) $\sum_{i=1}^n |G_i| < B$,

respectively.

Let $G(p, p^+)$, $G(p^+, p^+)$, $G(p^-, p)$ and $G(p^-, p^-)$ represent $\lim_{x \rightarrow p^+} G(p, x)$, $\lim_{x, y \rightarrow p^+} G(x, y)$, $\lim_{x \rightarrow p^-} G(x, p)$ and $\lim_{x, y \rightarrow p^-} G(x, y)$, respectively. Now, $G \in S_1$ on $[a, b]$ only if $G(p^+, p^+)$ exists and is zero for $a \leq p < b$ and $G(p^-, p^-)$ exists and is zero for $a < p \leq b$; and $G \in S_2$ on $[a, b]$ only if $G(p, p^+)$ exists for $a \leq p < b$ and $G(p^-, p)$ exists for $a < p \leq b$. Further, $G \in OL^\circ$ on $[a, b]$ only if $G(p, p^+)$ and $G(p^+, p^+)$ exist for $a \leq p < b$ and $G(p^-, p)$ and $G(p^-, p^-)$ exist for $a < p \leq b$.

For additional background on product integration, the reader is referred to papers by P. R. Masani [10], J. S. MacNerney [9], B. W. Helton [2] and the author [7].

Suppose F and G are functions on $R \times R$. If $\int_a^b F$ exists and $\int_a^b G$ exists, then it is easily shown that $\int_a^b (F + G)$ exists. However, if ${}_x\Pi^y(1 + F)$ and ${}_x\Pi^y(1 + G)$ exist for $a \leq x < y \leq b$, it does not necessarily follow that ${}_x\Pi^y(1 + F + G)$ exists for $a \leq x < y \leq b$. The purpose of this paper is to investigate the existence of such product integrals. In particular, with suitable restrictions on the functions involved, we interrelate the existence of ${}_x\Pi^y(1 + F)$, ${}_x\Pi^y(1 + G)$ and ${}_x\Pi^y(1 + F + G)$. However, before stating our results, we need several additional definitions.

First, $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$. Second, $G \in OM^\circ$ on $[a, b]$ only if ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$ and $\int_a^b |1 + G - \Pi(1 + G)| = 0$. Third, $G \in OM^*$ on $[a, b]$ only if (1) ${}_x\Pi^y(1 + G)$

exists for $a \leq x < y \leq b$, and (2) if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leq p < q \leq n$, then

$$\left| \underset{x}{\prod}_p^x (1 + G) - \prod_{i=p+1}^q (1 + G_i) \right| < \epsilon.$$

We now state the main results of this paper.

Theorem 1. *If F and G are functions from $R \times R$ to N , F is in OP° and $S_1 \cap S_2$ on $[a, b]$ and G is in OB° and S_2 on $[a, b]$, then any two of the following statements imply the other:*

- (1) $F + G \in OM^*$ on $[a, b]$,
- (2) $F \in OM^*$ on $[a, b]$, and
- (3) $G \in OM^*$ on $[a, b]$.

Theorem 2. *If F and G are functions from $R \times R$ to N , F is in OP° and $S_1 \cap S_2$ on $[a, b]$ and G is in OB° and S_2 on $[a, b]$, then any two of the following statements imply the other:*

- (1) $F + G \in OM^\circ$ on $[a, b]$,
- (2) $F \in OM^\circ$ on $[a, b]$, and
- (3) $G \in OM^\circ$ on $[a, b]$.

Theorems 1 and 2 are not the same. A function can belong to OM^* on $[a, b]$ without belonging to OM° on $[a, b]$. For example, if $G \in OB^\circ$ on $[a, b]$ and $\underset{x}{\prod}^y (1 + G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$ [7, Theorem 1]; but, it is possible to construct a function G such that $G \in OB^\circ$ on $[a, b]$, $\underset{x}{\prod}^y (1 + G)$ exists for $a \leq x < y \leq b$ and $G \notin OM^\circ$ on $[a, b]$ [4, pp. 153–154]. However, if G is in OM° and OP° on $[a, b]$, then $G \in OM^*$ on $[a, b]$.

Theorem 2 is proved for functions from $R \times R$ to R in a previous paper by the author [6, Theorem 1, p. 101]. However, that proof relies heavily on the commutativity of R and thus is not the same as the proof presented in this paper. In this presentation, the lack of commutativity is handled by using a series representation for products.

The classes OM^* and OM° are not as restricted as may initially appear. As noted before, if $G \in OB^\circ$ on $[a, b]$ and $\underset{x}{\prod}^y (1 + G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$ [7, Theorem 1]. For another example, suppose

$$F(x, y) = \begin{bmatrix} 0 & 0 \\ h(y) - h(x) & 0 \end{bmatrix}$$

for $a \leq x < y \leq b$, where h is a quasi-continuous function from R to N . Then, with a suitable norm, F is in OP° , OM° and $S_1 \cap S_2$ on $[a, b]$. Thus, F

satisfies the hypotheses of Theorems 1 and 2; however, it does not necessarily follow that $F \in OB^\circ$ on $[a, b]$. With Theorems 1 and 2 and functions such as F , it is possible to construct many functions in OM^* and OM° . A fundamental correspondence exists between sum and product integrals. In particular, if $G \in OB^\circ$ on $[a, b]$, then $\int_a^b G$ exists if and only if $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$ [7, Theorem 4], and $G \in OA^\circ$ on $[a, b]$ if and only if $G \in OM^\circ$ on $[a, b]$ [2, Theorem 3.4, p. 301]. If G is a function from $R \times R$ to R , then $G \in OA^\circ$ on $[a, b]$ [8, p. 669]. Further, there exist other systems such that the existence of $\int_a^b G$ is sufficient to imply that $G \in OA^\circ$ on $[a, b]$ [1, Theorem 2, p. 155], [2, Theorem 4.1, p. 304]. Thus, with the preceding results, many functions in OM^* and OM° can be obtained. In addition, if $H \in OL^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then $\prod_x^y (1 + HG)$ exists for $a \leq x < y \leq b$ if $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$ [7, Theorem 5], and $HG \in OM^\circ$ on $[a, b]$ if $G \in OM^\circ$ on $[a, b]$ [3, Theorem 2, p. 494]. Therefore, there exist many functions to which the theorems of this paper apply.

We now establish Theorem 1. Several lemmas are needed.

Lemma 1. *If H and G are functions from $R \times R$ to N , $H \in OL^\circ$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and either $\int_a^b G$ exists or $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and $\prod_x^y (1 + HG)$ and $\prod_x^y (1 + GH)$ exist for $a \leq x < y \leq b$ [7, Theorem 5].*

Lemma 2. *If f is a function from R to R such that $(LR) \int_a^b (-df) f^{n-1} f^i$ exists for $i = 0, 1, \dots, n$, then*

$$\sum_{i=0}^n (LR) \int_a^b (-df) f^{n-1} f^i = f^{n+1}(a) - f^{n+1}(b).$$

Proof. This result follows by applying the identity

$$(r - s) \sum_{i=0}^n r^{n-i} s^i = r^{n+1} - s^{n+1}$$

to the approximating sums of the integrals involved.

Lemma 3. *If $\{F_i\}_{i=m}^n$ and $\{G_i\}_{i=m}^n$ are sequences of elements of N , then*

$$\prod_{i=m}^n (1 + F_i + G_i) = \sum_{i=0}^{n+1-m} S_{imn},$$

where

$$S_{0pn} = \begin{cases} \prod_{j=p}^n (1 + F_j) & \text{if } 0 < p \leq n, \\ 1 & \text{if } p > n, \end{cases}$$

and

$$S_{ipn} = \begin{cases} \sum_{j=p}^n [\prod_{k=p}^{j-1} (1 + F_k)] G_j S_{i-1, j+1, n} & \text{if } 0 < p \leq n, \\ 0 & \text{if } p > n \end{cases}$$

for $i = 1, 2, \dots$

Proof. This lemma can be established by induction.

Lemma 4. *If F and G are functions from $R \times R$ to N , $F \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then there exist a subdivision D of $[a, b]$, a number B and a positive nondecreasing function g defined on $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D , j is a nonnegative integer and $0 < p \leq q \leq n$, then*

$$(\dagger) \quad |S_{jpq}| \leq B^{j+1} [g(x_q) - g(x_{p-1})]^{j/j!},$$

where S_{jpq} is defined in Lemma 3.

Proof. Since $F \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D , then

- (1) $|\prod_{i=p+1}^q (1 + F_i)| < B$ for $0 \leq p < q \leq n$, and
- (2) $\sum_{i=1}^n |G_i| < B$.

Let g be the function defined on $[a, b]$ such that

- (1) $g(a) = 1$, and
- (2) $g(x) = 1 + \text{lub} \{ \sum_j |G_j| : J \text{ a refinement of } \{x_i\}_{i=0}^{p-1} \cup \{x\} \}$, where $0 < p \leq n$ and $x_{p-1} < x \leq x_p$.

Thus, g is a positive nondecreasing function.

We use induction to establish the desired inequality. If $\{x_i\}_{i=0}^n$ is a refinement of D and $0 < p \leq q \leq n$, then

$$|S_{0pq}| = \left| \prod_{i=p}^q (1 + F_i) \right| \leq B.$$

Thus, the inequality is true for $j = 0$.

Suppose the inequality holds for the nonnegative integer j . That is, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 < p \leq q \leq n$, then (\dagger) holds.

We now establish that the inequality also holds for $j + 1$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D and $0 < p \leq q \leq n$. To simplify notation in the following manipulations, let

$$f(v) = g(x_q) - g(v)$$

for $x_p \leq v \leq x_q$. Now,

$$\begin{aligned}
 |S_{j+1,p,q}| &= \left| \sum_{i=p}^q \left[\prod_{k=p}^{i-1} (1 + F_k) \right] G_i S_{j,i+1,q} \right| \\
 &\leq B \sum_{i=p}^q |G_i| |S_{j,i+1,q}| \\
 &\leq B \sum_{i=p}^q \{g(x_i) - g(x_{i-1})\} \{B^{j+1}[g(x_q) - g(x_i)]^{j/j!}\} \\
 &\leq B \int_{x_{p-1}}^{x_q} dg \{B^{j+1}[g(x_q) - g(v)]^{j/j!}\} \\
 &= [B^{j+2}/j!] \left[(R) \int_{x_{p-1}}^{x_q} (-df) f^j \right] \\
 &\leq [B^{j+2}/(j+1)!] \sum_{k=0}^j (LR) \int_{x_{p-1}}^{x_q} (-df) f^{j-k} f^k \\
 &= [B^{j+2}/(j+1)!] [f^{j+1}(x_{p-1}) - f^{j+1}(x_q)] \quad (\text{Lemma 2}) \\
 &= B^{j+2}[g(x_q) - g(x_{p-1})]^{j+1}/(j+1)!.
 \end{aligned}$$

Thus, the inequality holds for $j + 1$. Hence, the inequality is valid for $j = 0, 1, 2, \dots$. Therefore, Lemma 4 is established.

Lemma 5. *If F and G are functions from $R \times R$ to N , $F \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then $F + G \in OP^\circ$ on $[a, b]$.*

Proof. This lemma follows as a corollary to Lemmas 3 and 4.

Lemma 5 is established in a previous paper by the author for functions from $R \times R$ to R [5, Theorem 1 (1 \rightarrow 2), p. 378]. However, the proof presented there is different from the proof employed in this paper.

Lemma 6. *If $\{F_i\}_{i=m}^n$ and $\{G_i\}_{i=m}^n$ are sequences of elements of N , then*

$$\begin{aligned}
 \prod_{i=m}^n (1 + F_i + G_i) &= \prod_{i=m}^n (1 + F_i) \\
 &\quad + \sum_{i=m}^n \left[\prod_{j=m}^{i-1} (1 + F_j) \right] G_i \left[\prod_{j=i+1}^n (1 + F_j + G_j) \right].
 \end{aligned}$$

Proof. This lemma can be established by induction.

Lemma 7. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:*

- (1) $\int_a^b G$ exists, and
- (2) $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$.

Further, if $G \in OB^\circ$ on $[a, b]$ and (1) or (2) is true, then $G \in OM^*$ on $[a, b]$.

Proof. If $G \in OB^\circ$ on $[a, b]$ and $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$ [7, Theorem 1]. Also, if $G \in OB^\circ$ on $[a, b]$, then $\int_a^b G$ exists if and only if $\prod_x^y (1 + G)$ exists for $a \leq x < y \leq b$ [7, Theorem 4]. Thus, the lemma follows.

We now establish Theorem 1.

Proof of Theorem 1 [(2), (3) \rightarrow (1)]. We initially establish that $\sum_{i=0}^\infty P_i(x, y)$ converges uniformly and absolutely for $a \leq x < y \leq b$, where

$$P_0(x, y) = \prod_x^y (1 + F)$$

and

$$P_i(x, y) = (LR) \int_x^y \prod_x^u (1 + F) G P_{i-1}(v, y)$$

for $a \leq x < y \leq b$ and $i = 1, 2, \dots$. The existence of these integrals follows by applying Lemma 1.

From Lemma 4, there exist a subdivision D_1 of $[a, b]$, a number B and a positive nondecreasing function g defined on $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1 , i is a nonnegative integer and $0 < p \leq q \leq n$, then

$$|S_{ipq}| \leq B^{i+1} [g(x_q) - g(x_{p-1})]^i / i!$$

and

$$|-S_{ipq}| \leq B^{i+1} [g(x_q) - g(x_{p-1})]^i / i!,$$

where S_{ipq} is defined as in Lemma 3.

It follows from the result stated in the preceding paragraph that

$$|P_i(x, y)| \leq B^{i+1} [g(y) - g(x)]^i / i!$$

for $a \leq x < y \leq b$ and $i = 0, 1, 2, \dots$. Therefore, $\sum_{i=0}^\infty P_i$ converges uniformly and absolutely on $[a, b]$.

Suppose $a \leq x < y \leq b$. We now establish that $\prod_x^y (1 + F + G)$ exists and is $\sum_{i=0}^\infty P_i(x, y)$. Let $\epsilon > 0$.

There exists a positive integer N such that

$$\sum_{i=N+1}^\infty B^{i+1} [g(b) - g(a)]^i / i! < \epsilon / 3.$$

Further, from the existence properties of the integrals involved, there exists a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 \leq p < q \leq n$, then

$$\left| \sum_{i=0}^N P_i(x_p, x_q) - \sum_{i=0}^N S_{i,p+1,q} \right| < \frac{\epsilon}{3}.$$

Let D denote a subdivision of $[x, y]$ which refines the intersection of $[x, y]$ and $D_1 \cup D_2$ and has at least $N + 1$ elements. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D . Now,

$$\begin{aligned} & \left| \sum_{i=0}^{\infty} P_i(x, y) - \prod_{i=1}^n (1 + F_i + G_i) \right| \\ &= \left| \sum_{i=0}^{\infty} P_i(x, y) - \sum_{i=0}^n S_{i1n} \right| \tag{Lemma 3} \\ &\leq \left| \sum_{i=0}^N P_i(x, y) - \sum_{i=0}^N S_{i1n} \right| + \left| \sum_{i=N+1}^{\infty} P_i(x, y) \right| + \left| - \sum_{i=N+1}^n S_{i1n} \right| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Hence, ${}_x\Pi^y(1 + F + G)$ exists and is $\sum_{i=0}^{\infty} P_i(x, y)$.

We now establish that $F + G \in OM^*$ on $[a, b]$. Since ${}_x\Pi^y(1 + F + G)$ exists for $a \leq x < y \leq b$, it is only necessary to establish the approximation part of the definition. Let $\epsilon > 0$. Further, let D_1, D_2 and N be defined as before.

Since F is in OM^*, OP^0 and S_2 on $[a, b]$ and G is in OB^0 and S_2 on $[a, b]$, there exists a subdivision D_3 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of $[a, b]$, $0 \leq p < q \leq n$ and $q - p \leq N$, then

$$\left| {}_x\Pi_p^q(1 + F + G) - \prod_{i=p+1}^q (1 + F_i + G_i) \right| < \epsilon.$$

Let D denote the subdivision $D_1 \cup D_2 \cup D_3$ of $[a, b]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leq p < q \leq n$. If $q - p \leq N$, then the desired inequality follows immediately from the definition of D_3 . If $q - p \geq N$, then

$$\begin{aligned} & \left| {}_x\Pi_p^q(1 + F + G) - \prod_{i=p+1}^q (1 + F_i + G_i) \right| \\ &= \left| \sum_{i=0}^{\infty} P_i(x_p, x_q) - \sum_{i=0}^{p-q} S_{i,p+1,q} \right| \tag{Lemma 3} \\ &\leq \left| \sum_{i=0}^N P_i(x_p, x_q) - \sum_{i=0}^N S_{i,p+1,q} \right| \\ &\quad + \left| \sum_{i=N+1}^{\infty} P_i(x_p, x_q) \right| + \left| - \sum_{i=N+1}^{p-q} S_{i,p+1,q} \right| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Therefore, $F + G \in OM^*$ on $[a, b]$. Thus, (2) and (3) imply (1).

Proof of Theorem 1 [(1), (2) \rightarrow (3)]. Since F and $F + G$ are in OM^* and OP° on $[a, b]$, the existence of

$$(LR) \int_x^y \prod_x^u (1 + F)G \prod_x^y (1 + F + G)$$

for $a \leq x < y \leq b$ can be established by employing Lemma 6.

Since F and $F + G$ are in S_1 on $[a, b]$, there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$, then

$$\left| \int_x^y \prod_x^y (1 + F) \right| < \frac{1}{2} \quad \text{and} \quad \left| \int_x^y \prod_x^y (1 + F + G) \right| < \frac{1}{2}.$$

Suppose $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$. Let J and K represent interval functions such that, if $x \leq u < v \leq y$, then

$$J(u, v) = \prod_x^u (1 + F) \quad \text{and} \quad K(u, v) = \prod_x^y (1 + F + G).$$

Since J and K are in OL° on $[a, b]$, it follows that J^{-1} and K^{-1} are also in OL° on $[a, b]$. Thus, from Lemma 1 and the existence of the integral in the preceding paragraph, we have that $\int_x^y G$ exists.

We have now established that, if $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$, then $\int_x^y G$ exists. From this and the fact that G is in OB° and S_2 on $[a, b]$, it follows that $\int_a^b G$ exists. Hence, $G \in OM^*$ on $[a, b]$ by Lemma 7. Thus, (1) and (2) imply (3).

Proof of Theorem 1 [(1), (3) \rightarrow (2)]. It follows from Lemma 5 that $F + G \in OP^\circ$ on $[a, b]$. Further, $-G \in OM^*$ by Lemma 7. We have already established that (2) and (3) imply (1). Now, since $F + G - G \equiv F$, it follows that $F \in OM^*$ on $[a, b]$. Thus, (1) and (3) imply (2).

The proof of Theorem 1 is now complete. We next establish Theorem 2. One additional lemma is needed.

Lemma 8. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:*

- (1) $G \in OA^\circ$ on $[a, b]$, and
- (2) $G \in OM^\circ$ on $[a, b]$ [2, Theorem 3.4, p. 301].

Proof of Theorem 2 [(2), (3) \rightarrow (1)]. Since F and G are in OP° and OM° on $[a, b]$, F and G are also in OM^* on $[a, b]$. Hence, it follows from Theorem 1 [(2), (3) \rightarrow (1)] that $\int_x^y \prod_x^y (1 + F + G)$ exists for $a \leq x < y \leq b$. Thus, it is only necessary to show that $\int_a^b |1 + F + G - \prod(1 + F + G)|$ exists and is zero in order to establish that $F + G \in OM^\circ$ on $[a, b]$. Let $\epsilon > 0$.

Since $F \in OM^\circ$ on $[a, b]$, there exists a subdivision D_1 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1 , then

$$\sum_{i=1}^n \left| 1 + F_i - \prod_{x_{i-1}}^{x_i} (1 + F) \right| < \frac{\epsilon}{2}.$$

We know that F is in OP° and OM^* on $[a, b]$. Further, $F + G \in OP^\circ$ on $[a, b]$ by Lemma 5 and $F + G \in OM^*$ on $[a, b]$ by Theorem 1 [(2), (3) \rightarrow (1)]. Now, since $G \in OB^\circ$ on $[a, b]$, it follows by using Lemma 6 that

$$(LR) \int_x^y \prod_x^u (1 + F) G \prod_v^y (1 + F + G)$$

exists and is

$${}_x \prod^y (1 + F + G) - {}_x \prod^y (1 + F)$$

for $a \leq x < y \leq b$.

Since F and $F + G$ are in S_1 and OM^* on $[a, b]$, for each positive number β there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$, then

$$\left| 1 - {}_x \prod^y (1 + F) \right| < \beta \quad \text{and} \quad \left| 1 - {}_x \prod^y (1 + F + G) \right| < \beta.$$

By Lemma 8, $G \in OA^\circ$ on $[a, b]$. Further, F and $F + G$ are in OP° on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$. From these facts, it follows that

$$\int_a^b \left| G(x, y) - (LR) \int_x^y \prod_x^u (1 + F) G \prod_v^y (1 + F + G) \right|$$

exists and is zero. Thus, there exists a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2 , then

$$\sum_{i=1}^n \left| G_i - (LR) \int_{x_{i-1}}^{x_i} \prod_{x_{i-1}}^u (1 + F) G \prod_v^{x_i} (1 + F + G) \right| < \frac{\epsilon}{2}.$$

Let D denote the subdivision $D_1 \cup D_2$ of $[a, b]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D . Now,

$$\begin{aligned} & \sum_{i=1}^n \left| 1 + F_i + G_i - \prod_{x_{i-1}}^{x_i} (1 + F + G) \right| \\ &= \sum_{i=1}^n \left| 1 + F_i + G_i - \left[\prod_{x_{i-1}}^{x_i} (1 + F) \right. \right. \\ & \quad \left. \left. + (LR) \int_{x_{i-1}}^{x_i} \prod_{x_{i-1}}^u (1 + F) G \prod_v^{x_i} (1 + F + G) \right] \right| \\ &\leq \sum_{i=1}^n \left| 1 + F_i - \prod_{x_{i-1}}^{x_i} (1 + F) \right| \\ & \quad + \sum_{i=1}^n \left| G_i - (LR) \int_{x_{i-1}}^{x_i} \prod_{x_{i-1}}^u (1 + F) G \prod_v^{x_i} (1 + F + G) \right| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore, $F + G \in OM^0$ on $[a, b]$. Thus, (2) and (3) imply (1).

Proof of Theorem 2 [(1), (2) \rightarrow (3)]. The proof of Theorem 2 [(1), (2) \rightarrow (3)] is similar to the proof of Theorem 1 [(1), (2) \rightarrow (3)]. The only difference is that it is necessary to use Lemma 8 rather than Lemma 7. Thus, (1) and (2) imply (3).

Proof of Theorem 2 [(1), (3) \rightarrow (2)]. The proof of Theorem 2 [(1), (3) \rightarrow (2)] is similar to the proof of Theorem 1 [(1), (3) \rightarrow (2)]. As before, the only difference is that it is necessary to use Lemma 8 rather than Lemma 7. Thus, (1) and (3) imply (2).

The proof of Theorem 2 is now complete.

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