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ON COVERINGS AND HYPERALGEBRAS OF AFFINE ALGEBRAIC GROUPS

BY

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ABSTRACT. Over an algebraically closed field of characteristic zero, the universal group covering of a connected affine algebraic group, if such exists, can be constructed canonically from its Lie algebra only. In particular the isomorphism classes of simply connected affine algebraic groups are in 1-1 correspondence with the isomorphism classes of finite dimensional Lie algebras of some sort. We shall consider the counterpart of these results (due to Hochschild) in case of a positive characteristic, replacing the Lie algebra by the "hyperalgebra". We show that the universal group covering of a connected affine algebraic group scheme can be constructed canonically from its hyperalgebra only and hence, in particular, that the category of simply connected affine algebraic group schemes is equivalent to a subcategory of the category of hyperalgebras of finite type which contains all the semisimple hyperalgebras.

Introduction. Let k be an arbitrary field of arbitrary characteristic. Let $\mathfrak{G}$ and $\mathfrak{H}$ be connected affine algebraic k-group schemes. If $\eta: \mathfrak{H} \to \mathfrak{G}$ is an epimorphism of k-group schemes whose kernel $\mathfrak{Ker}(\eta)$ (in the category of k-group schemes) is a *finite etale* k-group scheme, then the pair $(\mathfrak{H}, \eta)$ is called an *etale group covering* of $\mathfrak{G}$. The k-group scheme $\mathfrak{G}$ is *simply connected* (or (SC)), if it has no nontrivial etale group covering. An etale group covering $\gamma: \mathfrak{G}^* \to \mathfrak{G}$ is called a *universal group covering* of $\mathfrak{G}$, if the k-group scheme $\mathfrak{G}^*$ is (SC). Such a universal group covering $(\mathfrak{G}^*, \gamma)$, if it exists, should satisfy the following universal mapping property and hence will be determined uniquely up to a unique isomorphism.

For each etale group covering $(\mathfrak{H}, \eta)$ of $\mathfrak{G}$, there exists a unique homomorphism of k-group schemes $\eta^*: \mathfrak{G}^* \to \mathfrak{H}$ with $\eta \circ \eta^* = \gamma$.

The purpose of this article is to generalize the following result of Hochschild to the case of arbitrary perfect ground field of arbitrary characteristic:

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Theorem (Hochschild [2]). *If* k *is an algebraically closed field of characteristic zero, the following statements hold.*

(a) *A connected affine algebraic* k*-group scheme* $\mathfrak{G}$ *has a universal group covering if and only if the radical of* $\mathfrak{G}$ *is unipotent.*

(b) *Then the universal group covering* $\mathfrak{G}^*$ *of* $\mathfrak{G}$ *can be constructed canonically from its Lie algebra* $L = \mathrm{Lie}(\mathfrak{G})$ *only.*

(c) *In particular, the isomorphism classes of (SC)* k*-group schemes are in* 1-1 *correspondence with the isomorphism classes of finite dimensional Lie algebras* L *over* k *whose radical* A *is nilpotent.*

More precisely, let L be a finite dimensional Lie algebra over k. The universal enveloping algebra $U(L)$ of L has a unique Hopf algebra structure having L as primitive elements. Its *dual* Hopf algebra $H(L) = U(L)^0$ (see §0.1 for definition) is a commutative (but not always finitely generated) domain as an algebra. Let A be the radical of L and denote by $\langle A\rangle$ the ideal of $U(L)$ generated by A. Those elements of $H(L)$ which annihilate some power of $\langle A\rangle$ form a *finitely generated* sub-Hopf algebra of $H(L)$ denoted by $B(L)$. Let $\mathfrak{G}(L) = \mathrm{Spec}(B(L))$ denote the corresponding affine algebraic k-group scheme. Suppose that the radical A is *nilpotent*. Then the k-group scheme $\mathfrak{G}(L)$ is (SC) and its Lie algebra $\mathrm{Lie}(\mathfrak{G}(L))$ is canonically isomorphic with L. If in particular $L = [L, L]$, then the radical A is automatically nilpotent and we have $B(L) = H(L)$. Now let $\mathfrak{G}$ be a connected affine algebraic k-group scheme whose radical is *unipotent*. Let $\mathcal{O}(\mathfrak{G})$ denote its affine Hopf algebra. The radical A of the Lie algebra $L = \mathrm{Lie}(\mathfrak{G})$ is then nilpotent and the image of the canonical injective homomorphism $\mathcal{O}(\mathfrak{G}) \hookrightarrow U(L)^0 = H(L)$ is contained in $B(L)$. Therefore there results a canonical epimorphism of k-group schemes $\gamma\colon \mathfrak{G}(L) \longrightarrow \mathfrak{G}$ which proves to be a universal group covering of $\mathfrak{G}$. In particular every (SC) k-group scheme is of the form $\mathfrak{G}(L)$ with a uniquely determined finite dimensional Lie algebra L whose radical A is nilpotent.

When the characteristic of the field is positive, the *hyperalgebra* plays the same role as the Lie algebra does in case of characteristic zero. The hyperalgebra $\mathrm{hy}(\mathfrak{G})$ of an affine algebraic k-group scheme $\mathfrak{G}$ is by definition the irreducible component containing 1 of the dual Hopf algebra $\mathcal{O}(\mathfrak{G})^0$ of the affine Hopf algebra $\mathcal{O}(\mathfrak{G})$. Takeuchi $[\mathrm{T_I}]$, $[\mathrm{T_{II}}]$ develops the theory of hyperalgebras of algebraic groups which is completely analogous to the classical theory of Lie algebras of algebraic groups over a field of characteristic zero. The theory of hyperalgebras is briefly summarized in §0.3 for convenience of the reader. We have been able to characterize the (SC) k-group schemes by their hyperalgebra as follows:

Theorem. *Suppose the base field k is perfect with a positive characteristic p. For each connected affine algebraic (not necessarily smooth) k-group scheme $\mathfrak{G}$, the following conditions are equivalent to each other:*

(i) $\mathfrak{G}$ *is (SC).*

(ii) *The affine Hopf algebra $\mathcal{O}(\mathfrak{G})$ is canonically isomorphic with the dual Hopf algebra $\mathrm{hy}(\mathfrak{G})^0$ of the hyperalgebra $\mathrm{hy}(\mathfrak{G})$.*

(iii) *For each locally algebraic (not necessarily affine) k-group scheme $\mathfrak{H}$, the map*

$$\mathrm{Hom}_{k\text{-}gr}(\mathfrak{G}, \mathfrak{H}) \to \mathrm{Hopf}_k(\mathrm{hy}(\mathfrak{G}), \mathrm{hy}(\mathfrak{H}))$$

which sends each $\mathfrak{f} \in \mathrm{Hom}_{k\text{-}gr}(\mathfrak{G}, \mathfrak{H})$ to the induced homomorphism of hyperalgebras, $\mathrm{hy}(\mathfrak{f}) \in \mathrm{Hopf}_k(\mathrm{hy}(\mathfrak{G}), \mathrm{hy}(\mathfrak{H}))$, is bijective.

From this theorem, it follows that if a connected affine algebraic k-group scheme $\mathfrak{G}$ has a universal group covering $(\mathfrak{G}^*, \gamma)$, then the affine Hopf algebra $\mathcal{O}(\mathfrak{G}^*)$ is canonically isomorphic with the dual Hopf algebra $\mathrm{hy}(\mathfrak{G})^0$ of $\mathrm{hy}(\mathfrak{G})$, because $\mathrm{hy}(\gamma)$: $\mathrm{hy}(\mathfrak{G}^*) \xrightarrow{\sim} \mathrm{hy}(\mathfrak{G})$ and $\mathcal{O}(\mathfrak{G}^*) \xrightarrow{\sim} \mathrm{hy}(\mathfrak{G}^*)^0$. This means that the dual Hopf algebra $\mathrm{hy}(\mathfrak{G})^0$ is *finitely generated*, the corresponding k-group scheme $\mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^0)$ is (SC), the hyperalgebra of $\mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^0)$ is canonically isomorphic with $\mathrm{hy}(\mathfrak{G})$ and that the canonical homomorphism of k-group schemes $\mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^0) \to \mathfrak{G}$ is a universal group covering of $\mathfrak{G}$. Thus the universal group covering of $\mathfrak{G}$, if it exists, can be canonically constructed from its *hyperalgebra* $\mathrm{hy}(\mathfrak{G})$ *only*. On the other hand, we have

Theorem. *Suppose k is perfect and $p > 0$. If $\mathfrak{G}$ is an (SC) affine algebraic k-group scheme, then $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is a finite k-group scheme, that is its affine Hopf algebra is finite dimensional. If in particular $\mathfrak{G}$ is smooth and (SC), then $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}]$ and the radical of $\mathfrak{G}$ is unipotent.*

Thus our (SC) k-group schemes in case of a positive characteristic correspond with the k-group schemes $\mathfrak{G}(L)$ with $L = [L, L]$ in case of characteristic zero. In particular the 'if' part of the statement (a) of the theorem of Hochschild does *not* hold in case of a positive characteristic as it stands. (The additive k-group scheme $\mathfrak{G}_a$ is (SC) if $p = 0$ but not if $p > 0$.) But if we replace the condition (SC) by the following concept of simply connectedness relative to p, then statement (a) does hold in any characteristic (cf. Miyanishi [4] also).

Let p^* denote the *characteristic exponent* (= Max(1, p)) of k. Let $\mathfrak{H}$ and $\mathfrak{G}$ be connected affine algebraic k-group schemes. An epimorphism of k-group schemes η: $\mathfrak{H} \to \mathfrak{G}$ is called a *p-etale group covering* of $\mathfrak{G}$ if the kernel

$\mathfrak{Ker}(\eta)$ of η is a finite etale k-group scheme whose order (= the dimension of the affine ring over k) is relatively prime to p^*. The k-group scheme $\mathfrak{G}$ is called *p-simply connected* (or $(SC)_p$) if it has no nontrivial p-etale group covering. By a *p-universal group covering* of $\mathfrak{G}$ we mean a p-etale group covering $(\mathfrak{G}^*, \gamma)$ of $\mathfrak{G}$ with $\mathfrak{G}^*$ $(SC)_p$. Such a p-universal group covering of $\mathfrak{G}$, if it exists, satisfies the same universal mapping property as the usual universal group covering of $\mathfrak{G}$ (where of course the p-etale group coverings must take the place of the usual etale group coverings) and hence is uniquely determined up to a unique isomorphism. If $p = 0$, being $(SC)_0$ is equivalent to being (SC).

Theorem. *If k is perfect, a connected smooth affine algebraic k-group scheme $\mathfrak{G}$ has a p-universal group covering if and only if the radical of $\mathfrak{G}$ is unipotent.*

In order to obtain the p-universal group covering of $\mathfrak{G}$ whose radical is unipotent, we must first treat the *semisimple* k-group schemes. Indeed we shall show that every connected semisimple k-group scheme has a universal group covering which is at the same time a p-universal group covering. Hence if $\mathfrak{G}_u$ denotes the unipotent radical of $\mathfrak{G}$, the quotient group $\mathfrak{G}/\mathfrak{G}_u$, which is semisimple by assumption, has a p-universal group covering $(\mathfrak{G}/\mathfrak{G}_u)^*$. If we pull it back along $\mathfrak{G} \to \mathfrak{G}/\mathfrak{G}_u$, then we obtain a p-universal group covering $\mathfrak{G}^* = \mathfrak{G} \times_{\mathfrak{G}/\mathfrak{G}_u} (\mathfrak{G}/\mathfrak{G}_u)^*$ of $\mathfrak{G}$.

The case of connected semisimple k-group schemes goes as follows. First we consider the case where k is *algebraically closed.* It is well known that all connected semisimple k-group schemes are then described up to isomorphisms by their *root system* (cf. Satake [5], etc.). Let $\mathfrak{G} = \mathfrak{G}(X, \nabla)$ denote the connected semisimple k-group scheme (i.e., the Chevalley k-group scheme) determined by the root system (X, ∇). Let $X_0 = \{\nabla\}_{\mathbf{Z}}$ be the $\mathbf{Z}$-submodule of X generated by ∇ and X^0 be the *weight module* of (X, ∇), that is

$$X^0 = \{x \in X_{\mathbf{Q}} \ \langle \nabla^*, x\rangle \subset \mathbf{Z}\}$$

where ∇^* denotes the *coroot system* of (X, ∇). We have canonical inclusions $X_0 \subset X \subset X^0$. Traditionally the Chevalley k-group scheme $\mathfrak{G}$ is called "simply connected" if $X = X^0$ and adjoint if $X = X_0$. But the simply connectedness in this sense is *not* equivalent with our (SC)-ness. That is:

Theorem. *Suppose k is algebraically closed. Determine the subgroup $\overline{X}$ of X^0 containing X by the following condition:*

$$(p^*, [\overline{X}: X]) = 1 \quad \textit{and} \quad [X^0: \overline{X}] = \textit{a power of } p^*.$$

(If $p = 0$, then $\overline{X} = X^0$.) We have then:

(a) $\mathfrak{G} = \mathfrak{G}(X, \nabla)$ *is (SC) if and only if* $X = \overline{X}$.

(b) *The natural inclusion of reduced root systems* $(X, \nabla) \hookrightarrow (\overline{X}, \nabla)$ *induces an isogeny of Chevalley k-group schemes* $\gamma\colon \mathfrak{G}(\overline{X}, \nabla) \to \mathfrak{G}(X, \nabla)$ *(uniquely determined up to inner automorphisms by k-rational points of the maximal torus of* $\mathfrak{G}(\overline{X}, \nabla)$*) which is a universal group covering, as well as a p-universal group covering, of* $\mathfrak{G}$.

Thus all "simply connected" Chevalley k-group schemes are (SC) in our sense but the converse does not hold. For instance even the adjoint Chevalley k-group scheme $\mathfrak{PSL}_n = \mathfrak{SL}_n/_n\mu$ is (SC) when n is a power of p^*. Miyanishi [4] seems to have missed these circumstances. (Look at the "proof" of [4, Lemma 4].)

Coming back to the case where k is only perfect, if $\mathfrak{G}$ is a connected semisimple affine algebraic k-group scheme, then $\mathfrak{G} \otimes \overline{k}$ (where $\overline{k}$ = the algebraic closure of k), which is semisimple, has the universal group covering $(\mathfrak{G} \otimes \overline{k})^*$. We shall prove that the $\overline{k}$-group scheme $(\mathfrak{G} \otimes \overline{k})^*$ has a "k-form" $\overline{\mathfrak{G}}$ such that the epimorphism $(\mathfrak{G} \otimes \overline{k})^* \to \mathfrak{G} \otimes \overline{k}$ is "defined over k". Then $\overline{\mathfrak{G}}$ is easily seen to be a universal, as well as a p-universal, group covering of $\mathfrak{G}$. In other words the dual Hopf algebra $\mathrm{hy}(\mathfrak{G})^0$ is finitely generated and the hyperalgebra of $\overline{\mathfrak{G}} = \mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^0)$, which is "the" universal group covering of $\mathfrak{G}$, is canonically isomorphic with $\mathrm{hy}(\mathfrak{G})$. It is clear that the hyperalgebra $\mathrm{hy}(\mathfrak{G})$ is smooth and *semisimple*, in the sense of having no "radical". Conversely we can prove

Theorem. *Let J be a smooth semisimple hyperalgebra over k of finite type where k is perfect. Then J is the hyperalgebra of some (SC) semisimple k-group scheme $\mathfrak{G}$ which is uniquely determined, that is* $\mathfrak{G} = \mathrm{Spec}(J^0)$. *Thus the category of (SC) semisimple k-group schemes is equivalent to the category of smooth semisimple finite type hyperalgebras over k.*

Finally we shall conclude this paper by giving an example of (SC) k-group schemes which are *not reductive.*

This article is based on the theories of Hopf algebras, group schemes and hyperalgebras. They are prepared in §0.

To avoid confusion, all k-group schemes are denoted by German letters, but all Hopf algebras by Latin letters.

§0. Preliminaries. Throughout the paper k denotes a fixed ground field of characteristic p. The *characteristic exponent* of k is denoted by $p^* = \mathrm{Max}(1, p)$. If V is a k-vector space, the dual space $\mathrm{Hom}_k(V, k)$ is denoted by V^*. A subset T of V^* is *dense* if $T^\perp = \{v \in V^* | \langle T, v\rangle = 0\} = 0$. A subspace W of V is *cofinite* if V/W is finite dimensional.

For each homomorphism of fields ϕ: $k \to K$ and each k-vector space V, $V \otimes_\phi K$ denotes the scalar extension $V \otimes_k K$. In particular $V^{(p)}$ means the scalar extension $V \otimes_f k$, where f: $k \to k$, $\lambda \mapsto \lambda^{p^*}$. Inductively $V^{(p^r)} = (V^{(p^{r-1})})^{(p)}$.

0.1. Concerning coalgebras and Hopf algebras we freely use the notation and the terminology of Sweedler [6]. The structure maps of a k-coalgebra C will generally be denoted by Δ: $C \to C \otimes C$ and ϵ: $C \to k$. C^+ means $\mathrm{Ker}(\epsilon)$. The "sigma" notation $\Delta(x) = \Sigma_{(x)} x_{(1)} \otimes x_{(2)}$ for $x \in C$ will be used. The k-coalgebra C is *irreducible* if any two nonzero subcoalgebras of C have nonzero intersection [6, §8.0]. A maximal irreducible subcoalgebra of C is called an *irreducible component* of C. The k-coalgebra C is *cocommutative* if $\Sigma_{(x)} x_{(1)} \otimes x_{(2)} = \Sigma_{(x)} x_{(2)} \otimes x_{(1)}$ for all $x \in C$.

The dual space C^* becomes a k-algebra which is called the *dual algebra* of C, if the product is defined by $f * g = (f \otimes g) \circ \Delta$. It is commutative if C is cocommutative.

Conversely, for each k-algebra A, the *dual coalgebra* A^0 is defined to be $\varinjlim_I (A/I)^*$, where I runs through all the *cofinite twosided ideals* of A [6, §6.0]. The functors $C \mapsto C^*$ and $A \mapsto A^0$ are *adjoint* to each other in the sense:

$$\mathrm{Alg}_k(A, C^*) \simeq \mathrm{Coalg}_k(C, A^0).$$

A subspace I of a k-coalgebra C is a *coideal* if $\epsilon(I) = 0$ and $\Delta(I) \subset I \otimes C + C \otimes I$. The quotient space C/I then has a natural coalgebra structure. If f: $C \to D$ is a homomorphism of k-coalgebras, then the kernel $\mathrm{Ker}(f)$ is a coideal of C and the map f factors through $C \to C/\mathrm{Ker}(f)$.

The structure maps of a *Hopf algebra* H over k will generally be denoted by Δ: $H \to H \otimes H$, m: $H \otimes H \to H$, ϵ: $H \to k$, u: $k \to H$ and S: $H \to H$ (the antipode). The dual coalgebra H^0 of the k-algebra H is a subalgebra of the dual algebra H^* of the k-coalgebra H and is stable under the map ${}^tS$: $H^* \to H^*$, $f \mapsto f \circ S$. The induced algebra structure makes H^0 a Hopf algebra with the antipode ${}^tS|H^0$, called the *dual Hopf algebra* of H [6, §6.2]. The functor $H \mapsto H^0$ is *selfadjoint* in the following sense:

$$\mathrm{Hopf}_k(H, K^0) \simeq \mathrm{Hopf}_k(K, H^0)$$

for all k-Hopf algebras H and K.

The *bracket product* in a k-Hopf algebra H is defined by

$$[x, y] = \sum_{(x,y)} x_{(1)}y_{(1)}S(x_{(2)})S(y_{(2)})$$

for $x, y \in H$. If K and J are sub-Hopf algebras of H, $[K, J]$ denotes the *subalgebra* of H generated by the elements $[x, y]$ with $x \in K$ and $y \in J$. If H is *cocommutative*, this is a subbialgebra of H.

A subspace I of H is a *Hopf ideal* if it is a coideal and a twosided ideal of H and $S(I) \subset I$. The quotient space H/I then has a natural Hopf algebra structure. If $f: H \to H'$ is a homomorphism of Hopf algebras, then the kernel $\mathrm{Ker}(f)$ is a Hopf ideal of H and the map f factors through $H \to H/I$.

If K is a sub-Hopf algebra of H, then HK^+ is a coideal of H. The quotient coalgebra H/HK^+ is denoted by $H//K$. For each coideal I of H, we put $l(I) = \{x \in H \mid \Delta(x) - 1 \otimes x \in I \otimes H\}$. A sub-Hopf algebra K of H is *normal* if $\Sigma_{(x)} x_{(1)} y S(x_{(2)}) \in K$ for all $x \in H$ and $y \in K$ or equivalently if $[H, K] \subset K$. A Hopf ideal J of H is *normal* if $\Sigma_{(x)} x_{(1)} S(x_{(3)}) \otimes x_{(2)} \in H \otimes J$ for all $x \in J$. In some cases we have a "bijective correspondence" between some class of sub-Hopf algebras and some class of coideals of H as follows:

0.1.1. **Proposition** ([7] or [DG, III, §3, n°7]). *Let H be a commutative Hopf algebra.*

(a) *If K is a sub-Hopf algebra of H, then H is faithfully flat over K and $I = HK^+$ is a normal Hopf ideal of H. If H is finitely generated, then so is K.*

(b) *If I is a normal Hopf ideal of H, then $l(I)$ is a sub-Hopf algebra of H.*

(c) *The correspondences $K \mapsto HK^+$ and $I \mapsto l(I)$ establish a bijection between the sets of sub-Hopf algebras and of normal Hopf ideals of H.*

0.1.2. **Proposition** [T_{II}, 5.4.2.1, 4.2.2.1, 5.4.2.5, 5.4.3.7]. *Let H be a cocommutative Hopf algebra.*

(a) *If K is a sub-Hopf algebra of H, then H is a left and a right faithfully flat K-module* [7] *and an injective cogenerator in the category of right $H//K$-comodules. HK^+ is clearly a coideal and a left ideal of H.*

(b) *If I is a coideal and a left ideal of H, then $l(I)$ is a sub-Hopf algebra of H.*

(c) *The correspondences $K \mapsto HK^+$ and $I \mapsto l(I)$ establish a bijection between the sets of sub-Hopf algebras of H and of coideal-left-ideals of H.*

(d) *The coideal HK^+ is a Hopf ideal of H if and only if K is normal in H. In particular the bijective correspondence of* (c) *induces a bijection between the sets of normal sub-Hopf algebras of H and of Hopf ideals of H.*

If $f: H \to H'$ is a homomorphism of Hopf algebras, the *Hopf kernel* Hopf-ker(f) is defined to be the largest sub-Hopf algebra K of H with $K^+ \subset \mathrm{Ker}(f)$. This equals $l(\mathrm{Ker}(f))$ when H is cocommutative or when H is commutative and the Hopf ideal $\mathrm{Ker}(f)$ is normal in H.

Suppose $p > 0$. The *Verschiebung* map of a *cocommutative* k-coalgebra C, $V_C\colon C \to C^{(p)}$ is a unique k-coalgebra map such that the composite

$$C^{*(p)} \xrightarrow{\text{cano}} C^{(p)*} \xrightarrow{{}^tV} C^*$$

sends each element $X \otimes \lambda$ (with $X \in C^*$ and $\lambda \in k$) to $X^p\lambda$ [$\mathrm{T_I}$, 1.9.1], [$\mathrm{T_{II}}$, 5.5.3.1]. If H is a *cocommutative* Hopf algebra, the Verschiebung map $V_H\colon H \to H^{(p)}$ is a Hopf algebra map.

0.2. Concerning the basic theory of algebraic schemes and algebraic groups we refer the reader to [DG]. Here we briefly recall some of the fundamental relations between *affine algebraic* group schemes and commutative Hopf algebras.

A k-group scheme $\mathfrak{G}$ is *affine algebraic* if it is represented by some finitely generated commutative Hopf algebra A, that is $\mathfrak{G} = \mathrm{Spec}(A)$. The Hopf algebra A, which is uniquely determined by $\mathfrak{G}$, is denoted by $\mathcal{O}(\mathfrak{G})$. The following well-known relations between $\mathfrak{G} \leftrightarrow A$ are of particular importance:

$\mathfrak{G}$ is *smooth* $\Leftrightarrow A^{(p)}$ is reduced,

$\mathfrak{G}$ is *connected* $\Leftrightarrow A$ has no idempotents other than 0 and 1,

$\mathfrak{G}$ is *unipotent* $\Leftrightarrow A$ is irreducible as a coalgebra,

$\mathfrak{G}$ is *finite* $\Leftrightarrow [A\colon k] < \infty$,

$\mathfrak{G}$ is *etale* $\Leftrightarrow A$ is a finite product of finite separable extensions of k.

The *additive* and the *multiplicative* k-group schemes $\mathfrak{G}_a$ and $\mathfrak{G}_m$ are defined by

$$\mathfrak{G}_a = \mathrm{Spec}(k[T]), \qquad \Delta(T) = T \otimes 1 + 1 \otimes T,$$

$$\mathfrak{G}_m = \mathrm{Spec}(k[X, X^{-1}]), \qquad \Delta(X) = X \otimes X.$$

For each finitely generated abelian group Γ, the *diagonalizable* k-group scheme $\mathfrak{D}(\Gamma)$ is defined by

$$\mathfrak{D}(\Gamma) = \mathrm{Spec}(k[\Gamma]), \qquad \Delta(\gamma) = \gamma \otimes \gamma \quad \text{for all } \gamma \in \Gamma.$$

If K/k is an extension of fields, the scalar extension $\mathfrak{G} \otimes_k K$, where $\mathfrak{G} = \mathrm{Spec}(A)$, is $\mathrm{Spec}_K(A \otimes_k K)$.

The k-group schemes we shall treat are not necessarily smooth. In particular "finite connected" k-group schemes as follow will be taken into our consideration also:

$${}_q\alpha_k = \mathrm{Spec}(k[T]/T^q), \qquad \Delta(T) = T \otimes 1 + 1 \otimes T,$$

$${}_q\mu_k = \mathrm{Spec}(k[X]/(X^q - 1)), \qquad \Delta(X) = X \otimes X,$$

where $q =$ some power of p^*. The terms such as kernel, center, quotient, etc.

should be always interpreted in the sense of affine algebraic k-group schemes (not in the sense of $\bar{k}$-rational points).

Let $\mathfrak{G} = \mathrm{Spec}(A)$ be an affine algebraic k-group scheme. The k-group scheme of the form $\mathfrak{G}'' = \mathrm{Spec}(B)$ with B a sub-Hopf algebra of A, is called a *quotient* k-group scheme of $\mathfrak{G}$. The k-group scheme of the form $\mathfrak{H} = \mathrm{Spec}(A/I)$ with I a Hopf ideal of A, is called a *closed subgroup scheme* of $\mathfrak{G}$. The closed subgroup scheme $\mathfrak{H}$ is *normal* in $\mathfrak{G}$ if so is the Hopf ideal I in the sense of §0.1. It follows from 0.1.1 (c) that there is a natural bijective correspondence between the sets of normal closed subgroup schemes and of quotient group schemes of $\mathfrak{G}$. The quotient group scheme of $\mathfrak{G}$ associated with a normal closed subgroup scheme $\mathfrak{N}$ of $\mathfrak{G}$ is denoted by $\mathfrak{G}/\mathfrak{N}$. If $\mathfrak{f}\colon \mathfrak{G} \to \mathfrak{G}'$ is a homomorphism of affine algebraic k-group schemes, then the normal closed subgroup scheme of $\mathfrak{G}$ corresponding to the image of the induced Hopf algebra map $\mathcal{O}(\mathfrak{f})\colon \mathcal{O}(\mathfrak{G}') \to \mathcal{O}(\mathfrak{G}) = A$, which is a sub-Hopf algebra of A, is called the *kernel* of $\mathfrak{f}$ and denoted by $\mathfrak{Ker}(\mathfrak{f})$. The map $\mathfrak{f}$ factors through $\mathfrak{G} \to \mathfrak{G}/\mathfrak{Ker}(\mathfrak{f})$, via which the quotient group scheme $\mathfrak{G}/\mathfrak{Ker}(\mathfrak{f})$ can be viewed as a closed subgroup scheme of $\mathfrak{G}'$ naturally. The map $\mathfrak{f}$ is an *epimorphism* (resp. a *monomorphism*) if $\mathfrak{G}/\mathfrak{Ker}(\mathfrak{f}) \xrightarrow{\simeq} \mathfrak{G}'$ (resp. $\mathfrak{Ker}(\mathfrak{f}) = (e)$) or equivalently if the Hopf algebra map $\mathcal{O}(\mathfrak{f})$ is injective (resp. surjective). A sequence $1 \to \mathfrak{N} \xrightarrow{\mathfrak{i}} \mathfrak{G} \xrightarrow{\mathfrak{p}} \mathfrak{G}'' \to 1$ of affine algebraic k-group schemes is *exact* if $\mathfrak{i}\colon \mathfrak{N} \xrightarrow{\simeq} \mathfrak{Ker}(\mathfrak{p})$ and $\mathfrak{p}\colon \mathfrak{G}/\mathfrak{N} \xrightarrow{\simeq} \mathfrak{G}''$.

If $\mathfrak{H}$ is a closed subgroup scheme of $\mathfrak{G}$, we denote by $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})$ and $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})$ *the normalizer* and *the centralizer* of $\mathfrak{H}$ in $\mathfrak{G}$ respectively [DG, II, §1, n°3]. They are closed subgroup schemes of $\mathfrak{G}$ [DG, II, §1, 3.7]. In particular so is the *centre* $\mathfrak{cent}(\mathfrak{G})$.

The *derived* group $[\mathfrak{G}, \mathfrak{G}]$ is defined to be the kernel of the projection $\mathfrak{G} \to \mathrm{Spec}(B)$, where B denotes the *largest cocommutative* sub-Hopf algebra of $\mathcal{O}(\mathfrak{G})$. If $\mathfrak{G}$ is smooth, this definition coincides with [DG, II, §5, 4.8].

Let $\mathfrak{G}$ be a connected smooth affine algebraic k-group scheme. When k is *perfect*, we define the *radical* $\mathfrak{rad}(\mathfrak{G})$ (resp. the *unipotent radical* $\mathfrak{G}_u$) of $\mathfrak{G}$ to be the largest normal connected smooth solvable (resp. unipotent) closed subgroup scheme of $\mathfrak{G}$. If $\bar{k}$ denotes the algebraic closure of k, then $\mathfrak{rad}(\mathfrak{G}) \otimes_k \bar{k}$ (resp. $\mathfrak{G}_u \otimes_k \bar{k}$) is the radical (resp. the unipotent radical) of $\mathfrak{G} \otimes_k \bar{k}$ in the usual sense. The smooth connected k-group scheme $\mathfrak{G}$ is *semisimple* (resp. *reductive*) if $\mathfrak{rad}(\mathfrak{G}) = (e)$ (resp. $\mathfrak{G}_u = (e)$).

Suppose $p > 0$ in the rest of this §0.2. The *Frobenius* map $\mathfrak{F}\colon \mathfrak{G} \to \mathfrak{G}^{(p)}$ of an affine algebraic k-group scheme $\mathfrak{G} = \mathrm{Spec}(A)$ corresponds to the Hopf algebra map $A^{(p)} \to A$, $a \otimes \lambda \mapsto a^p\lambda$ with $a \in A$ and $\lambda \in k$. The k-group scheme $\mathfrak{G}$ is smooth (resp. etale) if and only if the Frobenius map $\mathfrak{F}$ is an epimorphism (resp. a monomorphism).

When $\mathfrak{G}$ is *commutative* (or equivalently if $\mathcal{O}(\mathfrak{G}) = A$ is *cocommutative*), the *Verschiebung* map of $\mathfrak{G}$, $\mathfrak{V}_{\mathfrak{G}}$: $\mathfrak{G}^{(p)} \to \mathfrak{G}$ [DG, IV, §3, n°6] corresponds to the Verschiebung map of A, V_A: $A \to A^{(p)}$ (defined in §0.1).

Let $k[F]$ denote the noncommutative polynomial ring over k defined by $F\lambda = \lambda^p F$ for all $\lambda \in k$. The category of commutative affine algebraic k-group schemes killed by the Verschiebung map is antiequivalent to the category of finitely generated left $k[F]$-modules [DG, IV, §3, 6.6]. We denote by $\mathfrak{U}(M)$ the k-group scheme determined by a left $k[F]$-module M. If we view M as a commutative p-Lie algebra over k, $U^{[p]}(M)$, the universal enveloping algebra of M, has a unique Hopf algebra structure having M as primitive elements (i.e., $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in M$) and we have $\mathfrak{U}(M) = \mathrm{Spec}(U^{[p]}(M))$ with this Hopf algebra structure. We have the following equivalence relations:

$\mathfrak{U}(M)$ is smooth $\Leftrightarrow 0 \to M^{(p)} \xrightarrow{F} M$,

$\mathfrak{U}(M)$ is finite $\Leftrightarrow [M: k] < \infty$,

$\mathfrak{U}(M)$ is etale $\Leftrightarrow M = kFM$,

$\mathfrak{U}(M)$ is connected $\Leftrightarrow$ each torsion element of M is killed by some power of F.

If k is algebraically closed, we have further

$\mathfrak{U}(M)$ is etale $\Leftrightarrow M \simeq (k[F]/F - 1)^s$,

$\mathfrak{U}(M)$ is connected smooth $\Leftrightarrow M \simeq k[F]^r$

[DG, IV, §3, 6.11].

0.3. As a final preliminary, we briefly recall the theroy of *hyperalgebras* developed in [T_I], [T_{II}].

A *hyperalgebra* means an *irreducible cocommutative* Hopf algebra. The *Lie algebra* Lie(J) of a hyperalgebra J is by definition the primitive elements of J, $P(J) = \{x \in J \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$. When Lie($J$) is finite dimensional, J is called *of finite type*.. If so is J, the dual algebra J^* is noetherian [T_I, 1.4.1] and has a *finite* Krull dimension which cannot exceed [Lie(J): k]. We define the Krull dimension dim(J) to be the Krull dimension of J^* [T_I, 1.4.4]. When the equality $\dim J = [\mathrm{Lie}(J): k]$ holds, the finite type hyperalgebra J is called *smooth* or *of Birkhoff-Witt type* [T_I, 1.6.1]. This is equivalent to saying $J \simeq B(U)$ as a coalgebra, for some finite dimensional k-vector space U with the notation of [6, §12.2]. If $p = 0$ all hyperalgebras are smooth and if $p > 0$, J is smooth if and only if the Verschiebung map $V_J: J \to J^{(p)}$ is surjective [T_I, 1.9.4]. From this it follows easily that an arbitrary finite type hyperalgebra J has *the largest smooth* subhyperalgebra J_{sm} called *the smooth part* of J if k is *perfect* [T_I, 1.9.5]. Then the quotient coalgebra $J//J_{sm}$ is *finite dimensional* over k and we have $\dim(J) = \dim(J_{\mathrm{sm}}) = [\mathrm{Lie}(J_{\mathrm{sm}}) : k]$ [$\mathrm{T_{II}}$, 5.5.3.8].

Among other things very important and useful is the fact that if $p > 0$ any *finite type* hyperalgebra J is *the union* of its *finite dimensional* normal subhyperalgebras [$\mathrm{T_{II}}$, 5.5.3.7, 5.5.3.9].

Let $\mathfrak{G}$ be a *locally algebraic* (not necessarily affine) k-group scheme [DG, II, §5]. Let $\mathcal{O}_e$ denote the stalk over the unit e of the structure sheaf $\mathcal{O}_{\mathfrak{G}}$. The multiplication $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ defines naturally a multiplication of the dual coalgebra $(\mathcal{O}_e)^0$ and makes it a *finite type* hyperalgebra [$\mathrm{T_I}$, 3.1.4, 3.3.1] which is denoted by $\mathrm{hy}(\mathfrak{G})$ and called *the hyperalgebra of* $\mathfrak{G}$.

If $\mathfrak{G}$ is affine, then $\mathrm{hy}(\mathfrak{G})$ coincides with the *irreducible component* containing 1 of the dual Hopf algebra $\mathcal{O}(\mathfrak{G})^0$. In particular, since the functor $H \mapsto H^0$ is selfadjoint (§0.1), the inclusion $\mathrm{hy}(\mathfrak{G}) \hookrightarrow \mathcal{O}(\mathfrak{G})^0$ corresponds to a natural Hopf algebra map $\mathcal{O}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G})^0$ which is *not necessarily injective*.

A homomorphism of locally algebraic k-group schemes $\mathfrak{f}: \mathfrak{G} \to \mathfrak{G}'$ induces clearly a homomorphism of hyperalgebras $\mathrm{hy}(\mathfrak{f}): \mathrm{hy}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G}')$. If $\mathfrak{G}$ is affine, it is easy to see that the natural map $\mathcal{O}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G})^0$ is *bijective* if and only if the natural map $\mathrm{Hom}_{k\text{-gr}}(\mathfrak{G}, \mathfrak{G}') \to \mathrm{Hopf}_k(\mathrm{hy}(\mathfrak{G}), \mathrm{hy}(\mathfrak{G}'))$ which sends each map $\mathfrak{f}$ to the induced map $\mathrm{hy}(\mathfrak{f})$ is *bijective* for all *affine* algebraic k-group schemes $\mathfrak{G}'$.

The functor $\mathfrak{G} \mapsto \mathrm{hy}(\mathfrak{G})$ has many interesting properties which are similar and reduce to the properties of the functor $\mathfrak{G} \mapsto \mathrm{Lie}(\mathfrak{G})$ in case $p = 0$:

0.3.1. **Proposition.** *Let* $\mathfrak{G}$ *be a locally algebraic k-group scheme.*

(a) [$\mathrm{T_I}$, 3.1.8, 3.3.1]. *The* Lie algebra *of* $\mathfrak{G}$, $\mathrm{Lie}(\mathfrak{G})$ [DG, II, §4, n°1], *equals* $\mathrm{Lie}(\mathrm{hy}(\mathfrak{G}))$. *The* dimension *of* $\mathfrak{G}$, $\dim \mathfrak{G}$ [DG, II, §5, 1.3], *equals* $\dim(\mathrm{hy}(\mathfrak{G}))$.

(b) [$\mathrm{T_I}$, 3.1.7]. *If K/k is an arbitrary extension of fields, then* $\mathrm{hy}(\mathfrak{G}) \otimes_k K$ *equals the K-hyperalgebra* $\mathrm{hy}_K(\mathfrak{G} \otimes_k K)$ *of the locally algebraic K-group scheme* $\mathfrak{G} \otimes_k K$.

(c) [$\mathrm{T_I}$, 2.2.9]. *If* $p > 0$, *the* Frobenius *map* $\mathfrak{F}_{\mathfrak{G}}: \mathfrak{G} \to \mathfrak{G}^{(p)}$ *induces the* Verschiebung *map* V: $\mathrm{hy}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G})^{(p)}$.

(d) [$\mathrm{T_I}$, 3.3.5, 3.3.11]. $\mathfrak{G}$ *is* smooth *if and only if* $\mathrm{hy}(\mathfrak{G})$ *is. When k is perfect, let* $\mathfrak{G}_{\mathrm{red}}$ *denote the reduced part of* $\mathfrak{G}$ [DG, II, §5, 2.3]. *Then the hyperalgebra* $\mathrm{hy}(\mathfrak{G}_{\mathrm{red}})$ *equals the smooth part* $\mathrm{hy}(\mathfrak{G})_{\mathrm{sm}}$ *of* $\mathrm{hy}(\mathfrak{G})$.

(e) [$\mathrm{T_I}$, 3.3.3]. $\mathfrak{G}$ *is* etale *if and only if* $\mathrm{hy}(\mathfrak{G}) = k$.

(f) [$\mathrm{T_I}$, 3.3.6]. *A subgroup scheme* $\mathfrak{H}$ *of* $\mathfrak{G}$ *is* open *if and only if* $\mathrm{hy}(\mathfrak{H}) = \mathrm{hy}(\mathfrak{G})$. *In particular if* $\mathfrak{G}$ *is* connected, *then* $\mathfrak{H} = \mathfrak{G}$ *if and only if* $\mathrm{hy}(\mathfrak{H}) = \mathrm{hy}(\mathfrak{G})$.

(g) *If $\mathfrak{G}$ is affine, the canonical map* $\mathcal{O}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G})^0$ *is* <u>*injective*</u> *if and only if $\mathfrak{G}$ is* <u>*connected.*</u>

(h) *If $p > 0$ and if $\mathfrak{G}$ is affine connected, then $\mathfrak{G}$ is* <u>*unipotent*</u> *if and only if each element of* $\mathrm{hy}(\mathfrak{G})^+$ *is* <u>*nilpotent.*</u>

0.3.2. **Proposition.** *Let $\mathfrak{f}\colon \mathfrak{G} \to \mathfrak{G}'$ be a homomorphism of locally algebraic k-group schemes.*

(a) $[\mathrm{T_I}$, 3.1.5]. *The hyperalgebra of the* <u>*kernel*</u> $\mathfrak{Ker}(\mathfrak{f})$ *equals the* <u>*Hopf kernel*</u> *of* $\mathrm{hy}(\mathfrak{f})$. *In particular if $\mathfrak{f}$ is a* <u>*monomorphism*</u>, *then* $\mathrm{hy}(\mathfrak{f})$ *is* <u>*injective.*</u>

(b) $[\mathrm{T_I}$, 3.3.2]. *If both $\mathfrak{G}$ and $\mathfrak{G}'$ are algebraic, there is a unique closed subgroup scheme $\mathfrak{f}(\mathfrak{G})^\sim$ of $\mathfrak{G}'$, called the* <u>*image-subgroup*</u> *of $\mathfrak{f}$, such that the induced map $\mathfrak{f}\colon \mathfrak{G} \to \mathfrak{f}(\mathfrak{G})^\sim$ is faithfully flat* [DG, III, §3, 5.2, 2.6, II, §5.5.1]. *Then we have*

$$\mathrm{hy}(\mathfrak{f}(\mathfrak{G})^\sim) = \mathrm{Im}(\mathrm{hy}(\mathfrak{f})) = \mathrm{hy}(\mathfrak{G})/\!/\text{Hopf-ker}(\mathrm{hy}(\mathfrak{f})).$$

(c) $[\mathrm{T_I}$, 3.3.3]. *The map $\mathfrak{f}$ is* <u>*nonramified*</u> [DG, I, §4, 3.2] *if and only if* $\mathrm{hy}(\mathfrak{f})$ *is* <u>*injective.*</u>

(d) $[\mathrm{T_I}$, 3.3.4]. *The map $\mathfrak{f}$ is* <u>*flat*</u> *if and only if* $\mathrm{hy}(\mathfrak{f})$ *is* <u>*surjective.*</u>

(e) $[\mathrm{T_I}$, 3.3.7]. *If $\mathfrak{G}'$ is connected and $\mathfrak{G}$ is algebraic, then $\mathfrak{f}(\mathfrak{G})^\sim = \mathfrak{G}'$ if and only if* $\mathrm{hy}(\mathfrak{f})$ *is* <u>*surjective.*</u>

(f) $[\mathrm{T_I}$, 3.3.9]. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be subgroup schemes of $\mathfrak{G}$. If $\mathfrak{H}$ is* <u>*connected*</u>, *then $\mathfrak{H} \subset \mathfrak{K}$ if and only if* $\mathrm{hy}(\mathfrak{H}) \subset \mathrm{hy}(\mathfrak{K})$. *In particular the correspondence $\mathfrak{H} \mapsto \mathrm{hy}(\mathfrak{H})$ from the set of* <u>*connected*</u> *(and hence closed) subgroup schemes of $\mathfrak{G}$ into the set of subhyperalgebras of* $\mathrm{hy}(\mathfrak{G})$ *is* <u>*injective.*</u>

(g) $[\mathrm{T_I}$, 3.3.10]. *Let $\mathfrak{f}'\colon \mathfrak{G} \to \mathfrak{G}'$ be another homomorphism. If $\mathfrak{G}$ is connected (and hence algebraic), then $\mathfrak{f} = \mathfrak{f}'$ if and only if* $\mathrm{hy}(\mathfrak{f}) = \mathrm{hy}(\mathfrak{f}')$.

Let $\mathfrak{G}$ be a locally algebraic k-group scheme. The inner automorphism action $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, $(g, h) \mapsto ghg^{-1}$ induces a linear representation $\mathfrak{Ad}\colon \mathfrak{G} \to \mathfrak{GL}(\mathrm{hy}(\mathfrak{G}))$, called the *adjoint representation* of $\mathfrak{G}$ $[\mathrm{T_I}$, 3.1.6, 3.4.13]. We view as usual each k-group scheme as a *k-group functor*, i.e., a functor from the category $\mathbf{M}_k$ of *k-models* to the category of groups [DG, I, §1, n°4].

0.3.3. **Proposition.** *Let $\mathfrak{H}$ be a closed subgroup scheme of a locally algebraic k-group scheme $\mathfrak{G}$. $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})$ and $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})$ denote the normalizer and the centralizer of $\mathfrak{H}$ in $\mathfrak{G}$.*

(a) $[\mathrm{T_I}$, 3.4.13]. $\mathrm{hy}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H}))$ *(resp.* $\mathrm{hy}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H}))$*) is the largest subcoalgebra C (resp. D) of* $\mathrm{hy}(\mathfrak{G})$ *satisfying*

$$\sum_{(c)} \mathfrak{Ad}(h)(c_{(1)})\mathcal{S}(c_{(2)}) \in R \otimes \mathrm{hy}(\mathfrak{H})$$

$$\left(\textit{resp.} \sum_{(d)} \mathfrak{Ad}(h)(d_{(1)})\mathcal{S}(d_{(2)}) \in R \otimes k\right)$$

for all $R \in \mathbf{M}_k$, $h \in \mathfrak{H}(R)$ *and* $c \in C$ (*resp.* $d \in D$), *where* S *denotes the antipode of* $\mathrm{hy}(\mathfrak{G})$.

(b) [T_I, 3.4.15]. *If* $\mathfrak{G}$ *is connected* (*and hence algebraic*) *then* $\mathfrak{H}$ *is normal* (*resp. central*) *in* $\mathfrak{G}$ *if and only if*

$$\sum_{(a)} \mathfrak{Ad}(h)(a_{(1)})S(a_{(2)}) \in R \otimes \mathrm{hy}(\mathfrak{H})$$

$$\left(\text{resp. } \sum_{(a)} \mathfrak{Ad}(h)(a_{(1)})S(a_{(2)}) \in R \otimes k\right)$$

for all $R \in \mathbf{M}_k$, $h \in \mathfrak{H}(R)$ *and* $a \in \mathrm{hy}(\mathfrak{G})$.

(c) [T_I, 3.4a.5]. *If* $\mathfrak{H}$ *is connected, then* $\mathfrak{H}$ *is normal* (*resp. central*) *in* $\mathfrak{G}$ *if and only if* $\mathrm{hy}(\mathfrak{H})$ *is* $\mathfrak{Ad}(\mathfrak{G})$*-stable* (*resp.* $\mathfrak{Ad}(\mathfrak{G})$ *acts trivially on* $\mathrm{hy}(\mathfrak{H})$).

(d) [T_I, 3.4.15]. *If* $\mathfrak{H}$ *and* $\mathfrak{G}$ *are both connected, then* $\mathfrak{H}$ *is normal* (*resp. central*) *in* $\mathfrak{G}$ *if and only if so is the subhyperalgebra* $\mathrm{hy}(\mathfrak{H})$ *in* $\mathrm{hy}(\mathfrak{G})$. (*A sub-Hopf algebra* J *of a Hopf algebra* H *is* <u>*central*</u> *if* $[H, J] = k$.)

Let $\mathfrak{G}$ be a locally algebraic k-group scheme. A subhyperalgebra J of $\mathrm{hy}(\mathfrak{G})$ is *algebraic* (or *closed*) if there is a subgroup scheme $\mathfrak{H}$ of $\mathfrak{G}$ with $J = \mathrm{hy}(\mathfrak{H})$. It follows from 0.3.2(f) that we can take then a *unique connected* (and hence closed algebraic) one as $\mathfrak{H}$. For each subhyperalgebra J of $\mathrm{hy}(\mathfrak{G})$ there is a unique *smallest* algebraic subhyperalgebra $A(J)$ of $\mathrm{hy}(\mathfrak{G})$ containing J [T_I, §3.6] called the *algebraic hull* of J.

0.3.4. **Proposition.** *Let* $\mathfrak{G}$ *be a locally algebraic k-group scheme and* J *a subhyperalgebra of* $\mathrm{hy}(\mathfrak{G})$.

(a) *If* $[J: k] < \infty$, *then* J *is closed.*

(b) *Let* K *be a subhyperalgebra of* J. *If* $J//K$ *is finite dimensional, then* K *is closed in* $\mathrm{hy}(G)$ *if and only if so is* J.

(c) *Let* l/k *be an arbitrary extension of fields. Then* J *is closed in* $\mathrm{hy}(\mathfrak{G})$ *if and only if so is* $J \otimes l$ *in* $\mathrm{hy}(\mathfrak{G}) \otimes l = \mathrm{hy}_l(\mathfrak{G}_l)$.

(d) [T_I, 3.6.2]. *Let* A *be a subalgebra and* C *a subcoalgebra of* $\mathrm{hy}(\mathfrak{G})$ *such that* $AC \subset C$. *If* $[J, C] \subset A$, *then* $[A(J), C] \subset A$.

(e) [T_I, 3.6.2]. *Let* K *be a subhyperalgebra of* J. *If* K *is* <u>*normal*</u> *in* J, *then so is* K *in* $A(J)$.

(f) [T_I, 3.6.3]. *We have* $[J, J] = [A(J), A(J)]$ (*which is* <u>*closed*</u> *in* $\mathrm{hy}(\mathfrak{G})$ *by the following* (g)). *In particular the quotient hyperalgebra* $A(J)//J$ *is* <u>*abelian.*</u>

(g) *If* J_1 *and* J_2 *are* <u>*closed*</u> *subhyperalgebras of* $\mathrm{hy}(\mathfrak{G})$, *then the commutator subhyperalgebra* $[J_1, J_2]$ *is also* <u>*closed*</u> *in* $\mathrm{hy}(\mathfrak{G})$.

(h) [T_I, 3.5.6]. *If* $\mathfrak{G}$ *is connected smooth, then* $\mathrm{hy}([\mathfrak{G}, \mathfrak{G}]) = [\mathrm{hy}(\mathfrak{G}), \mathrm{hy}(\mathfrak{G})]$.

(i) *If* $\mathfrak{G}$ *is connected affine, then* $\mathrm{hy}([\mathfrak{G}, \mathfrak{G}]) = [\mathrm{hy}(\mathfrak{G}), \mathrm{hy}(\mathfrak{G})]$.

The proof of (a), (b), (c), (f), (g) and (i) above will be published elsewhere. In particular if $\mathfrak{H}$ and $\mathfrak{K}$ are *connected* subgroup schemes of a locally algebraic k-group scheme $\mathfrak{G}$, we can define the *commutator* subgroup $[\mathfrak{H}, \mathfrak{K}]$ to be the *unique connected* subgroup of $\mathfrak{G}$ with $\mathrm{hy}([\mathfrak{H}, \mathfrak{K}]) = [\mathrm{hy}(\mathfrak{H}), \mathrm{hy}(\mathfrak{K})]$. This definition generalizes [DG, II, §5, 4.9].

1. The (SC) (or $(SC)_p$) k-group schemes. We have defined in the introduction the concepts of etale group covering, universal group covering, (SC) or $(SC)_p$ k-group scheme, etc.

1.1. Proposition. *Let* $\mathfrak{G}$ *and* $\mathfrak{H}$ *be connected affine algebraic k-group schemes and* $\eta: \mathfrak{H} \to \mathfrak{G}$ *a homomorphism of k-group schemes. Then* $(\mathfrak{H}, \eta)$ *is an etale group covering of* $\mathfrak{G}$ *if and only if* $\mathrm{hy}(\eta): \mathrm{hy}(\mathfrak{H}) \xrightarrow{\sim} \mathrm{hy}(\mathfrak{G})$.

This follows directly from 0.3.2(c) and (e).

1.2. Proposition. *Let* $\mathfrak{G}$ *be a connected affine algebraic k-group scheme and* $\mathfrak{K}$ *a normal connected closed subgroup scheme of* $\mathfrak{G}$. (a) *If* $\mathfrak{G}$ *is (SC) (resp.* $(SC)_p$*), then so is* $\mathfrak{G}/\mathfrak{K}$. (b) *If* $\mathfrak{G}/\mathfrak{K}$ *and* $\mathfrak{K}$ *are both (SC) (resp.* $(SC)_p$*), then so is* $\mathfrak{G}$.

The proof is easy.

1.3. Remark. All finite connected k-group schemes are clearly (SC).

1.4. Remark. The multiplicative group $\mathfrak{G}_m$ is *not* $(SC)_p$, in view of the canonical extension $1 \to {}_n\mu \to \mathfrak{G}_m \to \mathfrak{G}_m \to 1$ for all n relatively prime to p^*. More generally if $\mathfrak{H}$ is a k-*torus*, that is $\mathfrak{H} \otimes \bar{k} \simeq (\mathfrak{G}_m \otimes \bar{k})^r$ for some $r \in \mathbb{N}$, then $\mathfrak{H}$ is *not* $(SC)_p$ unless $r = 0$. Indeed if $\mathfrak{K}$ denotes the kernel of the morphism $\mathfrak{H} \to \mathfrak{H}$, $x \mapsto x^n$, then $\mathfrak{K} \otimes \bar{k} \simeq ({}_n\mu \otimes \bar{k})^r$ clearly and so $\mathfrak{K}$ is finite etale with the order n^r which is prime to p^* if n is so chosen. The additive group $\mathfrak{G}_a$ is *not* (SC) when $p > 0$, in view of the Artin-Schreier extension

$$1 \to (\mathbb{Z}/p\mathbb{Z})_k \to \mathfrak{G}_a \xrightarrow{\mathfrak{F}-1} \mathfrak{G}_a \to 1$$

where $\mathfrak{F}$ denotes the Frobenius map.

In general, each affine algebraic k-group scheme $\mathfrak{G}$ has a normal closed connected finite subgroup scheme $\mathfrak{N}$ such that $\mathfrak{G}/\mathfrak{N}$ is smooth [DG, III, §3, 6.10]. Since $\mathfrak{N}$ is always (SC), it follows that $\mathfrak{G}$ is (SC) (resp. $(SC)_p$) if and only if $\mathfrak{G}/\mathfrak{N}$ is. If $\mathfrak{G}/\mathfrak{N}$ has a (p-) universal group covering, then pulling it back along $\mathfrak{G} \to \mathfrak{G}/\mathfrak{N}$, we obtain a ($p$-) universal group covering of $\mathfrak{G}$. Hence all consideration concerning group coverings reduces to the case of smooth k-group schemes.

1.5. **Proposition.** *Connected affine algebraic unipotent k-group schemes are* $(SC)_p$.

Proof. We shall consider the case $p > 0$ first. Let $\mathfrak{G}$ be connected unipotent $\neq (e)$. Since $\mathfrak{G} \neq [\mathfrak{G}, \mathfrak{G}]$, it suffices to prove that $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is $(SC)_p$, by the induction hypothesis. Thus we can assume $\mathfrak{G}$ is commutative. Let $\mathfrak{V}_{\mathfrak{G}}: \mathfrak{G}^{(p)} \to \mathfrak{G}$ denote the Verschiebung map of $\mathfrak{G}$. Since $\mathfrak{G} \neq \mathfrak{V}_{\mathfrak{G}}(\mathfrak{G}^{(p)})$, we can assume that $\mathfrak{V}_{\mathfrak{G}} = 0$, by the induction hypothesis again. There is then a finitely generated left $k[F]$-module M with $\mathfrak{G} \simeq \mathfrak{U}(M)$ (§0.2). Let $1 \to \mathfrak{K} \to \mathfrak{H} \to \mathfrak{U}(M) \to 1$ be a p-etale group covering of $\mathfrak{U}(M)$. Since $\mathrm{hy}(\mathfrak{H}) \xrightarrow{\sim} \mathrm{hy}(\mathfrak{U}(M))$, it follows from 0.3.1(h) that $\mathfrak{H}$ is unipotent and from 0.3.1(g) that $\mathfrak{H}$ is commutative. Since $\mathrm{hy}(\mathfrak{V}_{\mathfrak{H}})$ is trivial, it follows from 0.3.2(g) that $\mathfrak{V}_{\mathfrak{H}} = 0$. In particular we have $\mathfrak{V}_{\mathfrak{K}} = 0$ and so $\mathfrak{K} \simeq \mathfrak{U}(N)$ for some *finite dimensional* (over k) left $k[F]$-module N. The order of $\mathfrak{K}$ $(= [U^{[p]}(N): k])$ is a power of p. But since the order of $\mathfrak{K}$ is relatively prime to p, it follows that $\mathfrak{K} = (e)$. Hence $\mathfrak{U}(M)$ is $(SC)_p$.

Next suppose that $p = 0$. Since each affine algebraic unipotent k-group scheme has a central series of closed subgroup schemes each of whose quotients is isomorphic to $\mathfrak{G}_a$ [DG, IV, §2, 3.9, 4.1], it suffices to show that $\mathfrak{G}_a$ is $(SC)_0$. But since any etale group covering of $\mathfrak{G}_a$ is clearly 1-dimensional unipotent and hence isomorphic to $\mathfrak{G}_a$ [DG, IV, §2, 2.10], the claim follows from the fact that $\mathfrak{G}_a$ has no nontrivial etale subgroup scheme [DG, IV, §2, 1.1]. Q.E.D.

1.6. **Theorem.** *If k is perfect, a connected smooth affine algebraic k-group scheme* $\mathfrak{G}$ *is* $(SC)_p$ *if and only if the radical* $\mathfrak{rad}(\mathfrak{G})$ *is unipotent and that* $\overline{\mathfrak{G}} = \mathfrak{G}/\mathfrak{rad}(\mathfrak{G})$ *is* (SC).

Proof. The 'if' part follows from Propositions 1.5 and 1.2. Suppose $\mathfrak{G}$ is $(SC)_p$. Let $\mathfrak{G}_u$ denote the unipotent radical of $\mathfrak{G}$. Since $\overline{\mathfrak{G}} = \mathfrak{G}/\mathfrak{G}_u$ is $(SC)_p$ and reductive, it follows that $\overline{\mathfrak{G}}/[\overline{\mathfrak{G}}, \overline{\mathfrak{G}}]$ is an $(SC)_p$ torus. But because any nontrivial k-torus is not $(SC)_p$, it follows that $\overline{\mathfrak{G}} = [\overline{\mathfrak{G}}, \overline{\mathfrak{G}}]$ or equivalently that $\overline{\mathfrak{G}}$ is semisimple. Since we show in §3 that all connected $(SC)_p$ semisimpel k-group schemes are (SC), it follows that $\overline{\mathfrak{G}}$ is (SC). Q.E.D.

1.7. **Theorem.** *If k is perfect, a connected smooth affine algebraic k-*

group scheme $\mathfrak{G}$ has a p-universal group covering if and only if the radical $\mathfrak{rad}(\mathfrak{G})$ is unipotent.

Proof. The 'only if' part follows from Theorem 1.6. Suppose conversely that the radical $\mathfrak{rad}(\mathfrak{G})$ is unipotent. The semisimple k-group scheme $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{rad}(\mathfrak{G})$ has a p-universal group covering $\bar{\mathfrak{G}}^*$ as we shall see in §3. The pull-back $\mathfrak{G}^* = \mathfrak{G} \times_{\bar{\mathfrak{G}}} \bar{\mathfrak{G}}^*$ then is a p-universal group covering of $\mathfrak{G}$ by Theorem 1.6. Q.E.D.

1.8. **Lemma.** *Let $\mathfrak{G}$ be a connected affine algebraic k-group scheme and consider the following three conditions:*

(a) *$\mathfrak{G}$ is (SC).*

(b) *For each locally algebraic (not necessarily affine) k-group scheme $\mathfrak{H}$, the natural map*

$$\mathrm{Hom}_{k\text{-}gr}(\mathfrak{G}, \mathfrak{H}) \to \mathrm{Hopf}_k(\mathrm{hy}(\mathfrak{G}), \mathrm{hy}(\mathfrak{H})), \quad \mathfrak{f} \mapsto \mathrm{hy}(\mathfrak{f})$$

is bijective.

(c) *The canonical map $\mathcal{O}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G})^0$ is bijective.*

We have then an implication (b) $\Rightarrow$ (c) $\Rightarrow$ (a). *If further $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is finite, then* (a) $\Rightarrow$ (b).

Proof. We pointed out in §0.3 that condition (c) is equivalent to the bijectivity of the maps of (b) for all *affine* algebraic k-group schemes $\mathfrak{H}$. In particular we have (b) $\Rightarrow$ (c). Let γ: $\bar{\mathfrak{G}} \to \mathfrak{G}$ be an etale group covering. Since $\mathrm{hy}(\gamma)$ is isomorphic, if (c) is valid, there is a unique homomorphism σ: $\mathfrak{G} \to \bar{\mathfrak{G}}$ with $\mathrm{hy}(\sigma) = \mathrm{hy}(\gamma)^{-1}$. Then $\gamma \circ \sigma = \mathrm{id}$, because $\mathrm{hy}(\gamma \circ \sigma) = \mathrm{id}$. Since σ is an epimorphism of affine algebraic k-group schemes, it follows that $\sigma \circ \gamma = \mathrm{id}$. Hence γ is an isomorphism, so we have (c) $\Rightarrow$ (a). Finally suppose that $\mathfrak{G}$ is (SC) with $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ *finite*. Let $\mathfrak{H}$ be an arbitrary locally algebraic k-group scheme. Let ω: $\mathrm{hy}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{H})$ be a homomorphism of hyperalgebras. The composite

$$\psi: \mathrm{hy}(\mathfrak{G}) \xrightarrow{\Delta} \mathrm{hy}(\mathfrak{G}) \otimes \mathrm{hy}(\mathfrak{G}) \xrightarrow{1 \otimes \omega} \mathrm{hy}(\mathfrak{G}) \otimes \mathrm{hy}(\mathfrak{H}) = \mathrm{hy}(\mathfrak{G} \times \mathfrak{H})$$

is an injective homomorphism of hyperalgebras. Put $\mathfrak{K} = \mathfrak{G} \times \mathfrak{H}$, $J = \mathrm{Im}(\psi)$ and $J' = [J, J]$. Then J' is closed in $\mathrm{hy}(\mathfrak{K})$ by 0.3.4(f) and $J//J' \simeq \mathrm{hy}(\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}])$ (0.3.2(b)) is finite dimensional by assumption and hence J is closed in $\mathrm{hy}(\mathfrak{G} \times \mathfrak{H})$ by 0.3.4(b). There exists a unique connected closed subgroup scheme $\mathfrak{G}^*$ of $\mathfrak{G} \times \mathfrak{H}$ with $\mathrm{hy}(\mathfrak{G}^*) = J$. The projection $\mathfrak{pr}_1$: $\mathfrak{G}^* \to \mathfrak{G}$, which is then an etale group covering of $\mathfrak{G}$, is an isomorphism because $\mathfrak{G}$ is (SC). The composite

$$\mathfrak{f}: \mathfrak{G} \xrightarrow{(\mathfrak{pr}_1)^{-1}} \mathfrak{G}^* \xrightarrow{\mathfrak{pr}_2} \mathfrak{H}$$

is easily seen to be a unique homomorphism of k-group schemes with $\mathrm{hy}(\mathfrak{f}) = \omega$. Q.E.D.

1.9. **Theorem.** *If k is perfect and $p > 0$, a connected affine algebraic k-group scheme $\mathfrak{G}$ is (SC) if and only if* $\mathcal{O}(\mathfrak{G}) \xrightarrow{\sim} \mathrm{hy}(\mathfrak{G})^0$. *Then $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is finite.*

Proof. By virtue of Lemma 1.8, it suffices to show that $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is finite when $\mathfrak{G}$ is (SC). Suppose $\mathfrak{G}$ is (SC) and $p > 0$. We can assume that $\mathfrak{G}$ is smooth. Let $\mathfrak{G}_u$ denote the unipotent radical of $\mathfrak{G}$. Since $\mathfrak{G}/\mathfrak{G}_u$ is semisimple (1.6), it follows that $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}]\mathfrak{G}_u$. Hence $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is (SC), unipotent and smooth. Since it has a central series of closed connected subgroups each of whose quotients is isomorphic with $\mathfrak{G}_a$ [DG, IV, §2, 3.9], it follows from Remark 1.4 that $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}]$. Q.E.D.

2. **The case of Chevalley group schemes.** In this section we assume k to be *algebraically closed* and consider the problem of group coverings for connected semisimple k-group schemes. A connected *smooth* affine algebraic k-group scheme is *semisimple* if its *radical* is (e). Let $\mathfrak{G}$ be a connected semisimple k-group scheme with a maximal torus $\mathfrak{T}$. The character group $X = X(\mathfrak{T}) = \mathrm{Hom}_{k\text{-}gr}(\mathfrak{T}, \mathfrak{G}_m)$ is a free $\mathbf{Z}$-module of finite rank and has a natural root system ∇ in it. For each root $\alpha \in \nabla$, the associated coroot α^* is uniquely determined in the space $\hat{X} = \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$. Let ∇^* denote the set of coroots α^*, $\alpha \in \nabla$. We then have the following subgroups of $X_{\mathbf{Q}} = X \otimes_{\mathbf{Z}} \mathbf{Q}$:

$$X_0 = \{\nabla\}_{\mathbf{Z}} \subset X \subset X^0 = \{\nabla^*\}_{\mathbf{Z}} = \{x \in X_{\mathbf{Q}} | \langle \nabla^*, x \rangle \subset \mathbf{Z}\}.$$

The group X^0 is called the weight module of (X, ∇). The quotient group X^0/X_0 is finite. See Iwahori [3, Vol. 2, p. 58] for the table of X^0/X_0 for the irreducible root systems. For any subgroup Y between X_0 and X^0, the pair (Y, ∇) is a reduced root system, say, in the sense of Satake [5, p. 44] and if we identify $Y_{\mathbf{Q}} = X_{\mathbf{Q}}$, then Y has the same coroot system as X and we have $Y_0 = X_0$ and $Y^0 = X^0$.

Let $\mathfrak{f}: \mathfrak{G} \to \mathfrak{G}'$ be an "isogeny" of connected affine algebraic k-group schemes, by which we mean an epimorphism of affine algebraic k-group schemes whose kernel $\mathfrak{Ker}(\mathfrak{f})$ is a finite k-group scheme. If $\mathfrak{G}'$ is smooth and $\mathfrak{Ker}(\mathfrak{f})$ is etale, then $\mathfrak{G}$ is smooth too. Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are both smooth. If one of $\mathfrak{G}$ and $\mathfrak{G}'$ is reductive (resp. semisimple), then so is the other. Hence suppose that both $\mathfrak{G}$ and $\mathfrak{G}'$ are semisimple. If $\mathfrak{T}$ is a maximal torus of $\mathfrak{G}$, then $\mathfrak{T}' = \mathfrak{f}(\mathfrak{T})$ is a maximal torus of $\mathfrak{G}'$ and the restricted isogeny $\mathfrak{f}: \mathfrak{T} \to \mathfrak{T}'$ induces an injection of abelian groups ${}^t\mathfrak{f}: X' = X(\mathfrak{T}) \hookrightarrow X = X(\mathfrak{T})$. This is a *special homomorphism* in the following sense:

(i) $[X: {}^t\mathfrak{f}(X')] < \infty$.

(ii) If ∇ and ∇' denote the root systems of $(\mathfrak{G}, \mathfrak{T})$ and $(\mathfrak{G}', \mathfrak{T}')$ respectively, then there are a bijection $\beta: \nabla \xrightarrow{\sim} \nabla'$ and a family $(q_\alpha)_{\alpha\in\nabla}$ of powers of p^* with ${}^t\mathfrak{f}(\beta(\alpha)) = q_\alpha \alpha$ for all $\alpha \in \nabla$.

The famous uniqueness theorem of Chevalley [5, p. 53] tells us conversely that each special homomorphism $\psi: X' \hookrightarrow X$ comes from an isogeny $\mathfrak{f}: \mathfrak{G} \to \mathfrak{G}'$ such that $\mathfrak{f}(\mathfrak{T}) = \mathfrak{T}'$, which is determined uniquely up to inner automorphisms by the k-rational points of $\mathfrak{T}$. On the other hand for each reduced root system (X, ∇) there exists a connected semisimple k-group scheme $(\mathfrak{G}, \mathfrak{T})$, determined uniquely up to isomorphisms, having (X, ∇) as its root system (the existence theorem of Chevalley). The connected semisimple k-group scheme determined by the root system (X, ∇) is denoted by $\mathfrak{G}(X, \nabla)$.

An isogeny of connected semisimple k-group schemes $\mathfrak{f}: \mathfrak{G} \to \mathfrak{G}'$ (or the corresponding special homomorphism ${}^t\mathfrak{f}: X' \hookrightarrow X$) is called *standard* if all the indices q_α are equal to 1. The following facts concerning standard isogenies are well known. Recall that $\mathfrak{D}(\Gamma)$ denotes the *diagonalizable* k-group scheme represented by a finitely generated $\mathbf{Z}$-module Γ (§0.2).

2.1. Lemma. *Let $\mathfrak{f}: \mathfrak{G} \to \mathfrak{G}'$ be a standard isogeny of connected semisimple k-group schemes, $\mathfrak{T}$ a maximal torus of $\mathfrak{G}$, $\mathfrak{T}' = \mathfrak{f}(\mathfrak{T})$, $X = X(\mathfrak{T})$ and $X' = X(\mathfrak{T}')$. Via the injection ${}^t\mathfrak{f}: X' \hookrightarrow X$, we view X' as a subgroup of X.*

(1) $\mathfrak{Ker}(\mathfrak{f}) = \mathfrak{Ker}(\mathfrak{f}|\mathfrak{T}) = \mathfrak{D}(X/X')$.

(2) $\mathfrak{Ker}(\mathfrak{f}) \subset \mathfrak{Cent}(\mathfrak{G})$.

(3) $\mathfrak{Cent}(\mathfrak{G}) = \mathfrak{D}(X/X_0)$.

(4) $\mathfrak{Ker}(\mathfrak{f})$ *is etale if and only if* $(p^*, [X: X']) = 1$. *Hence* $(\mathfrak{G}, \mathfrak{f})$ *is a p-etale group covering of* $\mathfrak{G}'$ *in this case.*

(5) *If* $(\mathfrak{H}, \gamma)$ *is an etale group covering of* $\mathfrak{G}'$, *then* $\mathfrak{H}$ *is smooth connected semisimple and* γ *a standard isogeny.*

Let $(\mathfrak{G}, \mathfrak{T})$ be a connected semisimple k-group scheme having (X, ∇) as its root system. Let $\overline{X}/X$ denote the largest subgroup of X^0/X whose order is relatively prime to p^*. Hence $[X^0: \overline{X}]$ = a power of p^*.

2.2. Theorem. *Let k be algebraically closed.* (i) *The standard isogeny $\mathfrak{G}(\overline{X}, \nabla) \to \mathfrak{G}(X, \nabla) = \mathfrak{G}$ induced from the inclusion of root systems $(X, \nabla) \hookrightarrow (\overline{X}, \nabla)$ is a universal group covering, as well as a p-universal group covering, of $\mathfrak{G}$.*

(ii) *In particular the following conditions are equivalent to each other:* (a) $\mathfrak{G}$ *is* (SC); (b) $\mathfrak{G}$ *is* $(SC)_p$; (c) $X = \overline{X}$; (d) $[X^0 : X]$ = *a power of* p^*.

Proof. This theorem follows driectly from Lemma 2.1, since every etale group covering of $\mathfrak{G}$ is (essentially) obtained as the standard isogeny $\mathfrak{G}(Y, \nabla) \to \mathfrak{G}(X, \nabla)$ induced from the inclusion $(X, \nabla) \hookrightarrow (Y, \nabla)$ for some subgroup $X \subset Y \subset X^0$ with $(p^*, [Y: X]) = 1$. Q.E.D.

2.3. **Corollary.** *If k is algebraically closed, for a connected semisimple k-group scheme $\mathfrak{G}$, the following facts hold:*

(1) *The dual Hopf algebra* $\mathrm{hy}(\mathfrak{G})^0$ *is finitely generated.*

(2) $\overline{\mathfrak{G}} = \mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^0)$ *is smooth connected semisimple and* (*SC*).

(3) *The canonical epimorphism* $\overline{\mathfrak{G}} \to \mathfrak{G}$ *is a p-etale group covering.*

(4) $\mathrm{hy}(\overline{\mathfrak{G}}) \simeq \mathrm{hy}(\mathfrak{G})$.

(5) $\mathrm{hy}(\mathfrak{G})^0 // \mathcal{O}(\mathfrak{G})$ *is finite dimensional over k.*

(6) $\overline{\mathfrak{G}}$ *is a universal group covering, as well as a p-universal group covering, of* $\mathfrak{G}$.

This follows directly from Lemma 1.8 and Theorem 2.2.

3. **The case of semisimple group schemes.** In this section we generalize the results of §2 to the case of *perfect* ground field. Suppose k is perfect throughout §3. Let $\mathfrak{G}$ be a connected smooth *semisimple* affine algebraic k-group scheme. This means that the $\bar{k}$-group scheme $\mathfrak{G} \otimes \bar{k}$ is a Chevalley $\bar{k}$-group scheme, where $\bar{k}$ = the algebraic closure of k. But of course $\mathfrak{G}$ itself is not necessarily of Chevalley type.

If H is an arbitrary Hopf algebra over k, the dual $\bar{k}$-Hopf algebra $(H \otimes \bar{k})^0$ contains $H^0 \otimes \bar{k}$ as a $\bar{k}$-sub-Hopf algebra. If H is *cocommutative* and $(H \otimes \bar{k})^0$ is finitely generated over $\bar{k}$, then it follows from 0.1.1(a) that H^0 is finitely generated over k. In particular put $H = \mathrm{hy}(\mathfrak{G})$ the hyperalgebra of $\mathfrak{G}$. Then $H \otimes \bar{k} = \mathrm{hy}_{\bar{k}}(\mathfrak{G} \otimes \bar{k})$ is the $\bar{k}$-hyperalgebra of the $\bar{k}$-group scheme $\mathfrak{G} \otimes \bar{k}$. Since $(H \otimes \bar{k})^0$ is finitely generated over $\bar{k}$ by Corollary 2.3(1), it follows that H^0 is finitely generated over k. Let $\overline{\mathfrak{G}} = \mathrm{Spec}(H^0)$ and $(\mathfrak{G} \otimes \bar{k})^* = \mathrm{Spec}_{\bar{k}}((H \otimes \bar{k})^0)$. The canonical inclusions of $\bar{k}$-Hopf algebras

$$\mathcal{O}(\mathfrak{G}) \otimes \bar{k} \hookrightarrow H^0 \otimes \bar{k} \hookrightarrow (H \otimes \bar{k})^0$$

induce epimorphisms of affine algebraic $\bar{k}$-group schemes

$$\mathfrak{G} \otimes \bar{k} \leftarrow \overline{\mathfrak{G}} \otimes \bar{k} \leftarrow (\mathfrak{G} \otimes \bar{k})^*$$

whose composite is a universal, as well as a p-universal, group covering of $\mathfrak{G} \otimes \bar{k}$ by Corollary 2.3(6). Applying the functor $\mathrm{hy}_{\bar{k}}(?)$, one can easily conclude that the canonical epimorphism of affine algebraic k-group schemes $\overline{\mathfrak{G}} \to \mathfrak{G}$ (induced from $\mathcal{O}(\mathfrak{G}) \hookrightarrow H^0$) is a *$p$-etale group covering.*

This means in particular that $\mathrm{hy}(\overline{\mathfrak{G}}) \simeq \mathrm{hy}(\mathfrak{G})$, so $\mathcal{O}(\overline{\mathfrak{G}}) \simeq \mathrm{hy}(\overline{\mathfrak{G}})^0$. Therefore $\overline{\mathfrak{G}}$ is (SC) by Lemma 1.8. It follows that $\overline{\mathfrak{G}}$ is a universal, as well as a

p-universal, group covering of $\mathfrak{G}$. Notice that $\overline{\mathfrak{G}}$ is connected smooth *semisimple* (since isogenous to $\mathfrak{G}$).

In particular if $\mathfrak{G}$ is $(SC)_p$, then $\overline{\mathfrak{G}} \simeq \mathfrak{G}$, so $\mathfrak{G}$ is (SC). Summarizing we have

3.1. Theorem. *Suppose k is perfect and let $\mathfrak{G}$ be a connected smooth semisimple affine algebraic k-group scheme. Then the following facts hold.*

(1) *The dual Hopf algebra* $\mathrm{hy}(\mathfrak{G})^0$ *is finitely generated.*

(2) $\overline{\mathfrak{G}} = \mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^0)$ *is smooth connected semisimple and* (SC).

(3) *The canonical epimorphism* $\overline{\mathfrak{G}} \to \mathfrak{G}$ *is a p-etale group covering.*

(4) $\mathrm{hy}(\overline{\mathfrak{G}}) \simeq \mathrm{hy}(\mathfrak{G})$.

(5) $\mathrm{hy}(\mathfrak{G})^0 /\!/ \mathcal{O}(\mathfrak{G})$ *is finite dimensional over k.*

(6) $\overline{\mathfrak{G}}$ *is a universal, as well as a p-universal, group covering of* $\mathfrak{G}$.

(7) $\mathfrak{G}$ *is* $(SC)_p$ *if and only if* (SC).

The purpose of the rest of this section is to prove that the canonical epimorphism $(\mathfrak{G} \otimes \overline{k})^* \to \overline{\mathfrak{G}} \otimes \overline{k}$ is isomorphic.

In general if $\overline{\mathfrak{H}}$ is a $\overline{k}$-group scheme, we mean by a *k-form* of $\overline{\mathfrak{H}}$, a k-group scheme $\mathfrak{H}$ with $\mathfrak{H} \otimes \overline{k} \simeq \overline{\mathfrak{H}}$. If $\overline{\mathfrak{f}}: \overline{\mathfrak{H}} \to \mathfrak{G} \otimes \overline{k}$ is a homomorphism of $\overline{k}$-group schemes, we say that $\overline{\mathfrak{f}}$ is *defined over k* if there are a k-form $\mathfrak{H}$ of $\overline{\mathfrak{H}}$ and a homomorphism of k-group schemes $\mathfrak{f}: \mathfrak{H} \to \mathfrak{G}$ with

$$\overline{\mathfrak{f}}: \overline{\mathfrak{H}} \simeq \mathfrak{H} \otimes \overline{k} \xrightarrow{\mathfrak{f} \otimes \overline{k}} \mathfrak{G} \otimes \overline{k}.$$

3.2. Lemma. *The canonical epimorphism of $\overline{k}$-group schemes, $(\mathfrak{G} \otimes \overline{k})^* \to \mathfrak{G} \otimes \overline{k}$, is defined over some finite extension field l of k.*

Proof. Let k_0 be the prime field in k. Each Chevalley $\overline{k}$-group scheme $\mathfrak{G}(X, \nabla)$ has a canonical k_0-form [5, p. 53] and the standard isogeny $\mathfrak{G}(\overline{X}, \nabla) \to \mathfrak{G}(X, \nabla)$ induced from the inclusion $X \hookrightarrow \overline{X}$ can be taken to be defined over k_0 [5, p. 60]. Now that $\mathfrak{G}$ is semisimple, there is a unique root system (X, ∇) with $\mathfrak{G} \otimes \overline{k} \simeq \mathfrak{G}(X, \nabla)$. By the universal mapping property of universal group coverings, we have a commutative diagram

$$\begin{array}{ccc} (\mathfrak{G} \otimes \overline{k})^* & \simeq & \mathfrak{G}(\overline{X}, \nabla) \\ \mathrm{cano} \downarrow & & \downarrow \mathrm{cano} \\ \mathfrak{G} \otimes \overline{k} & \simeq & \mathfrak{G}(X, \nabla) \end{array}$$

where the right vertical arrow is defined over k_0. Since $\mathfrak{G}$ and $\mathfrak{G}(X, \nabla)$ are both algebraic, there is a finite extension l/k such that the isomorphism $\mathfrak{G} \otimes \overline{k} \simeq \mathfrak{G}(X, \nabla)$ is defined over l. The lemma follows from this directly. Q.E.D.

If we put $H = \mathrm{hy}(\mathfrak{G})$ as before, the above lemma implies that there is an l-sub-Hopf algebra A of $(H \otimes \bar{k})^0$ with

$$\mathcal{O}(\mathfrak{G}) \otimes_k l \subset A \quad \text{and} \quad A \otimes_l \bar{k} = (H \otimes \bar{k})^0.$$

If we put $\mathfrak{L} = \mathrm{Spec}_l(A)$, then the canonical epimorphism of l-group schemes $\mathfrak{L} \to \mathfrak{G} \otimes_k l$ is an etale group covering, since so is

$$(\mathfrak{G} \otimes_k \bar{k})^* = \mathfrak{L} \otimes_l \bar{k} \to (\mathfrak{G} \otimes_k l) \otimes_l \bar{k} = \mathfrak{G} \otimes_k \bar{k}.$$

Hence $\mathrm{hy}_l(\mathfrak{L}) \xrightarrow{\sim} H \otimes_k l$, so $\mathrm{hy}_l(\mathfrak{L})^0 \xleftarrow{\sim} H^0 \otimes_k l$. (In general, if l/k is finite, $(B \otimes_k l)^0 = B^0 \otimes_k l$ for all k-algebras B.) Thus we get a commutative diagram:

$$\begin{array}{ccc} \mathrm{hy}_l(\mathfrak{L})^0 & \xleftarrow{\sim} & H^0 \otimes_k l \\ \cup & & \cup \\ A & \supset & \mathcal{O}(\mathfrak{G}) \otimes_k l. \end{array}$$

Applying the functor $? \otimes_l \bar{k}$, we obtain

$$\begin{array}{ccc} \mathrm{hy}_l(\mathfrak{L})^0 \otimes_l \bar{k} & \xleftarrow{\sim} & H^0 \otimes_k \bar{k} \\ \cup & & \cup \\ A \otimes_l \bar{k} & \supset & \mathcal{O}(\mathfrak{G}) \otimes_k \bar{k}. \end{array}$$

But since $A \otimes_l \bar{k} = (H \otimes_k \bar{k})^0$, this means that $(H \otimes_k \bar{k})^0 = H^0 \otimes_k \bar{k}$ and hence that $(\mathfrak{G} \otimes \bar{k})^* \xrightarrow{\sim} \bar{\mathfrak{G}} \otimes \bar{k}$. Thus we proved

3.3. **Theorem.** *Suppose k is perfect. Let $\mathfrak{G}$ be a connected smooth semisimple affine algebraic k-group scheme. If $\bar{\mathfrak{G}}$ is the universal group covering of $\mathfrak{G}$, then $\bar{\mathfrak{G}} \otimes \bar{k}$ is the universal group covering of the $\bar{k}$-scheme $\mathfrak{G} \otimes \bar{k}$. In particular $\mathfrak{G}$ is (SC) (or equivalently $(SC)_p$) if and only if $\mathfrak{G} \otimes \bar{k}$ is (SC) (or equivalently $(SC)_p$) as a $\bar{k}$-group scheme.*

4. **Semisimple hyperalgebras.** In this section assume k is *perfect*. We prove that every *semisimple* hyperalgebra of finite type is the hyperalgebra of some connected semisimple k-group scheme. Since connected semisimple k-group schemes have the universal group covering, it will follow from Theorem 3.1 that the category of (SC) semisimple k-group schemes is equivalent to the category of semisimple hyperalgebras of finite type.

A hyperalgebra of finite type J is called *representable* if the dual Hopf algebra J^0 is dense in J^*, or equivalently if there is an affine algebraic k-group scheme $\mathfrak{G}$ with $J \subset \mathrm{hy}(\mathfrak{G})$. Here the k-group scheme $\mathfrak{G}$ can be taken so that J is *dense* in $\mathrm{hy}(\mathfrak{G})$ (i.e., $\mathcal{O}(\mathfrak{G}) \subset J^0$) and hence in particular that $\mathfrak{G}$ is connected. Each finite type hyperalgebra J contains the *smallest normal sub-hyperalgebra* K such that the quotient hyperalgebra $J//K$ is representable;

K is the Hopf kernel of $J \to J^{00}$. We shall denote $K = K_{\mathrm{rep}}(J)$. It follows from Dieudonné [1, Proposition 20, p. 365] that $K_{\mathrm{rep}}(J)$ is contained in the center of J.

4.1. **Lemma.** *Let J be a smooth hyperalgebra of finite type and K a* <u>*finite*</u> *normal subhyperalgebra of J. If $J//K$ is representable, then so is J.*

Proof. We can assume $p > 0$. With a sufficiently large positive integer r, let V^r: $J \to J^{(p^r)}$ denote the r-times iterated *Verschiebung* map of J (§0.1). Then V^r is surjective (§0.3) with a finite Hopf kernel containing K and hence it induces a surjective homomorphism $J//K \to J^{(p^r)}$ which has also a finite Hopf kernel. If, in general, $l|k$ is a field extension and if L is a representable hyperalgebra over k, then the l-hyperalgebra $l \otimes L$ is representable too. Since now k is perfect, we have $J = (J^{(p^r)})^{(p^{-r})}$. Hence it suffices to show the representability of $J^{(p^r)}$, which will follow from the next lemma. Q.E.D.

4.2. **Lemma.** *If J is a representable hyperalgebra of finite type and K a finite normal subhyperalgebra of J, then the quotient $J//K$ also is representable.*

Proof. Imbed J into hy$(\mathfrak{G})$ as a *dense* subhyperalgebra for some connected affine algebraic k-group scheme $\mathfrak{G}$. Since $[J, K] \subset K$, it follows from 0.3.4(e) that $[\mathrm{hy}(\mathfrak{G}), K] \subset K$. Hence K is a normal subhyperalgebra of hy$(\mathfrak{G})$. Since any finite dimensional subhyperalgebra of hy$(\mathfrak{G})$ is closed (0.3.4(a)), it follows that there is a unique closed normal connected subgroup scheme $\mathfrak{H}$ of $\mathfrak{G}$ with $K = \mathrm{hy}(\mathfrak{H})$. The induced inclusion

$$J//K \hookrightarrow \mathrm{hy}(\mathfrak{G})//\mathrm{hy}(\mathfrak{H}) = \mathrm{hy}(\mathfrak{G}/\mathfrak{H})$$

proves the representability of $J//K$. Q.E.D.

4.3. **Proposition.** *Let J be a* <u>*smooth*</u> *hyperalgebra of finite type. Then $K_{\mathrm{rep}}(J)$ is smooth too.*

Proof. Let K_{sm} denote the *smooth part* of $K = K_{\mathrm{rep}}(J)$ (§0.3). Since K is central, we can consider the quotient hyperalgebra $J//K_{\mathrm{sm}}$ which has a finite dimensional normal subhyperalgebra $K//K_{\mathrm{sm}}$ (§0.3). Since the quotient $(J//K_{\mathrm{sm}})//(K//K_{\mathrm{sm}}) = J//K$ is representable, it follows from Lemma 4.1 that so is $J//K_{\mathrm{sm}}$. By the definition of $K_{\mathrm{rep}}(J)$, this proves that $K = K_{\mathrm{sm}}$. Q.E.D.

For each hyperalgebra J, we have defined the derived subhyperalgebra $[J, J]$ in §0.1. Hence we can define the derived series $\{J^{(\nu)}\}$ of J by $J^{(\nu)} = [J^{(\nu-1)}, J^{(\nu-1)}]$ and $J^{(0)} = J$. The hyperalgebra J is called *solvable* if $J^{(N)} = k$ for some N.

Let J be a *smooth* hyperalgebra of finite type. The set of *normal smooth* *solvable* subhyperalgebras of J contains clearly the largest element denoted by $\mathrm{rad}(J)$, which we shall call the *radical* of J. J is called *semisimple* if $\mathrm{rad}(J) = k$. In general $J//\mathrm{rad}(J)$ is semisimple for any smooth hyperalgebra J of finite type.

4.4. **Proposition.** *Semisimple smooth hyperalgebras of finite type are representable.*

Proof. If J is such a hyperalgebra, then $K_{\mathrm{rep}}(J)$ is smooth and central by Proposition 4.3 and hence is trivial. Q.E.D.

Let J be a smooth semisimple hyperalgebra of finite type. Embed as usual J into the hyperalgebra $\mathrm{hy}(\mathfrak{G})$ as a dense subhyperalgebra for some connected affine algebriac smooth k-group scheme $\mathfrak{G}$. Let $\mathfrak{rad}(\mathfrak{G})$ denote the radical of $\mathfrak{G}$.

4.5. **Lemma.** *The hyperalgebra* $J \cap \mathrm{hy}(\mathfrak{rad}(\mathfrak{G})) = K$ *is finite deminsional.*

Proof. K is clearly normal solvable in J. Since $[J, K] \subset K$, it follows from $J = J_{\mathrm{sm}}$ that $[J, K_{\mathrm{sm}}] \subset K_{\mathrm{sm}}$, where $(\ )_{\mathrm{sm}}$ denotes the smooth part of the hyperalgebra. Thus K_{sm} is normal smooth solvable in J and is trivial by assumption. This means that K is finite dimensional. Q.E.D.

Notice that $J//K \subset \mathrm{hy}(\mathfrak{G}/\mathfrak{rad}(\mathfrak{G}))$ is also dense. Hence it follows from 0.3.4(f) that

$$[J//K,\ J//K] = [\mathrm{hy}(\mathfrak{G}/\mathfrak{rad}(\mathfrak{G})),\ \mathrm{hy}(\mathfrak{G}/\mathfrak{rad}(\mathfrak{G}))].$$

Since $\mathfrak{G}/\mathfrak{rad}(\mathfrak{G})$ is a semisimple k-group scheme, it coincides with the derived group. This means that

$$[J//K,\ J//K) = \mathrm{hy}(\mathfrak{G}/\mathfrak{rad}(\mathfrak{G}))$$

and hence, in particular, that $J//K = [J//K, J//K]$, or equivalently that $J = [J, J]K$. Then the quotient hyperalgebra $J//[J, J]$ is finite and smooth and therefore $J = [J, J]$. This proves that J is a *closed* subhyperalgebra of $\mathrm{hy}(\mathfrak{G})$ (0.3.4(g)). Since J is dense in $\mathrm{hy}(\mathfrak{G})$, it follows that $J = \mathrm{hy}(\mathfrak{G})$. Now $\mathrm{hy}(\mathfrak{rad}(\mathfrak{G}))$ is a normal smooth solvable subhyperalgebra of J. Since J is semisimple, it follows that $\mathrm{hy}(\mathfrak{rad}(\mathfrak{G})) = k$ and hence that $\mathfrak{rad}(\mathfrak{G}) = (e)$ or that $\mathfrak{G}$ is *semisimple*. Thus we have proved

4.6. **Theorem.** *If* k *is perfect each smooth semisimple hyperalgebra of finite type is the hyperalgebra of some connected semisimple* k*-group scheme.*

Conversely we have

4.7. **Proposition.** *Let* $\mathfrak{G}$ *be a connected smooth affine algebriac* k*-group scheme with radical* $\mathfrak{rad}(\mathfrak{G})$. *Then* $\mathrm{rad}(\mathrm{hy}(\mathfrak{G})) = \mathrm{hy}(\mathfrak{rad}(\mathfrak{G}))$. *In particular* *the k-group scheme $\mathfrak{G}$ is semisimple if and only if the hyperalgebra* $\mathrm{hy}(\mathfrak{G})$ *is.*

Proof. The inclusion $\mathrm{hy}(\mathfrak{rad}(\mathfrak{G})) \subset \mathrm{rad}(\mathrm{hy}(\mathfrak{G}))$ is clear. Let K be a normal smooth solvable subhyperalgebra of $\mathrm{hy}(\mathfrak{G})$ and $A(K)$ the algebraic hull of K in $\mathrm{hy}(\mathfrak{G})$ (§0.3). Since K is contained in the smooth part $A(K)_{\mathrm{sm}}$, which is also closed, it follows that $A(K)$ is smooth. Since $[A(K), A(K)] = [K, K]$, the solvability of $A(K)$ follows. Similarly the equality $[\mathrm{hy}(\mathfrak{G}), A(K)] = [\mathrm{hy}(\mathfrak{G}), K]$ (0.3.4(d)) implies that $A(K)$ is normal in $\mathrm{hy}(\mathfrak{G})$. Let $\mathfrak{H}$ be a unique closed connected normal subgroup scheme of $\mathfrak{G}$ with $\mathrm{hy}(\mathfrak{H}) = A(K)$. Then $\mathfrak{H}$ is solvable smooth and hence contained in $\mathfrak{rad}(\mathfrak{G})$. Therefore $K \subset \mathrm{hy}(\mathfrak{rad}(\mathfrak{G}))$. Q.E.D.

By virtue of Theorem 3.1, Theorem 4.6 has the following corollary:

4.8. **Theorem.** *Suppose k is perfect. If J is a smooth semisimple hyperalgebra of finite type, then the dual Hopf algebra J^0 is finitely generated and the corresponidng affine k-group scheme* $\mathrm{Spec}(J^0)$ *is (SC) semisimple and has J as its hyperalgebra. The functors $J \mapsto \mathrm{Spec}(J^0)$ and $\mathfrak{G} \mapsto \mathrm{hy}(\mathfrak{G})$ give rise to an equivalence between the categories of smooth semisimple hyperalgebras of finite type and of (SC) semisimple k-group schemes.*

5. **An example of nonreductive (SC) k-group schemes.** To conclude this paper we shall provide a simple example of (SC) affine algebraic k-group schemes which are *not reductive*.

In this section we shall assume $k = \bar{k}$ with $p > 0$. For each left finitely generated $k[F]$-module M, let $\mathfrak{U}(M) = \mathrm{Spec}(U^{[p]}(M))$ denote the corresponding commutative unipotent affine algebraic k-group scheme killed by the Verschiebung map (see §0.2). Let $\bar{\mathfrak{G}}$ denote an arbitrary affine algebraic k-group scheme.

5.1. **Definition.** Let M be a finitely generated left $k[F]$-module. View the affine ring $\mathcal{O}(\bar{\mathfrak{G}})$ as a left $k[F]$-module via $Fa = a^p$, $a \in \mathcal{O}(\bar{\mathfrak{G}})$. A right $\mathcal{O}(\bar{\mathfrak{G}})$-comodule structure on M, $\rho\colon M \to M \otimes \mathcal{O}(\bar{\mathfrak{G}})$, which is $k[F]$-linear, is said to be compatible with the $k[F]$-module structure. A left *$k[F]$-$\bar{\mathfrak{G}}$-module* means a finitely generated left $k[F]$-module with a right $\mathcal{O}(\bar{\mathfrak{G}})$-comodule structure compatible with the $k[F]$-module structure.

5.2. **Lemma.** *The k-group scheme actions as automorphisms of k-group schemes $\mathfrak{U}(M) \times \bar{\mathfrak{G}} \to \mathfrak{U}(M)$ correspond bijectively with the right $\mathcal{O}(\bar{\mathfrak{G}})$-comodule structures on M which are compatible with the $k[F]$-module structure.*

Proof. The k-group scheme actions of the above mentioned type can be identified with those right comodule structures

$$\rho\colon\ U^{[p]}(M) \to U^{[p]}(M) \otimes \mathcal{O}(\bar{\mathfrak{G}})$$

which are compatible with the Hopf algebra structure on $U^{[p]}(M)$. This means in particular that $\rho(M) \subset M \otimes \mathcal{O}(\bar{\mathfrak{G}})$ and that the restricted coaction $\rho|M$ is compatible with the $k[F]$-module structure. Conversely if M is a left $k[F]$-$\bar{\mathfrak{G}}$-module with coaction $\sigma\colon M \to M \otimes \mathcal{O}(\bar{\mathfrak{G}})$ then the extended algebra map $\sigma\colon U^{[p]}(M) \to U^{[p]}(M) \otimes \mathcal{O}(\bar{\mathfrak{G}})$ is easily seen to be compatible with the Hopf algebra structure of $U^{[p]}(M)$. Q.E.D.

In the following let M be a (finitely generated) left $k[F]$-$\bar{\mathfrak{G}}$-module such that $\mathfrak{U}(M)$ is *connected* (see §0.2). The isomorphism classes of exact sequences of homomorphisms of affine k-group schemes, $1 \to \mathfrak{U}(M) \to \mathfrak{G} \to \bar{\mathfrak{G}} \to 1$, the right action of $\bar{\mathfrak{G}}$ on $\mathfrak{U}(M)$ induced from which coincides with the original action, form an abelian group in a usual manner, which we shall write as $\operatorname{Ext}(\bar{\mathfrak{G}}, \mathfrak{U}(M))$. The neutral element is supplied by the semidirect product $\bar{\mathfrak{G}} \times_s \mathfrak{U}(M)$.

5.3. **Definition.** The $k[F]$-$\bar{\mathfrak{G}}$-module M is of type (#) if the following condition is satisfied:

Let $k = k[F]/(F-1)$ be a trivial left $k[F]$-$\bar{\mathfrak{G}}$-module. Each exact sequence of $k[F]$-$\bar{\mathfrak{G}}$-modules of the form $0 \to M \to N \to k \to 0$ splits *as* $k[F]$-*modules*.

5.4. **Proposition.** *If* $\bar{\mathfrak{G}}$ *is (SC) and the* $k[F]$-$\bar{\mathfrak{G}}$-*module* M *is of type* (#), *then each k-group scheme* $\mathfrak{G}$ *in* $\operatorname{Ext}(\bar{\mathfrak{G}}, \mathfrak{U}(M))$ *is (SC) too.*

Proof. Let $\mathfrak{G} \in \operatorname{Ext}(\bar{\mathfrak{G}}, \mathfrak{U}(M))$, $(\mathfrak{H}, \gamma)$ an etale group covering of $\mathfrak{G}$, and $\mathfrak{H}_u$ the connected component of $\gamma^{-1}(\mathfrak{U}(M))$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \mathfrak{H}_u & \to & \mathfrak{H} & \to & \mathfrak{H}/\mathfrak{H}_u & \to & 1 \\ & & \alpha\downarrow & & \gamma\downarrow & & \beta\downarrow & & \\ 1 & \to & \mathfrak{U}(M) & \to & \mathfrak{G} & \to & \bar{\mathfrak{G}} & \to & 1 \end{array}$$

where the homomorphisms α, β and γ are all etale group coverings. Since $\bar{\mathfrak{G}}$ is (SC), β is an isomorphism, via which $\bar{\mathfrak{G}}$ operates on $\mathfrak{H}_u$ which is commutative unipotent and killed by the Verschiebung, because $\operatorname{hy}(\alpha)\colon \operatorname{hy}(\mathfrak{H}_u) \xrightarrow{\sim} \operatorname{hy}(\mathfrak{U}(M))$. Hence we can write $\mathfrak{H}_u = \mathfrak{U}(N)$ for some left $k[F]$-$\bar{\mathfrak{G}}$-module N. The homomorphism α induces an injection of $k[F]$-$\bar{\mathfrak{G}}$-modules $M \hookrightarrow N$ and the quotient module N/M, which represents the kernel of α, which is etale, is therefore a direct sum of some finite number of copies of the trivial $k[F]$-$\bar{\mathfrak{G}}$-module $k = k[F]/(F-1)$. Say $N/M \simeq k^r$. The exact sequence of $k[F]$-$\bar{\mathfrak{G}}$-modules $0 \to M \to N \to k^r \to 0$ splits as $k[F]$-modules, because M is of type (#). But since $\mathfrak{U}(N)$ is connected, this means that $r = 0$ and hence that α is an isomorphism. Hence γ is an isomorphism and $\mathfrak{G}$ is (SC). Q.E.D.

Suppose that the affine algebraic k-group scheme $\overline{\mathfrak{G}}$ is connected smooth and put $A = \mathcal{O}(\overline{\mathfrak{G}})$. Let V be a finite dimensional right A-comodule with the structure map ρ: $V \longrightarrow V \otimes A$ and put $M = k[F] \otimes V$. Extending ρ $k[F]$-linearly, we make M into a $k[F]$-$\overline{\mathfrak{G}}$-module.

5.5. **Proposition.** *Suppose that $\overline{\mathfrak{G}}$ is connected smooth. Then the $k[F]$-$\overline{\mathfrak{G}}$-module $M = k[F] \otimes V$ is of type (#) if and only if $V^{\overline{\mathfrak{G}}} = 0$.*

Proof. The 'only if' part is easy. Suppose that $V^{\overline{\mathfrak{G}}} = 0$. Let $V \otimes_i A$ be the tensor product of V and A over k with the defining relation

$$\lambda v \otimes_i a = v \otimes_i \lambda^{p^i} a \quad \text{for } \lambda \in k,\ v \in V,\ a \in A.$$

The space $M \otimes A$ is the direct sum of $kF^i \otimes (V \otimes_i A)$, $i \geq 0$. Since A is smooth, the k-linear maps

$$V \otimes_i A \hookrightarrow V \otimes_{i+j} A, \quad v \otimes_i a \mapsto (v \otimes_i a)^{(j)} \underset{\text{def}}{=} v \otimes_{j+j} a^{p^j}$$

are injective. Let $0 \longrightarrow M \longrightarrow N \longrightarrow k[F]/(F-1) \longrightarrow 0$ be an exact sequence of $k[F]$-$\overline{\mathfrak{G}}$-modules. Thus we can write $N = M \oplus ke$ with $Fe - e = \Sigma_i F^i \otimes v_i$ ($v_i \in V$) and $\rho(e) - e \otimes 1 = \Sigma_i F^i \otimes x_i$ ($x_i \in V \otimes_i A$). Since ρ is $k[F]$-linear, it follows that $x_i = \partial(v_i)^{(i)} + x_{i-1}^{(1)}$ for $i > 0$ and $x_0 = \partial(v_0)$, where $\partial(v) = v \otimes 1 - \rho(v) \in V \otimes A$ $(= V \otimes_0 A)$ for $v \in V$. Hence if we put $u_i = v_i + \cdots + v_0 \in V$, then $x_i = \partial(u_i)^{(i)}$. Since the k-linear map ∂: $V \longrightarrow V \otimes A$ is injective by assumption, it follows that u_i are equal to zero for almost all i. Hence we can well define an element of M $m = -\Sigma F^i \otimes u_i$. It is easy to see $Fm - m = Fe - e$. This means that the $k[F]$-$\overline{\mathfrak{G}}$-module M is of type (#). Q.E.D.

If we take $\overline{\mathfrak{G}}$ to be an (SC) semisimple k-group scheme and V a nontrivial irreducible k-$\overline{\mathfrak{G}}$-module, then $V^{\overline{\mathfrak{G}}} = 0$ clearly and hence each element of $\mathrm{Ext}(\overline{\mathfrak{G}}, \mathfrak{D}_a(V))$ provides an example of nonreductive (SC) k-group schemes, where $\mathfrak{D}_a(V) = \mathrm{Spec}(S(V))$ represented by the symmetric algebra $S(V)$ on V, which is a Hopf algebra having V as primitive elements.

Added in proof. All subhyperalgebras of a finite type hyperalgebra is of finite type by definition. Each quotient hyperalgebra, which must be of the form $J//K$ with K a normal subhyperalgebra by (0.1.2), is also of finite type, when J is [T_{II}, 5.5.2.1]. This is implicitly used in §4.

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