THE STRUCTURE OF SEMIPRIMARY AND
NOETHERIAN HEREDITARY RINGS

BY

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ABSTRACT. In the first portion of this paper a structure theorem for
semiprimary hereditary rings is given in terms of $M \times M$ "triangular" row-finite
matrices over a division ring D. This structure theorem differs from previous
theorems of this type in that the representation is explicit in terms of matrices
over a division ring. In the second portion of this paper we are able to apply
the results of Gordon and Small to obtain a structure theorem for semiheredi-
tary and left hereditary rings which are left orders in a semiprimary ring. In
the case of the left hereditary rings, the representation is explicit in terms of ma-
trices over left hereditary Goldie prime rings and their respective classical left
quotient rings. As an application we obtain, by a different method, a non-
Noetherian generalization of a result of Chatters which states that a two-sided
hereditary Noetherian ring is a ring direct sum of an Artinian ring and a semi-
prime ring.

1. **Preliminaries.** Throughout this paper all rings R are assumed to be as-
sociative with multiplicative identity, and all R-modules will be unitary. A right
or left R-module will be denoted by M_R or $_RM$ respectively. The socle of an R-
module M will be denoted by soc (M).

If R is a ring, the Jacobson radical of R will be denoted by $J(R)$, or just J
if there is no ambiguity. The prime radical of a ring R is denoted by $N(R)$ or
simply N. Semisimple shall mean semisimple Artinian. A ring R is semiperfect
[3] if R/J is semisimple and idempotents can be lifted modulo J. A ring R is
semiprimary if R is semiperfect and J is nilpotent. In this case $N = J$.

The module M is of finite (Goldie) dimension over R if every direct sum of
nonzero submodules of M contains only a finite number of summands. We say
that R is left finite dimensional if $_RR$ is of finite dimension.

If R is left nonsingular, then R possesses a maximal left quotient ring

(MLQ) T which is a regular ring [21]. If R is left finite dimensional, then the MLQ of R is semisimple [28].

A nonzero element of a ring R is regular if it has zero left and right annihilators. A ring Q is said to be a classical left quotient ring of R (R is called a left order in Q) if (i) $R \subset Q$, (ii) every element of Q has the form $c^{-1}a$ where $a, c \in R$ with c regular in R, and (iii) every regular element of R has an inverse in Q.

For all homological notions, we refer the reader to [4].

Let R be a semiperfect ring. Then $R = \bigoplus_{1 \leqslant i \leqslant n} \Sigma_{1 \leqslant j \leqslant m_i} Re_{ij}$ where the e_{ij} are primitive idempotents and $Re_{ij} \simeq Re_{i'j'}$ if and only if $i = i'$. Let $e = e_{11} + \cdots + e_{n1}$. Then the ring $eRe = R'$ is a direct sum of pairwise nonisomorphic primitive left ideals and is semiperfect. R' is called the *basic ring* of R and R' is unique up to isomorphism [15]. The ring R may be recovered from the ring R'. In fact, R may be taken to be the ring of all $n \times n$ blocked matrices

$$\begin{bmatrix} B_{11} & \cdot & \cdot & \cdot & B_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ B_{n1} & \cdot & \cdot & \cdot & B_{nn} \end{bmatrix}$$

where B_{ij} is the $e_iR'e_i - e_jR'e_j$ bimodule of $m_i \times m_j$ matrices with entries in $e_iR'e_j$ [15]. Furthermore it is well known [15] that R and R' are Morita equivalent. Hence we may reduce the study of the representation theory of semiperfect rings to that of semiperfect basic rings.

If D is a division ring and if I and J are sets, then an $I \times J$ matrix over D is a function from $I \times J$ to D. Multiplication of row-finite $I \times I$ matrices can be defined in a natural way, and there is the usual correspondence between linear transformations and matrices. The interested reader may consult Jacobson [20, Chapter IX].

A ring is said to have *unmixed characteristic* if as an abelian group it is p-primary or torsionfree. A ring R is said to have *uniform characteristic* if every nonzero element of R has the same additive order.

PROPOSITION 1.1. *Let R be an indecomposable semiprimary ring. Then R has unmixed characteristic.*

PROOF. If $(R, +)$ is a torsion group, then it has bounded order since R has only finitely many nonisomorphic simple modules. Since the p-primary

components of $(R, +)$ are ideals of R, the group primary decomposition of R yields a ring decomposition of R. R is indecomposable so that R must have just one p-primary component and hence has unmixed characteristic. Hence we may assume that $(R, +)$ is not a torsion group.

Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive idempotents for R. Consider a simple right module S; S is isomorphic to $e_i R/e_i N$ for some i. S is a left vector space over the division ring $e_i Re_i/e_i Ne_i$, and hence the characteristic of S is the same as that of $e_i Re_i/e_i Ne_i$. The ring $e_i Re_i$ is a local semiprimary ring whose only simple module is $e_i Re_i/e_i Ne_i$ and hence $e_i Re_i$ has unmixed characteristic as does every module over $e_i Re_i$.

Let e be the sum of those e_i's for which $e_i R/e_i N$ has characteristic zero and let $f = 1 - e$ be the sum of the remainder. The ring eRe is a semiprimary ring over which every simple module has characteristic zero; hence every module over eRe is torsionfree as an abelian group. The ring fRf, however, is a semiprimary ring over which every simple module has prime characteristic; hence every module over fRf is torsion as an abelian group. This forces the bimodules fRe and eRf to be 0. Thus R is the ring direct sum of eRe and fRf, and since R is an indecomposable ring for which $(R, +)$ is not torsion, it must be that $R = eRe$ is torsionfree and R has unmixed characteristic. □

COROLLARY 1.2. *A nonsingular indecomposable semiprimary ring has uniform characteristic.*

PROOF. If R has characteristic zero, then the result follows from Proposition 1.1. If not, then by Proposition 1.1, R must have characteristic p^k. Let Q be the maximal quotient ring of R; Q is semiprime. If $k \neq 1$, then the ideal $(p1)Q$ is nilpotent of index k since $p1$ is an element of the center of Q. This is a contradiction since Q is semiprime; thus char $R = p$ and R has uniform characteristic. □

It is also true that an indecomposable PWD (see §2) has uniform characteristic, the proof being very similar to that of Proposition 1.1.

We will need the following result due to P. M. Cohn [10].

THEOREM 1.3. *Two division rings K, L can be imbedded in a common division ring if and only if they have the same characteristic.* □

2. **Hereditary semiprimary rings.** The purpose of this section is to prove Theorem 2.1 which describes the structure of a hereditary semiprimary ring R in terms of $M \times M$ "triangular" row-finite matrices over a division ring D. There have been several results in the past (Chase [8], Harada [19], Gordon [15]) concerning triangular representation of a hereditary ring. Theorem 2.1 differs from

these mainly in that it gives an explicit representation in terms of matrices over a division ring. The results of this section will be used in §3, which is concerned with semihereditary and hereditary rings. Some of the results are broader than needed, but they will be of use in a subsequent paper on splitting rings.

THEOREM 2.1. *Let R be a basic indecomposable hereditary semiprimary ring. Then*: (i) *There is a division ring D.* (ii) *There are ordinal numbers* $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_n$ *(let* $M = l_1 + \cdots + l_n$*).* (iii) *For each* $i = 1, \ldots, n$ *there is a division subring* D_i *of the ring of* $l_i \times l_i$ *row-finite matrices over D.* (iv) *For each i there is a sequence of integers* $i < i(1) < i(2) < \cdots < i(k) \leqslant n$ *and a sequence of ordinal numbers* $v(1), v(2), \ldots, v(k)$ *for which* $\Sigma_{\alpha=1}^{k} l_{i(\alpha)} v(\alpha) = l_i$*; such that R is isomorphic to the following blocked lower triangular subring of the ring Q of* $M \times M$ *row-finite matrices over D*:

(1)

where each N_i *is defined recursively in terms of* N_j *for* $j > i$ *by*

(2) $\qquad N_i = \begin{bmatrix} C_1^1 & C_2^1 & \cdots & C_{v(1)}^1 & \cdots & C_1^\alpha & \cdots & C_{v(\alpha)}^\alpha & \cdots & C_{v(k)}^k \end{bmatrix}$

and where C_β^α *is the collection of all matrices of the following form*

(3) $\qquad\qquad\qquad C_\beta^\alpha = {}^t[0 \mathbin{\vdots} D_{i(\alpha)} \mathbin{\vdots} N_{i(\alpha)}]$

where the zero denotes the appropriate sized zero matrix, and ^{t} *denotes transpose.*

Conversely, any such matrix ring is a basic hereditary semiprimary ring and Q contains a copy of the maximal left quotient ring of R.

The ring R will be left Artinian exactly when the ordinal numbers $v(\alpha)$ *are integers for each i.* □

As a simple example of the theorem let R be the ring of matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & b+c & d \end{bmatrix}$$

where a, b, c, and d are real numbers. Then R is an indecomposable hereditary semiprimary ring and R is isomorphic to the ring S of matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & d \end{bmatrix}$$

The theorem states that there always is such an isomorphism.

The theorem is stated for an indecomposable ring since a semiprimary ring is a direct product of finitely many indecomposable semiprimary rings, and since a finite product of rings is hereditary if and only if each factor ring is hereditary. The theorem also assumes that R is basic; in view of the results stated in §1 concerning basic rings and Morita equivalence this gives no loss of generality. While the assumption that R is basic does not alter any ideas in the proof of Theorem 2.1, it does help to simplify notation.

The remainder of this section is concerned with the proof of Theorem 2.1. The proof will consist of imbedding R in a full $M \times M$ row-finite matrix ring Q over a division ring D so that Q is flat as a right R-module; using the flatness of Q_R and the projectivity of left ideals of R to draw conclusions concerning the matrices which represent R; and finally viewing Q as the ring of linear transformations of M-dimensional row space over D so that a change of basis can be used to obtain the desired representation for R.

PROPOSITION 2.2. *Let R be a semiprimary left nonsingular indecomposable ring. Then there is a cardinal number M and a division ring D such that R can be imbedded in the ring Q of $M \times M$ row-finite matrices over D such that $_R Q$ is nonsingular. If R is left hereditary, then this can be done so that $_R Q$ is R-injective and Q_R is R-flat.*

PROOF. Since R is left nonsingular and semiprimary by Johnson [22] the maximal left quotient ring T of R is a ring direct sum $T_1 \times \cdots \times T_k$ where each T_i can be identified with the $c_i \times c_i$ row-finite matrices over a division ring K_i. Let M be the sum of the cardinal numbers $c_1, \ldots, c_k$. Since R is indecomposable, R has uniform characteristic by Corollary 1.2; hence, the division rings $K_1, \ldots, K_k$ have the same characteristic. Hence by Theorem 1.3 there is a divi-

sion ring D containing $K_1, \ldots, K_k$. Each T_i is then a subring of the ring of $c_i \times c_i$ row-finite matrices over D. T and R are then imbedded in Q by the map which imbeds the sum of the $c_i \times c_i$ matrix rings in Q as a diagonal block matrix ring.

If I is an essential left ideal of R and if $q \in Q$ for which $Iq = 0$, then $TIq = 0$. The left R-module ${}_R R$ is an essential submodule of ${}_R T$ since $T = E({}_R R)$; therefore, ${}_R I$ is essential in ${}_R T$, and finally TI is thus an essential left ideal of T. As a result we have that ${}_R Q$ is R-nonsingular if ${}_T Q$ is T-nonsingular. Let I be an essential left ideal of T and let $q \in Q$ for which $Iq = 0$. Let E_{hh} be the $M \times M$ matrix with a 1 in the h, h position and zeroes elsewhere (for $h \in M$). The left ideal TE_{hh} is a minimal left ideal of T; hence TE_{hh} is contained in the essential left ideal I and $E_{hh} \in I$. It follows that $E_{hh}q = 0$ for all $h \in M$, and thus that $q = 0$; that is, ${}_T Q$ is nonsingular.
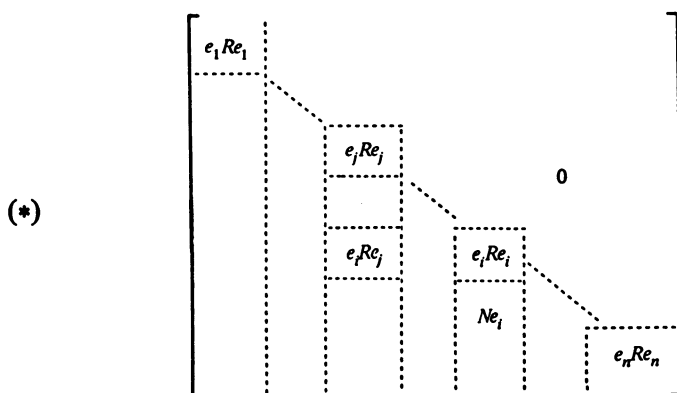
R is hereditary semiprimary; therefore, by Cateforis [5, p. 246] T_R is flat. Since ${}_R R$ is nonsingular, T is a regular ring; hence it follows from [4, p. 123, Problem 10] that Q_R must be flat. Since ${}_Q Q$ is Q-injective and since Q_R is R-flat, we have that ${}_R Q$ is R-injective, also by [4, p. 123, Problem 10]. $\square$

A ring R is called a PWD (see Gordon and Small [18]) if there is a complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ with the property that $xy = 0$ for $x \in e_i Re_k$ and $y \in e_k Re_j$ implies $x = 0$ or $y = 0$. All PWD's are nonsingular [17], and the class of PWD's properly includes the class of hereditary semiprimary rings.

Let R be a basic semiprimary PWD with respect to the complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$. As in [18] we can order the idempotents $\{e_1, \ldots, e_n\}$ in a manner so that $e_i Re_j = \mathrm{Hom}_R(Re_i, Re_j) = 0$ if $i < j$. Since N is nilpotent, it also follows that $e_i Ne_i = \mathrm{Hom}_R(Re_i, Ne_i) = 0$ and hence $e_i Re_i$ is a division ring D_i.

Let K be a set and D a division ring. A K-rspace (K-cspace) over D is the vector space over D consisting of those K-rowtuples (columntuples) which are zero for all but finitely many k in K.

By Proposition 2.2 R can be identified with a subring of the ring Q of $M \times M$ row-finite matrices over a division ring D, and the e_i's will be a set of orthogonal idempotents of Q. The ring Q is the ring of linear transformations of M-rspace over D; the set of idempotents $\{e_1, \ldots, e_n\}$ will then give a direct sum decomposition of M-rspace so that e_i will be the natural projection on the ith subspace. Choose a basis for the ith subspace indexed by a set A_i; this can be done so that $A_1 \cup \cdots \cup A_n = M$. Partially order M so that if $x \in A_i$ and $y \in A_j$ then $x < y$ if $i < j$. Represent each of the linear transformations with respect to the basis consisting of the union of the bases of the subspaces; e_i will then be a matrix with an $A_i \times A_i$ identity block along the diagonal and zeroes elsewhere. In fact R will then consist of a collection of matrices having the following form:

(∗)

$$
\begin{bmatrix}
\boxed{e_1 Re_1} & & & & & \\
& \boxed{e_j Re_j} & & & 0 & \\
& & \boxed{e_iRc_j} & \boxed{e_iRe_i} & & \\
& & & \boxed{Ne_i} & & \\
& & & & & \boxed{e_nRe_n}
\end{bmatrix}
$$

Since R is basic, each e_iRe_i can be identified (as in the diagram (∗)) with a division subring of the ring of $A_i \times A_i$ row-finite matrices over D; as before denote it by D_i. Also, each e_iRe_j can be naturally identified with a left D_i (and right D_j) subspace of the space of $A_i \times A_j$ row-finite matrices over D. Throughout this section the elements of e_iRe_j will be viewed both as $M \times M$ and $A_i \times A_j$ matrices.

We will need the following lemma which appears in Cartan-Eilenberg [4, p. 122, Problem 5].

LEMMA 2.3. *Let R be a ring and let A be a right R-module. If I is a left ideal of R, then $A \otimes_R I \longrightarrow A \otimes_R R$ is a monomorphism if and only if whenever $\Sigma\, a_i\mu_i = 0$ for $\{a_i\} \subset A$ and $\{\mu_i\} \in I$, there exist $\{b_j\} \subset A$ and $\{\lambda_{ij}\} \subset R$ such that $\Sigma_j\, b_j\lambda_{ij} = a_i$ for all i and $\Sigma_i\, \lambda_{ij}\mu_i = 0$ for all j.* □

Let E_{hk} be the $M \times M$ matrix with a 1 in the h, k position and zeroes elsewhere. It follows that $Q = \Pi_{h \in M}E_{hh}Q$. Intuitively $E_{hh}Q$ is the hth row of Q, while QE_{hh} is the hth column.

The next three lemmas will be technical lemmas concerned with the flatness of Q_R.

Let I and J be sets. An $I \times J$ row-finite matrix will be called *faithful* if it is a monomorphism when considered as a linear transformation from I-rspace to J-rspace.

LEMMA 2.4. *Let R be a PWD with structure as in (∗). If Q_R is R-flat, then:* (1) *if x is a nonzero element of e_iRe_j, then x is faithful;* (2) *if $e_iRe_j \neq 0$ and $j < i$, then $|A_i| \leqslant |A_j|$;* (3) *the e_i's can be reordered in a manner consistent with the structure in (∗) so that $|A_j| \geqslant |A_i|$ whenever $j < i$.*

PROOF. (1) Suppose that x is a nonzero element of e_iRe_j and suppose that $qx = 0$ for $0 \neq q \in A_i$-rspace. Identify q with that element of $E_{aa}Q$ for which $qe_i = q$ where $a \in A_i$. Since $E_{aa}Q$ is flat, it follows from Lemma 2.3 that there

are elements $\{b_\alpha\} \subset E_{aa}Q$ and $\{\lambda_\alpha\} \subset R$ for which $\Sigma \, b_\alpha \lambda_\alpha = q$ and $\lambda_\alpha x = 0$ for all α. Since $q = qe_i \neq 0$, for at least one α, say α_0, $\lambda_{\alpha_0} e_i \neq 0$. Since Re_i is indecomposable and since R is a PWD, we have that every element of $\mathrm{Hom}_R(Re_i, Re_i)$ is a monomorphism. Right multiplication by x is such a homomorphism; therefore $(\lambda_{\alpha_0} e_i)x \neq 0$. But this is a contradiction since $(\lambda_{\alpha_0} e_i)x = \lambda_{\alpha_0}(e_i x) = \lambda_{\alpha_0} x = 0$. Hence there can be no such q and x is faithful.

(2) Let x be a nonzero element of $e_i Re_j$. Consider x as a linear transformation from A_i-rspace to A_j-rspace. By (1) dim range $x = |A_i|$ which then must be less than $|A_j|$.

(3) Suppose that the e_i's are not ordered in the desired manner; then there is an i for which $|A_i| < |A_j|$ for some $j > i$. Let i be the first such i and let j be the first such j. Let $i \leqslant k < j$, then $|A_j| > |A_i| \geqslant |A_k|$; hence by (1) $\mathrm{Hom}_R(Re_j, Re_k) = 0$. As a result the ordering $\{e_1, \ldots, e_{i-1}, e_j, e_i, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n\}$ will retain the property that $e_k Re_m = 0$ whenever $m > k$. This reordering of the e_i's will retain the structure (∗). Continuing this process will give the desired ordering. □

For the rest of this section assume that the e_i's are ordered as in Lemma 2.4 (3).

LEMMA 2.5. *Let R be semiprimary. A right R-module A_R is flat if and only if $0 \longrightarrow A \otimes_R Ne_i \longrightarrow A \otimes_R Re_i$ is exact for all i.*

PROOF. The ring R is semiprimary so that by Proposition 7 of Auslander [2], A_R will be flat if and only if $\mathrm{Tor}_1^R(A_R, S_R) = 0$ for all simple right R-modules S_R. The set $\{Re_1/Ne_1, \ldots, Re_n/Ne_n\}$ is a complete set of representatives for the isomorphism classes of simple right modules over R; therefore, by a long exact sequence of Tor we have that A_R is flat if and only if $0 \longrightarrow A \otimes Ne_i \longrightarrow A \otimes Re_i$ is exact for all i. □

Let $\{V_\alpha = \alpha \in A\}$ be a family of vector spaces over a division ring D and let $T_\alpha \in \mathrm{Hom}(V_\alpha, V)$ for some vector space V over D. The family $\{T_\alpha\}$ will be called *codirect* if their images sum directly as subspaces of V. The family $\{T_\alpha\}$ will be called *full* if the sum of their images is all of V. A family of matrices will be called *codirect* (*full*) if the family of linear transformations they represent is codirect (full). A family of matrices will be called *faithful* if each matrix in the family is faithful.

The following lemma will give a criterion for the codirectness of a family of matrices.

LEMMA 2.6. *Let A be a well-ordered set and let $\{x_\alpha : \alpha \in A\}$ be a faithful family of $A_\alpha \times M$ row-finite matrices where the A_α's and M are sets. The family $\{x_\alpha\}$ will be codirect if and only if the matrix C*

$$C = \begin{bmatrix} \hline x_1 \\ \hline x_2 \\ \hline \vdots \end{bmatrix}$$

is a faithful linear transformation from $\bigoplus_{\alpha \in A}(A_\alpha\text{-rspace})$ to M-rspace.

PROOF. The $\{x_\alpha\}$ are codirect if and only if $\Sigma\, q_\alpha x_\alpha = 0$ for $q_\alpha \in A_\alpha$-rspace implies each $q_\alpha x_\alpha = 0$. Since each x_α is faithful, this happens exactly when each q_α is zero. $\square$

The following proposition will constitute the main part of the proof of Theorem 2.1.

PROPOSITION 2.7. *Let R be a semiprimary ring with matrix structure as in* (∗). *The ring R will be hereditary with Q_R flat if and only if for each i, Ne_i has a faithful codirect family of generators.*

PROOF. Since R is semiprimary and $\text{Hom}(Re_j, Ne_i) = 0$ for $j \leqslant i$, it follows that for each i there is a sequence of positive integers $i < i(1) < i(2) < \cdots < i(k) \leqslant n$ and a set A_α for $\alpha = 1, \ldots, k$ such that $Ne_i/N^2 e_i \simeq \bigoplus\{(Re_{i(\alpha)}/Ne_{i(\alpha)})^{A_\alpha}:$ $\alpha = 1, \ldots, k\}$ where $(Re_{i(\alpha)}/Ne_{i(\alpha)})^{A_\alpha}$ denotes the direct sum of copies of $Re_{i(\alpha)}/Ne_{i(\alpha)}$ indexed by the set A_α. It follows then that Ne_i has a set of generators $\{u_\beta^\alpha\}$ where $u_\beta^\alpha \in e_{i(\alpha)}Re_i$ and $Ru_\beta^\alpha/Nu_\beta^\alpha \simeq Re_{i(\alpha)}/Ne_{i(\alpha)}$ for $\alpha = 1, \ldots, k$ and $\beta \in A_\alpha$.

($\Rightarrow$) Let R be hereditary and let Q_R be flat. Consider Ne_i and $\{u_\beta^\alpha\}$ chosen as above. The projective cover of $Ne_i/N^2 e_i$ is isomorphic to $P = \bigoplus(Re_{i(\alpha)})^{A_\alpha}$. Since R is hereditary, Ne_i is also a projective cover of $Ne_i/N^2 e_i$ and it follows easily from a natural isomorphism of P onto $\Sigma\, Ru_\beta^\alpha$ that the Ru_β^α's sum directly.

To show $\{u_\beta^\alpha\}$ is codirect, suppose that there are q_β^α's in A_α-rspace for which $\Sigma_{\alpha,\beta}\, q_\beta^\alpha u_\beta^\alpha = 0$. Identify each q_β^α in the natural manner with that element of $E_{mm}Q$ for which $q_\beta^\alpha e_{i(\alpha)} = q_\beta^\alpha$ for some fixed m in M. The module Q_R is flat, hence so is $E_{mm}Q$; therefore by Lemma 2.3 there exist b_γ's in $E_{mm}Q$ and $\{\lambda_{\beta\gamma}^\alpha\} \subset R$ such that $\Sigma_\gamma\, b_\gamma \lambda_{\beta\gamma}^\alpha = q_\beta^\alpha$ for each β and α, and $\Sigma_{\alpha,\beta}\, \lambda_{\beta\gamma}^\alpha u_\beta^\alpha = 0$ for all γ. Since the Ru_β^α's sum directly, $\lambda_{\beta\gamma}^\alpha u_\beta^\alpha = 0$ for all α, β, γ. Since $e_{i(\alpha)}u_\beta^\alpha = u_\beta^\alpha$, we have that $(\lambda_{\beta\gamma}^\alpha e_{i(\alpha)})u_\beta^\alpha = \lambda_{\beta\gamma}^\alpha(e_{i(\alpha)}u_\beta^\alpha) = \lambda_{\beta\gamma}^\alpha u_\beta^\alpha = 0$. Each u_β^α is faithful by Lemma 2.4; this implies $\lambda_{\beta\gamma}^\alpha e_{i(\alpha)} = 0$ for all α, β, γ. Therefore

$$q_\beta^\alpha = q_\beta^\alpha e_{i(\alpha)} = \left(\sum_\gamma b_\gamma \lambda_{\beta\gamma}^\alpha\right) e_{i(\alpha)} = \sum_\gamma b_\gamma \lambda_{\beta\gamma}^\alpha e_{i(\alpha)} = 0.$$

Thus all the q_β^α's are zero, and the family $\{u_\beta^\alpha\}$ must be codirect.

($\Leftarrow$) By Proposition 7 of Auslander [2] and by Lemma 2.5 it will be enough to show that for each i, Ne_i is projective and $0 \to Q \otimes Ne_i \to Q \otimes Re_i$ is exact.

Consider Ne_i and let $\{u_\beta^\alpha\}$ as above, be a faithful codirect family. The family is faithful so each u_β^α is faithful and $Ru_\beta^\alpha = (Re_{i(\alpha)})u_\beta^\alpha \simeq Re_{i(\alpha)}$ is projective. Then Ne_i will be projective provided it can be shown that the Ru_β^α's sum directly. Let $0 = \Sigma_{\alpha,\beta} \, r_\beta^\alpha u_\beta^\alpha$ for $\{r_\beta^\alpha\} \subset R$. This will be the case if and only if $E_{mm}(\Sigma \, r_\beta^\alpha u_\beta^\alpha) = 0$ for all $m \in M$; but

$$E_{mm}\left(\sum r_\beta^\alpha u_\beta^\alpha\right) = \sum_{\alpha,\beta} E_{mm} r_\beta^\alpha u_\beta^\alpha = \sum E_{mm} r_\beta^\alpha e_{i(\alpha)} u_\beta^\alpha = 0$$

since $e_{i(\alpha)} u_\beta^\alpha = u_\beta^\alpha$. For each α and β, $E_{mm} r_\beta^\alpha e_{i(\alpha)}$ can be identified with an element of A_α-rspace; hence $E_{mm} r_\beta^\alpha e_{i(\alpha)} = 0$ since $\{u_\beta^\alpha\}$ is codirect. Thus we have $0 = r_\beta^\alpha e_{i(\alpha)}$ and $r_\beta^\alpha e_{i(\alpha)} u_\beta^\alpha = r_\beta^\alpha u_\beta^\alpha = 0$ for all α and β; hence the Ru_β^α's sum directly and Ne_i is projective. This shows that R is left (and right) hereditary.

Since R is hereditary and since $Q_R = \Pi_{m \in M} E_{mm} Q_R$, Q_R will be flat if and only if $E_{mm} Q_R$ is flat for all m by Theorem 2.1 of Chase [7]. The Ru_β^α's sum directly; therefore, every element of $E_{mm} Q \otimes Ne_i$ can be written uniquely in the form $\Sigma_{\alpha,\beta} E_{mm} q_\beta^\alpha \otimes u_\beta^\alpha$. Suppose that $\Sigma_{\alpha,\beta} \, q_\beta^\alpha u_\beta^\alpha = 0$ where $q_\beta^\alpha \in E_{mm} Q$; then $0 = \Sigma_{\alpha,\beta} q_\beta^\alpha u_\beta^\alpha = \Sigma \, q_\beta^\alpha e_{i(\alpha)} u_\beta^\alpha$ since $e_{i(\alpha)} u_\beta^\alpha = u_\beta^\alpha$. Each $q_\beta^\alpha e_{i(\alpha)}$ however can be naturally identified with an element of an A_α-rspace; in which case $q_\beta^\alpha u_\beta^\alpha = q_\beta^\alpha e_{i(\alpha)} u_\beta^\alpha = 0$ for each α and β since $\{u_\beta^\alpha\}$ is codirect. Since each u_β^α is faithful, we further have that $q_\beta^\alpha e_{i(\alpha)} = 0$ for each α and β. Let $b_\beta^\alpha = q_\beta^\alpha$ and let $\lambda_\beta^\alpha = 1 - e_{i(\alpha)}$. Then we have

$$b_\beta^\alpha \lambda_\beta^\alpha = q_\beta^\alpha (1 - e_{i(\alpha)}) = q_\beta^\alpha - q_\beta^\alpha e_{i(\alpha)} = q_\beta^\alpha - 0 = q_\beta^\alpha;$$

also,

$$\lambda_\beta^\alpha u_\beta^\alpha = (1 - e_{i(\alpha)}) u_\beta^\alpha = u_\beta^\alpha - e_{i(\alpha)} u_\beta^\alpha = u_\beta^\alpha - u_\beta^\alpha = 0.$$

Therefore

$$\sum_{\alpha,\beta} q_\beta^\alpha \otimes u_\beta^\alpha = \sum_{\alpha,\beta} b_\beta^\alpha \lambda_\beta^\alpha \otimes u_\beta^\alpha = \sum b_\beta^\alpha \otimes \lambda_\beta^\alpha u_\beta^\alpha = \sum b_\beta^\alpha \otimes 0 = 0$$

and hence $0 \to E_{mm} Q \otimes Ne_i \to E_{mm} Q \otimes Re_i$ is exact. Thus $E_{mm} Q_R$ is flat and hence Q_R is flat. $\square$

We can now give the proof to Theorem 2.1.

PROOF. Let R be an indecomposable hereditary basic semiprimary ring. By Proposition 2.2 we can imbed R with triangular structure (*) in the ring Q of $M \times M$ row-finite matrices over a division ring D. Choose for each i a codirect family of generators for Ne_i as in Proposition 2.7. Let V be M-space over D; we

will view Q as the ring of linear transformations of V, and then choose a basis for V which will give each u_β^α a representation which will then give R the representation of Theorem 2.1.

The orthogonal set of idempotents $\{e_1, \ldots, e_n\}$ gives a direct sum decomposition of V; $V = Ve_1 \oplus \cdots \oplus Ve_n$. We will inductively choose a basis for each Ve_i.

Choose any basis for Ve_n. Well-order this basis and let its order type be that of an ordinal l_n.

Inductively assume that a basis of order type l_j has been chosen for all $j > i$. Using the notation of Proposition 2.7, well-order each A_α and let the order type of A_α be $v(\alpha)$. The set $\{u_\beta^\alpha\}$ will then be well ordered under the lexicographic order. By Proposition 2.6 the matrix

$$C = {}^t[u_1^1 \vdots \cdots \vdots u_{v(k)}^k]$$

will represent a faithful linear transformation from $U = \bigoplus_{\alpha,\beta} Ve_{i(\alpha)}^{(\beta)}$ into Ve_i where $i < i(1) < \cdots < i(k) \leqslant n$ and where $Ve_{i(\alpha)}^{(\beta)}$ denotes a copy of $Ve_{i(\alpha)}$. Let B_β^α be the copy in $Ve_{i(\alpha)}^{(\beta)}$ of the basis chosen for $Ve_{i(\alpha)}$; $B = \bigcup_{\alpha,\beta} B_\beta^\alpha$ will then be a basis for U, and if B is ordered lexicographically it will have order type $\Sigma \, l_{i(\alpha)} v(\alpha)$. Since C is faithful, $(B)C$ will be a linearly independent subset of Ve_i; extend it to an ordered basis in a manner which extends the ordering of $(B)C$ induced by that of B. Denote the resulting order type by l_i. Continuing this process will give an ordered basis for V; as a result we can consider M to be the ordinal $\Sigma_{i=1}^n l_i$.

We will now show that R will have the desired representation with respect to this basis. The structure of (1) follows from the fact that R has a structure as in (*). Consider Ne_i and inductively assume that Ne_j has structure in (2) and (3) for all $j > i$. Let $L = \Sigma \, l_{i(\alpha)} v(\alpha)$. With respect to this basis C will have the form $[I \vdots 0]$ where I is an $L \times L$ identity matrix. In this case u_β^α (considered as an $l_{i(\alpha)} \times l_i$ matrix) will have the form $[W \vdots I \vdots Z]$ where W is an $l_{i(\alpha)} \times (\Sigma_{\gamma < \alpha} l_{i(\gamma)} v(\gamma) + l_{i(\alpha)}(\beta - 1))$ zero matrix, I is an $l_{i(\alpha)} \times l_{i(\alpha)}$ identity matrix, and Z is the zero matrix consisting of the remaining columns. It is now easy to see that R has the desired representation. Let $C_\beta^\alpha = Re_{i(\alpha)} u_\beta^\alpha$; u_β^α is an $l_{i(\alpha)} \times l_{i(\alpha)}$ identity matrix so that C_β^α consists precisely of all matrices over $Re_{i(\alpha)}$. Hence C_β^α has the structure indicated in (3). By the above representation of u_β^α we see that u_β^α's form a sequence of nonoverlapping subidentity matrices from left to right; therefore, the C_β^α's are positioned as in (2). We have

$$Ne_i = \bigoplus_{\alpha,\beta} Ru_\beta^\alpha \doteq \bigoplus_{\alpha,\beta} Re_{i(\alpha)} u_\beta^\alpha = \bigoplus_{\alpha,\beta} C_\beta^\alpha$$

so that the C_β^α's will represent everything in Ne_i. Suppose now that $\Sigma_{\alpha=1}^k l_{i(\alpha)} v(\alpha) \neq l_i$; then the right-hand column of Ne_i will be identically zero.

Suppose that this is the mth column. We will then have

$$(\mathrm{Soc}_R R)E_{mm}Q = ((\mathrm{Soc}_R R)E_{mm})Q \subset (Ne_i)E_{mm}Q = 0$$

since $e_j E_{mm} = 0$ for $j \neq i$; but this contradicts that $_R Q$ is nonsingular as in Proposition 2.2. We have thus shown that each Ne_i has the desired structure.

Conversely, assume R is a ring with structure as in (1), (2) and (3). For each i pick a family $\{u_\beta^\alpha\}$ where u_β^α is the $l_{i(\alpha)} \times l_{i(\alpha)}$ subidentity matrix of C_β^α for which $e_{i(\alpha)}u_\beta^\alpha = u_\beta^\alpha$ ($e_{i(\alpha)}$ is the identity element of $D_{i(\alpha)}$). By choice $\{u_\beta^\alpha\}$ will be a faithful codirect family of generators for Ne_i. Hence, by Proposition 2.7, R will be hereditary and Q_R will be flat. It is easy to see that N is nilpotent of index at most n, so that R is semiprimary.

We show that the maximal left quotient ring T of R is a subring of Q. If $q \in Q$ for which $\mathrm{Soc}(Re_n)q = (Re_n)q = 0$, then clearly $e_n q = 0$. Suppose that $q \in Q$ for which $\mathrm{Soc}(Re_i)q = 0$. Inductively assume that if for $j > i$, $\mathrm{Soc}(Re_j)p = 0$ for $p \in Q$, then $e_j p = 0$. We have that

$$\mathrm{Soc}(Re_i) = \bigoplus_{\alpha,\beta} \mathrm{Soc}(C_\beta^\alpha) = \bigoplus_{\alpha,\beta} [\mathrm{Soc}(Re_{i(\alpha)})] u_\beta^\alpha;$$

whence $[\mathrm{Soc}(Re_{i(\alpha)})]u_\beta^\alpha q = 0$ for all α, β and thus by the induction hypothesis $e_{i(\alpha)}u_\beta^\alpha q = u_\beta^\alpha q = 0$. It follows that $E_{mm}q = 0$ whenever $u_\beta^\alpha E_{mm} \neq 0$. Suppose that $e_i E_{mm} \neq 0$; then $u_\beta^\alpha E_{mm} \neq 0$ for some α, β since $\Sigma\, l_{i(\alpha)}v(a) = l_i$; as a result we have $E_{mm}q = 0$ whenever $e_i E_{mm} \neq 0$ and $e_i q = 0$. Now suppose that $\mathrm{Soc}(_R R)q = 0$ for $q \in Q$; then $\mathrm{Soc}(Re_i)q = 0$ for each i, and $e_i q = 0$ for each i. It follows that $q = 0$ since $e_1 + \cdots + e_n = 1$. We have shown $Z(_R Q) = \mathrm{Ann}_Q(\mathrm{Soc}(_R R)) = 0$ and $_R Q$ is nonsingular. Since $_Q Q$ is injective and Q_R flat, it follows from Cartan-Eilenberg [4, p. 123, Exercise 10] that $_R Q$ is injective. Using the fact that $_R Q$ is a nonsingular injective module, we have that T is a subring of Q. In fact, T is exactly the subring of Q which leaves $\mathrm{Soc}(_R R)$ invariant under right multiplication.

It is clear from the proof that R will be left Artinian if and only if the $v(\alpha)$'s are integers. $\square$

If R is any matrix ring of the form described in Theorem 2.1, then it follows immediately that each Ne_i is isomorphic to a direct sum of indecomposable left ideals of R and that R is semiprimary. It is then easy to see that R is hereditary. However the theorem gives additional information concerning Q and R; namely that Q_R is R-flat and that Q contains a copy of the maximal left quotient ring of R.

REMARK 2.8. Theorem 2.1 can be restated somewhat. Let Q be the ring of $M \times M$ matrices over D. Then R is a semiprimary hereditary basic subring of Q with Q_R flat if and only if R is conjugate to a subring of Q having structure (1), (2) and (3). Furthermore, R's maximal left quotient ring is contained as an overring of R in Q if and only if $\Sigma\, l_{i(\alpha)}v(a) = l_i$ for each i.

The complicated use of flatness in Theorem 2.1 cannot be avoided entirely as is seen by considering the ring R of all matrices of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ where a and c are rational numbers while b is a complex number. R is a hereditary semi-primary subring of the ring of 2 by 2 matrices over the complex numbers, C_2. But R cannot be put in the form of Theorem 2.1 as a subring of C_2 since R is not finite dimensional. This also shows that $(C_2)_R$ is not R-flat.

It should be noted that the proof of Theorem 2.1 can be expanded to give an explicit structure for the maximal left quotient ring T of R as an overring of R in Q. This also gives the indecomposable nonsingular injective modules over R. Since every indecomposable injective module over a semiprimary hereditary ring is a direct summand of a quotient of an indecomposable nonsingular injective module, this gives an explicit structure for indecomposable injectives over R. Let us give an outline of that expansion.

Since T is the unique injective hull of $_RR$ contained in $_RQ$, T will also be the unique injective hull of $\text{Soc}(_RR)$ contained in Q; hence we shall first compute $E(Re_j)$ where Re_j is simple. By Gordon [16, Theorem 1.2], the Re_j-homogeneous component of $\text{Soc}(_RR)$ is e_jR; hence $e_jR = \bigoplus_{\gamma \in C_j} Rx_\gamma$ where $Rx_\gamma \simeq Re_j$. Using Theorem 2.1 we can take Rx_γ to be the collection of all matrices of the following form:

$$
B_\gamma^j \;=\; \begin{bmatrix} & & 0 & \\ \mu_\gamma & & l_j & \\ \hdashline 0 & \vdots & A_\gamma^j & \vdots & 0 \\ \hdashline & & 0 & \end{bmatrix} \begin{array}{l} \left.\rule{0pt}{20pt}\right\} \sum_{i<j} l_i \\ \left.\rule{0pt}{12pt}\right\} l_j \end{array}
$$

where A_γ^j is the collection of all matrices over $D_j = e_j Re_j$; furthermore x_γ can be taken as the $l_j \times l_j$ identity matrix. The matrices $\{B_\gamma^j : \gamma \in C_j\}$ will have nonoverlapping columns for different γ's. Let D_γ^j be the collection of all matrices of the following form:

$$
D_\gamma^j \;=\; \begin{bmatrix} & \vdots & 0 & \vdots & \\ \hdashline 0 & \vdots & A_\gamma^j & \vdots \\ \hdashline & \vdots & 0 & \vdots & \end{bmatrix} \begin{array}{l} \left.\rule{0pt}{12pt}\right\} \mu_\gamma \\ \left.\rule{0pt}{12pt}\right\} l_j \end{array}
$$

$$\underbrace{\sum_{i<j} l_i} \quad \underbrace{l_j}$$

Each element of D_γ^j is almost the transpose of an element of B_γ^j.

Let $E^j = \Pi_{\gamma \in C_j} D_\gamma^j$. Since the columns of the B_γ^j's are nonoverlapping, the rows of the D_γ^j's are nonoverlapping; hence E^j can be identified with the obvious subset of Q.

CLAIM. $E^j = E(Re_j)$.

Let y be a nonzero element of E^j, then the γ component of y is nonzero for some γ. In this case $x_\gamma y$ will be a nonzero element of Re_j. Hence E^j is essential over Re_j and is thus contained in the unique injective hull $E(Re_j)$ contained in Q.

Conversely, $_R Qe_j$ is an injective submodule of $_R Q$ which contains Re_j; therefore, $E(Re_j) \subset Qe_j$. Let $y \in E(Re_j)$, then right multiplication by y is an R-homomorphism; therefore $x_\gamma y \in \text{Soc}(E(Re_j))$ for each γ. As a result, we have that the A_γ^j position of y must be an element of D_j. Now suppose that y has a nonzero row β which is not covered by one of the A_γ^j's. Since $_R Q$ is nonsingular, there is an element z of $\text{Soc}(_R R)$ for which Rz is simple and for which $zE_{\beta\beta} \neq 0$. Since the β-row is not covered by an A_γ^j, Rz is not isomorphic to Re_j. As a result we have that Rzy is a nonzero simple submodule of $E(Re_j)$ which is not isomorphic to Re_j. This is a contradiction, and it must be that $y \in E^j$ and $E^j = E(Re_j)$.

Let E_γ^j be the collection of matrices of the following form:

$$E_\gamma^j = \begin{bmatrix} & \vdots & & \vdots & \\ 0 & \vdots & E^j & \vdots & 0 \\ & \vdots & & \vdots & \end{bmatrix}.$$

$$\underbrace{}_{\mu_\gamma}$$

It follows that $E_\gamma^j = E(Rx_\gamma)$. We then have $E(e_j R) = Q \cap (\Pi_{\gamma \in C_j} E_\gamma^j)$ which is isomorphic (canonically) to the ring of row-finite matrices over the division ring D_j, as it should be. Doing this for each j for which Re_j is simple, and taking the direct sum yields T.

EXAMPLE. Let R be the ring of matrices of the form

$$\begin{bmatrix} a & & & & & \\ 0 & a & & & & \\ 0 & 0 & a & & & \\ b & 0 & 0 & d & & \\ 0 & b & 0 & 0 & d & \\ 0 & 0 & c & 0 & 0 & e \end{bmatrix}$$

where $a \in Q$, b, $d \in \mathbf{R}$, and c, $e \in \mathbf{C}$. By Theorem 2.1 we have that R is hereditary semiprimary. By the previous discussion T is the ring of matrices of the form

$$\begin{bmatrix} x_1 & 0 & 0 & x_2 & 0 & 0 \\ 0 & x_1 & 0 & 0 & x_2 & 0 \\ 0 & 0 & y_1 & 0 & 0 & y_2 \\ x_3 & 0 & 0 & x_4 & 0 & 0 \\ 0 & x_3 & 0 & 0 & x_4 & 0 \\ 0 & 0 & y_3 & 0 & 0 & y_4 \end{bmatrix}$$

where $x_i \in \mathbf{R}$, $y_j \in \mathbf{C}$. □

By Theorem 2.1 it would seem that a two-sided hereditary Artinian ring would be very similar to the ring of lower triangular matrices over a division ring and should share many properties with such a ring. The following example shows that one property not shared is the finiteness of indecomposable injective modules.

EXAMPLE. P. M. Cohn [11] has shown that there are division rings $D_1 \subset D_2$ for which $[D_2 : D_1]_r = 2$ and $[D_2 : D_1]_l = \infty$. Let

$$R = \begin{bmatrix} D_1 & 0 \\ D_2 & D_2 \end{bmatrix}.$$

The ring R is a left and right Artinian hereditary ring by Theorem 2.1; but

$$E = E\left(\begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & D_2 \\ 0 & D_2 \end{bmatrix}$$

is not finitely generated as a left R-module. □

In the proof of Theorem 2.1 the structure of Ne_i was completely determined by the isomorphism type of the semisimple module $Ne/N^2 e_i$ (which is given numerically by the $v(\alpha)$'s). As a result it would be reasonable to conjecture that the diagonal division rings D_i together with the isomorphism type of each $Ne_i/N^2 e_i$ would give a complete set of invariants for basic semiprimary hereditary rings. This is not the case as the following example shows.

EXAMPLE. Let

$$R = \left\{ \begin{bmatrix} a + ib & 0 & 0 \\ 0 & a + ib & 0 \\ c & d & e \end{bmatrix} \right\} \quad \text{and} \quad S = \left\{ \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ c & d & e \end{bmatrix} \right\}$$

for a, $b \in \mathbf{R}$; c, d, $e \in \mathbf{C}$. Each of these rings is a basic hereditary Artinian ring

having isomorphic diagonal division rings and the same Ne_i/N^2e_i structure, but they are not isomorphic (every minimal right ideal of $N(R)$ is a two-sided ideal, while this is not the case for $N(S)$). □

3. **Semihereditary and hereditary rings.** In this section we first apply the results of Gordon and Small [18] and Gordon [17] to obtain a structure theorem (Theorem 3.7) for semihereditary rings which are left orders in a semiprimary ring. Our results are motivated by Small's example [29] $R = \begin{bmatrix} Q & 0 \\ Q & Z \end{bmatrix}$ of a semihereditary ring which is an order in an Artinian ring. In fact, we show that any semihereditary ring which is a left order in a semiprimary ring is of the same general form.

Then we apply the results of §2 along with our structure theorem for semihereditary rings which are left orders in a semiprimary ring to obtain a structure theorem for hereditary rings which are left orders in a semiprimary ring. These results differ from those of Small [30], Gordon and Small [18] and Gordon [15] in that the representation is explicit in terms of matrices over left hereditary left Goldie prime rings and their respective classical left quotient rings.

When a semihereditary ring is a two-sided order in a semiprimary ring, our results take on a particularly simplified form. In particular, we obtain Chatters' result [9] that a two-sided hereditary Noetherian ring is a direct sum of an Artinian ring and a semiprime ring. When the maximal quotient ring is two-sided, we recover a result of Goodearl's [13, pp. 38–44].

Before proceeding, we need some terminology. An R-module M is said to be *torsionfree* [25] if for any $x \in M$, the annihilator of x does not contain a regular element of R. A ring R with prime radical N is said to satisfy the *regularity condition* if $a + N$ is regular in R/N implies a is regular in R. A ring is said to have enough idempotents [18] if it has a complete set of orthogonal primitive idempotents. Finally, R is said to satisfy the left (right) *essentiality condition* [17] if Ra (aR) is essential in R for every regular element a of R.

Gordon [17] gives the following characterization of a left semihereditary ring possessing a semiprimary classical left quotient ring.

THEOREM 3.1. *A left semihereditary ring R possesses a semiprimary classical left quotient ring if and only if R has enough idempotents and R satisfies the left essentiality condition.* □

By Small [32], any left semihereditary ring possessing a semiprimary classical left quotient ring is right semihereditary. Also note that a ring R is a left order in a semiprimary ring Q if and only if Q is the classical left quotient ring of R. These facts will be used repeatedly in this section.

Let R be a semihereditary ring possessing a semiprimary classical left quotient ring. Then R is a PWD and has a representation:

$$(1) \qquad R \simeq \begin{bmatrix} P_1 & & & & \\ P_{21} & P_2 & & & \\ P_{31} & P_{32} & P_3 & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ P_{n1} & P_{n2} & P_{n3} & \cdots & & P_n \end{bmatrix}$$

where each P_i is a prime ring and each P_{ij} is a $P_i - P_j$ bimodule. We will let e_k be the matrix unit which has the identity of P_k in the (k, k)th block and zeroes elsewhere. Note that $P_i = e_i Re_i$ is semihereditary by [27] and $P_{ij} = e_i Re_j$.

Since R has a semiprimary classical left quotient ring Q, R/N is a left order in the semisimple ring $Q/N(Q)$.

LEMMA 3.2. *If R is indecomposable, then Q is indecomposable.*

PROOF. To show that Q is indecomposable, we need to show that Q has no proper central idempotents. The ring Q is semiprimary and $N(Q) = J(Q)$, so that any orthogonal lifting of a complete set of orthogonal idempotents for $Q/N(Q)$ will be a complete set for Q. By Gordon [17, Proposition 1], $R/N(R)$ is a direct sum of uniform left ideals; hence a complete set of orthogonal idempotents for $Q/N(Q)$ may be chosen in $R/N(R)$. The ideal $N(R)$ is a nilpotent ideal of R, so that any such set can be lifted to R. Therefore any central idempotent of Q belongs to R, and Q is indecomposable since R is. $\square$

The ring Q is thus an indecomposable hereditary semiprimary ring, and its structure will be given in detail by Theorem 2.1 and the discussion of basic rings in §1. In general, however, Q will have the following structure:

$$(2) \qquad Q \simeq \begin{bmatrix} K_1 & & & & \\ K_{21} & K_2 & & & \\ K_{31} & K_{32} & K_3 & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ K_{n1} & K_{n2} & K_{n3} & \cdots & & K_n \end{bmatrix}.$$

Since in the proof of Lemma 3.2 it was shown that a complete set of orthogonal idempotents for Q could be chosen in R, we can assume that $K_{ij} =$

$e_i Q e_j \supseteq e_i R e_j = P_{ij}$. Furthermore, each P_i will be a left order in the simple Artinian ring K_i.

LEMMA 3.3. *Each P_{ij} is a torsionfree left P_i-module and a torsionfree right P_j-module.*

PROOF. P_{ij} is a right P_j-submodule of the K_j-module K_{ij}. Since any K_j-module is torsionfree as a P_j-module, and since submodules of torsionfree modules are torsionfree, we have that P_{ij} is a torsionfree right P_j-module. Similarly P_{ij} is a torsionfree left P_j-module. □

LEMMA 3.4. *Each P_{ij} is a right K_j-module.*

PROOF. Let $e = e_i + e_j$. The ring eRe is naturally isomorphic to

$$\begin{bmatrix} P_i & 0 \\ P_{ij} & P_j \end{bmatrix}.$$

By [27], eRe is semihereditary. Furthermore eRe is a left order in eQe [14], so eRe has the left essentiality condition. Hence we may assume that $R = \begin{bmatrix} S & 0 \\ M & T \end{bmatrix}$ and show that M is an S-divisible right S-module. Let d be a regular element of S and suppose that $Md \neq M$. Let $m \in M - Md$ and consider the left ideal

$$R \begin{bmatrix} d & 0 \\ m & 0 \end{bmatrix} = \begin{bmatrix} Sd & 0 \\ Md + Tm & 0 \end{bmatrix}.$$

It is not projective, however, as we will show that the natural map η of R onto it does not split. Let $K = \ker \eta$; it is easy to see that

$$K = \left\{ \begin{bmatrix} 0 & 0 \\ x & t \end{bmatrix} : xd = -tm \right\}.$$

M_S is torsionfree by Lemma 3.2. Therefore if

$$\begin{bmatrix} 0 & 0 \\ x_1 & t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ x_2 & t \end{bmatrix}$$

are in K, $x_1 d = x_2 d$ or $x_1 = x_2$. Then K is naturally isomorphic to a left ideal of T under the map which sends $\begin{bmatrix} 0 & 0 \\ x & t \end{bmatrix}$ to $\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}$. Call the image of this map K'.

Since R satisfies the left essentiality condition, $\begin{bmatrix} 0 & 0 \\ Md & 0 \end{bmatrix}$ is essential in $\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix}$ and Md is essential in M as a left T-module. Consequently $K' = \{ t \in T: tm \in Md \} = (m: Md)$ is an essential left ideal of T. Note that $1 \notin (m: Md)$ since $m \notin Md$.

The natural map η cannot be a splitting map since K is not generated by an

idempotent; for if K is generated by an idempotent, then K' must be generated by an idempotent. But K', being essential in T, can be generated by an idempotent if and only if the idempotent is the identity of T, an impossibility since $1 \notin K'$. Hence M is S-divisible and P_{ij} is P_j-divisible.

Now since P_{ij} is a divisible torsionfree P_j-submodule of the K_j-module K_{ij}, P_{ij} must be K_j-invariant and hence a right K_j-module. $\square$

The following lemma gives more information concerning the relationship between R and Q.

LEMMA 3.5. $K_{ij} = K_i P_{ij}$.

PROOF. Since $K_{ij} = e_i Q e_j$, $K_i = e_i Q e_i$, and $P_{ij} = e_i R e_j$, this is a result of Small as generalized by Gordon [14, Theorem A]. $\square$

We will also need the following technical lemma.

LEMMA 3.6. *Let P be a prime left Goldie ring with simple Artinian classical left quotient ring K such that $P \neq K$. Then*: (i) *K_P is not finitely generated*; (ii) *if P is left Noetherian, then $_P K$ is not finitely generated*; (iii) *if P is left hereditary, then $_P K$ is not projective.*

PROOF. (i) Suppose that K_P is finitely generated with generating set $\{x_1, \ldots, x_k\}$. A common left denominator d can be chosen so that $x_i = d^{-1} a_i$ where each $a_i \in P$ and d is a regular element of P. Then it follows that $dK \in P$, and in particular, $d^{-1} = dd^{-2} \in P$. This is a contradiction since $P \neq K$.

(ii) Let d be a regular element in P for which $d^{-1} \notin P$. If $_P K$ is finitely generated, then $_P K$ will be Noetherian since P is left Noetherian. Hence there is a positive integer m for which $Pd^{-m} = Pd^{-(m+1)}$. Then there exists an $x \in P$ for which $xd^{-m} = d^{-(m+1)}$; we then have $d^{-1} = x$ an element of P, which is a contradiction.

(iii) Since $_P P$ has finite Goldie dimension, it follows from Albrecht [1] that P is left Noetherian. If $_P K$ is projective, then $_P K$ will be finitely generated by Albrecht [1] since it also must have finite Goldie dimension. This contradicts (ii). $\square$

If $\{e_1, e_2, \ldots, e_n\}$ is an ordered set of orthogonal idempotents for a ring R, for notational purposes we will let $f_i = e_i + \cdots + e_n$.

We are now in a position to state the main theorem on semihereditary rings. If R has a semiprimary classical quotient ring, then R will be a direct sum of indecomposable rings. Hence it is sufficient in the following theorem to assume that R is indecomposable.

THEOREM 3.7. *Let R be an indecomposable semihereditary ring with semiprimary classical left quotient ring Q; let R and Q have general form as in* (1) *and* (2). *Then each P_i is a semihereditary left Goldie prime ring, each P_{ij} is a*

right K_j-module, and for each i, $f_i Re_{i-1}$ is a flat left $f_i Rf_i$-module.
  Conversely, any such ring is semihereditary.

PROOF. By Gordon and Small [18] each P_i is a semihereditary prime left
Goldie ring, while Lemma 3.4 shows that P_{ij} is a right K_j-module. It remains to
be shown that $f_i Re_{i-1}$ is a flat left $f_i Rf_i$-module for $i = 2, \ldots, n$.

The ring $f_{i-1} Rf_{i-1}$ is again semihereditary by [27], so it is sufficient to
assume that R is of the form $R = \begin{bmatrix} S & 0 \\ M & T \end{bmatrix}$, and then show that $_T M$ is flat. But
$_R M = {}_T M$; therefore since $_R M$ is flat, $_T M$ is also flat.

To prove the converse we need only show that R is semihereditary. To do
this we induct on n, the number of orthogonal central idempotents of $Q/N(Q)$.
If $n = 1$, the result is trivial. Now assume the result holds for $n = r - 1$ and let

$$W = \begin{bmatrix} P_1 & & & \\ P_{21} & P_2 & & 0 \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot \\ P_{r1} & P_{r2} & \cdots & P_r \end{bmatrix}$$

satisfy the conditions of the theorem.
  Let

$$U = \begin{bmatrix} K_1 & \vdots & & & \\ P_{21} & P_2 & & & \\ \cdot & \vdots & \cdot & \cdot & \\ \cdot & \vdots & \cdot & & \cdot \\ \cdot & \vdots & \cdot & & \cdot \\ P_{r1} & P_{r1} & \cdots & & P_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

be partitioned as indicated. By the assumptions of the theorem, $_C B$ is flat; hence
by Fields [12, p. 348] and the induction hypothesis we have that WGD$(U) \leq 1$.
Since Q is semiprimary, it follows from Jondrup [23, Corollary 3.2] that finitely
generated flat U-modules are projective; whence U is semihereditary.
  Now let

$$
V = \begin{bmatrix}
P_1 & & & & & & \\
 & \cdot & \cdot & & & & \\
 & & \cdot & & \cdot & & \\
 & & & \cdot & & & 0 \\
 & & & & \cdot & & \\
P_{r-1,1} & \cdots & & P_{r-1} & & \\
\hline
K_r P_{r,1} & \cdots & & K_r P_{r,r-1} & & K_r
\end{bmatrix} = \begin{bmatrix} D & 0 \\ E & F \end{bmatrix}
$$

be partitioned as indicated. Let

$$
X = \begin{bmatrix}
K_1 & & & \\
K_2 P_{21} & K_2 & & 0 \\
\cdot & & \cdot & \\
\cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot \\
K_r P_{r,1} & K_r P_{r,2} & \cdots & K_r
\end{bmatrix}
\quad \text{and} \quad
Y = \begin{bmatrix}
K_1 & & & \\
\cdot & \cdot & & 0 \\
\cdot & & \cdot & \\
\cdot & & & \cdot \\
K_{r-1} P_{r-1,1} & \cdots & & K_{r-1}
\end{bmatrix} \cdot
$$

Now Y is the classical left quotient ring of D so Y_D is flat. E is a right ideal of X, and X is hereditary; hence E_X is flat. However $E_X = E_Y$, so E_Y is flat. Then by [4, Problem 10, p. 123], E_D is flat. Again by [12, p. 348], the induction hypothesis, and [23, Corollary 3.2], it follows that V is semihereditary.

Since W has a semiprimary classical left quotient ring X, to show W is semihereditary it suffices to show W is right semihereditary [32]. To do this, it is sufficient to show that any finitely generated right ideal I_W contained in $e_k W$ is projective for $1 \leqslant k \leqslant r$. If $1 \leqslant k < r$, $e_k W = e_k V$. Since V is semihereditary, I_V is projective so I_W is projective. Now let I_W be a submodule of $e_r W$ and let $I = \Sigma_{k=1}^m x_k W$ where $e_r x_k e_1 = 0$ for $k = 1, \ldots, t$ and $e_r x_k e_1 = x_k$ for $k = t+1, \ldots, m$. If $j \leqslant t$, then $x_j W = x_j U$. Let $J = \Sigma_{k=1}^t x_k W = \Sigma_{k=1}^t x_k U$. U is semihereditary, so J_U is projective. Let $P = \Sigma_{k>j} e_k W$ and let T_U be the trace of P in J. If $T_U \neq J$, then $\mathrm{Hom}_U(P, J/T) = 0$ since P_U is projective. Therefore

$$
J/T \simeq \mathrm{Hom}_U(U, J/T) \simeq \mathrm{Hom}_U(P, J/T) \oplus \mathrm{Hom}_U(K_1, J/T) = \mathrm{Hom}_U(K_1, J/T).
$$

Hence J/T is a semisimple module of homogeneous type K_1, and consequently is U-projective. Therefore $J \simeq T \oplus J/T$ and J has a direct summand isomorphic to K_1. If $P_1 \neq K_1$, then $(K_1)_{P_1}$ is not finitely generated by Lemma 3.6 (if $P_1 = K_1$, then $W = U$ is already semihereditary); hence we have a contradiction to the fact that J_W is finitely generated. Hence $T = J$ and J being U-projective is a direct summand of a direct sum of copies of P; J is thus W-projective.

Now Ie_1 has the same right W-structure as it has right P_1 structure. Je_1 is a right U-submodule of J; hence it is a module over K_1. Since K_1 is semisimple, Je_1 is a direct summand of the K_1-module P_{r1}. Since P_1 is a subring of K_1, Je_1 is a P_1 direct summand of any submodule of P_{r1} containing Je_1. In particular, Je_1 is a direct summand of Ie_1; so $Ie_1 = Je_1 \oplus L$ for some submodule L of Ie_1. But $I = J + L$ and $J \cap L = Je_1 \cap L = 0$, so $I = J \oplus L$. V is semihereditary so L is V-projective. Therefore L is P_1-projective and hence W-projective. Finally we have I_W is projective and W is semihereditary. Hence it follows that R is semihereditary. $\square$

REMARK. Note that many of our arguments used only the fact that R was a (left) p.p. ring with semiprimary classical left quotient ring Q. In fact, assuming that R is (left) p.p. and that R and Q are as in (1) and (2), then each P_i is a left p.p. Goldie prime ring, each P_{ij} is a right K_j-module and for each i, $f_i Re_{i-1}$ has projective left principal $f_i Rf_i$-submodules. In particular this may shed some light on Gordon's question[14] about left p.p. rings possessing a semiprimary classical left quotient ring.

If R is a semihereditary PWD, then the P_{ij}'s are flat left P_i-modules. This can be seen by considering the subring eRe where $e = e_i + e_j$ and recalling that eRe is semihereditary. Unfortunately the condition in Theorem 3.7, $f_i Re_{i-1}$ is a flat left $f_i Rf_i$-module, cannot be replaced by the simpler condition, P_{ij} is a flat left P_i-module, as seen by the following example. Let $\mathbf{C}_{(x)}[x]$ be the localization of the polynomial ring $\mathbf{C}[x]$ over the complex numbers at the prime ideal (x). Then $\mathbf{C}_{(x)}[x]$ will be a local principal ideal ring with completion the ring of formal power series over $\mathbf{C}$. Then by Kaplansky [24, p. 46] there exists an indecomposable torsionfree module M of rank two over $\mathbf{C}_{(x)}[x]$. Now let

$$R = \begin{bmatrix} \mathbf{R} & 0 & 0 \\ \mathbf{C} & \mathbf{R} & 0 \\ M & M & \mathbf{C}_{(x)}[x] \end{bmatrix}$$

where $\mathbf{R}$ denotes the field of real numbers. The ring R has a semiprimary classical left quotient ring and the respective P_{ij}'s are flat P_i-modules, but the ring is not semihereditary. To see this, note that $\begin{bmatrix} \mathbf{C} \\ M \end{bmatrix}$ needs two generators over the ring

$$S = \begin{bmatrix} \mathbf{R} & 0 \\ M & \mathbf{C}_{(x)}[x] \end{bmatrix}$$

and hence $\begin{bmatrix} \mathbf{C} \\ M \end{bmatrix}$ cannot be S-flat (equivalently projective) since it is indecomposable. Consequently R is not semihereditary.

Notice that the ring S in the previous paragraph is a semihereditary ring by Theorem 3.7; however, M is not a direct sum of uniform $\mathbf{C}_{(x)}[x]$ ideals. If R

is a left hereditary ring possessing a semiprimary classical left quotient ring, we will show that this cannot occur.

Let R be a semihereditary ring with semiprimary classical left quotient ring Q as in (1) and (2). Now let $e_i = e_{i1} + \cdots + e_{i\mu(i)}$ where $\mu(i)$ is a positive integer for $i = 1, \ldots, n$ and each e_{ij} is a primitive idempotent of Q. The following lemma shows that we can reduce the study of projective modules in our situation to indecomposable projective modules.

LEMMA 3.8. *Let R and Q be as in the previous paragraph. If I is a finitely generated indecomposable left ideal of R, then QI is an indecomposable projective Q-module. Furthermore, every projective R-module is a direct sum of indecomposable finitely generated projective modules.*

PROOF. To prove the first statement, let I be a finitely generated indecomposable left ideal of R and let i_0 be the largest i for which there is a nonzero projection of I into Re_{ij}. Since I is indecomposable, there is an l such that I projects monomorphically into $Re_{i_0 l}$. Identify I with this image. Then $QI \subset QRe_{i_0 l} = Qe_{i_0 l}$. Since $e_{i_0} I \neq 0$ (otherwise a larger i could have been chosen), $QI \not\subset N(Q)e_{i_0 l}$ so $QI = Qe_{i_0 l}$ is indecomposable.

To prove the second statement, by Albrecht [1] each projective left R-module is isomorphic to a direct sum of finitely generated left ideals of R; therefore it is sufficient to show that each finitely generated left ideal is a direct sum of indecomposable left ideals. Let I be a finitely generated left ideal of R and let the number of indecomposable direct summands of QI be k. The integer k is uniquely determined since Q is semiprimary. Suppose $I = I_1 \oplus \cdots \oplus I_t$ where each $I_j \neq 0$. Then by [31, Corollary 1.6], $QI = QI_1 \oplus QI_2 \oplus \cdots \oplus QI_t$. Thus $t \leqslant k$ and I can be written as a finite direct sum of indecomposable finitely generated left ideals. $\square$

The next proposition is a key result in that it characterizes indecomposable projective modules over left hereditary PWD's possessing a semiprimary classical left quotient ring.

PROPOSITION 3.9. *Let R be a left hereditary ring possessing a semiprimary classical left quotient ring Q with structure as in (1) and (2). If F is an indecomposable projective left R-module, then*

$$F \simeq {}^t[U_\alpha \mathbin{\vdots} N(R)e_{\alpha 1}]$$

where U_α is a finitely generated uniform projective submodule of $_{P_\alpha}K_\alpha$ for some α.

PROOF. By Lemma 3.8 and a simple induction on the triangular structure of R, we may assume without loss of generality that F is a finitely generated submodule of Qe_{11} for which $e_1 F \neq 0$; i.e., $F \not\subset N(Q)e_{11}$.

Since $e_1 F \neq 0$ and $e_1 F$ is a submodule of the uniform P_1-module $e_1 Q e_{11}$, we may take U to be $e_1 F$. Hence $R\begin{bmatrix} U \\ 0 \end{bmatrix} \subset F$; that is,

$$^t[U \,\vdots\, P_{21} U \,\vdots\, \cdots \,\vdots\, P_{21} U] \subset F.$$

Now by Lemma 3.4, P_{i1} is a right K_1-module; therefore $P_{i1} = P_{i1} K_1 U = P_{i1} K_1 e_{11} = P_{i1} e_1$. Thus $N(R) e_{11} \subset F$. Letting f_2 be as in Theorem 3.7, we have $N(R) e_{11} \subset f_2 F$. We claim that $N(R) e_{11} = f_2 F$.

Let $x \in f_2 F$; we want to show that $x \in N(R) e_{11}$. Since P_1 is a prime ring, $e_{11} F \cap P_1 \neq 0$ for otherwise $e_{11} F \cap P_1$ is a nilpotent left ideal of P_1. So let y be a nonzero element of $e_{11} F \cap P_1$ and then let $z = x + y$; note that $x = f_2 z$. Rz is a cyclic submodule of the projective R-module F; hence Rz is projective. As before $N(R) e_{11} \subset Rz$ and $y \neq 0$; therefore Rz must be indecomposable. Thus Rz is isomorphic to a principal left ideal I of Re_1. Let the image of z under this isomorphism be w. Since $e_{11} z \neq 0$, $e_{11} w \neq 0$ so $w = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ where R is partitioned as

$$R = \begin{bmatrix} e_1 R e_1 & 0 \\ f_2 R e_1 & f_2 R f_2 \end{bmatrix}^{.}$$

Since $e_{11} a \neq 0$, there exists an i such that $e_{11} a e_{1i} \neq 0$. By Gordon [17, Theorem 2] there exists $0 \neq u \in e_{1i} R e_{11}$. Then $au = wu \neq 0$ since R is a PWD; so Rwu and $P_1 au$ are nonzero homomorphic images of Rw and $P_1 a$ respectively. Since Rwu is projective and both Rw and $P_1 a$ are indecomposable, it follows that $Rwu \simeq Rw$ and $P_1 au \simeq P_1 a$. Thus we may assume that $e_{11} a e_{11} = a$.

Now $x \in f_2 Q e_1 \backslash f_2 R e_1$, so $x = d^{-1} c$ for d regular in $f_2 R f_2$, $c \in f_2 R e_1$. Since $x \notin R$, $c \notin d f_2 R e_1$. Now $z = \begin{bmatrix} y & 0 \\ x & 0 \end{bmatrix}$ and $N(R) e_{11} \subset Ry$. In particular, there is a $p \in f_2 R e_1$ for which $py = c$. Note that $p \notin d_2 f_2 R e_1$ since $c \notin d f_2 R e_1$. Now

$$\begin{bmatrix} 0 & 0 \\ p & -d \end{bmatrix} \begin{bmatrix} y & 0 \\ d^{-1} c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ py - dd^{-1}c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ p & -d \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ pa - db & 0 \end{bmatrix}^{.}$$

Since $df_2 R e_1$ is a right K_1-module, $df_2 R e_{11}$ is a right $e_{11} K_1 e_{11} = D_1$-module. Thus if $pa \in df_2 R e_{11}$, we must have $pe_{11} a e_{11} = pa \in df_2 R e_{11}$. Since the elements of $D_1 = e_{11} K_1 e_{11}$ are invertible, $p \in df_2 R e_{11}$, a contradiction. Thus $x \in f_2 R e_1$ as desired. $\square$

Let P be a prime left hereditary left Goldie ring with classical left quotient ring K and e a primitive idempotent of R. A finitely generated submodule of

Ke containing Pe will be called a *uniform fractional left P-ideal*.

We now state our main structure theorem for left hereditary PWD's which are left orders in a semiprimary classical left quotient ring. The P_i's, P_{ij}'s, K_i's, K_{ij}'s and e_{ij}'s are as defined in this section.

THEOREM 3.10. *Let R be an indecomposable, left hereditary ring possessing a semiprimary classical left quotient ring Q with structure as in* (1) *and* (2). *Then*

(a) *each P_i is a left hereditary, left Noetherian prime ring*;

(b) *each P_{ij} is a right K_j-module*;

(c) *there is a sequence of integers $i < i(1) < \cdots < i(k) \leqslant n$ and a sequence of ordinal numbers $v(1), \ldots, v(k)$ such that*

$$N(R)e_{ij} = \left[\, T_1^1 \;\vdots\; T_2^1 \;\vdots\; \cdots \;\vdots\; T_{v(1)}^1 \;\vdots\; \cdots \;\vdots\; T_1^\alpha \;\vdots\; \cdots \;\vdots\; T_{v(\alpha)}^\alpha \;\vdots\; \cdots \;\vdots\; T_{v(k)}^k \,\right]$$

where

$$T_\alpha^\beta = {}^t[0 \;\vdots\; U_{i(\alpha)} \;\vdots\; N(R)e_{i(\alpha)}]$$

and $U_{i(\alpha)}$ is a uniform projective fractional left $P_{i(\alpha)}$-ideal.

Conversely, any such matrix ring is a left hereditary ring possessing a semiprimary classical left quotient ring.

PROOF. As in the proof of Theorem 3.7, $P_i = e_i R e_i$ is a left hereditary prime Goldie ring, so P_i is left Noetherian by Levy [25]. By Lemma 3.4, P_{ij} is a right K_j-module.

Now consider $N(R)e_{ij}$. R is left hereditary so $N(R)e_{ij}$ is projective; hence $N(R)e_{ij} = \bigoplus_{\delta \in \Delta} I_\delta$ where each I_δ is a finitely generated indecomposable projective R-module. Then $N(Q)e_{ij} = QN(R)e_{ij} = \bigoplus_{\delta \in \Delta} QI_\delta$, where each QI_δ will be indecomposable and Q-projective by Lemma 3.8. Thus QI_δ is isomorphic to Qe^δ for some primitive idempotent e^δ of Q. Since $e^\delta QI_\delta \neq 0$, $e^\delta I_\delta \neq 0$ so $e^\delta I_\delta$ is isomorphic to a uniform submodule of $K_\delta e^\delta \cdot$ Letting $u^\delta \neq 0 \in e^\delta I_\delta$, it follows from the proof of Proposition 3.9 that $Qu^\delta \simeq Qe^\delta = QI_\delta$. As in Theorem 2.1, a "new basis" can be chosen so that each u^δ is a small identity matrix. Furthermore, there are integers $i < i(1) < \cdots < i(k) \leqslant n$ and ordinal number $v(1), \ldots, v(k)$ such that the u^δ's can be indexed as u_β^α's as in Theorem 2.1. In this case

$$N(Q)e_{ij} = \left[\, C_1^1 \;\vdots\; C_2^1 \;\vdots\; \cdots \;\vdots\; C_{v(1)}^1 \;\vdots\; \cdots \;\vdots\; C_1^\alpha \;\vdots\; \cdots \;\vdots\; C_{v(\alpha)}^\alpha \;\vdots\; \cdots \;\vdots\; C_{v(k)}^k \,\right]$$

where each C_β^α is the collection of matrices of the form

$$^t[0 \vdots Qe_{i(\alpha)1}].$$

This induces a representation for R so that

$$N(R)e_{ij} = \left[\begin{array}{ccccccccccc} T_1^1 & T_2^1 & \cdots & T_{v(1)}^1 & \cdots & T_1^\alpha & \cdots & T_{v(\alpha)}^\alpha & \cdots & T_{v(k)}^k \end{array} \right]$$

where each T_β^α is a finitely generated indecomposable projective submodule of C_β^α containing $Re_{i(\alpha)}$. Then by Proposition 3.9,

$$T_\alpha^\beta = {}^t[0 \vdots U_{i(\alpha)} \vdots N(R)e_{i(\alpha)1}]$$

where $U_{i(\alpha)}$ is a uniform projective fractional left $P_{i(\alpha)}$-ideal.

To prove the converse, suppose R and Q have the indicated structure. As in the proof of the converse of Theorem 3.7, we induct on the number of central idempotents of R/N. If $n = 1$, the result is trivial. Now assume that the result holds for $k < n$. Let I be a left ideal of R and show that I is projective. We may assume that $I \subset Re_{ij}$. If $I \subset N(R)e_{ij}$, then I is a projective left $f_{i+1}Rf_{i+1}$-module by the induction hypothesis when $f_{i+1} = e_{i+1} + \cdots + e_n$; hence I is R-projective. If $I \not\subset N(R)e_{ij}$, then e_iI is a uniform left P_i-ideal. Hence by the proof of Proposition 3.9, $N(R)e_{ij} \subset I$ and $I/N(R)e_{ij}$ is finitely generated since P_i is left Noetherian. But $N(R)e_{ij}$ is small in I so I is finitely generated. Therefore, I is projective, since by Theorem 3.7, R is semihereditary. Thus R is left hereditary. $\square$

The following example illustrates the power of Theorem 3.10.

EXAMPLE. Let F be a field, $F[x]$ the ring of polynomials over F and $F(x)$ the quotient field of $F[x]$. Then the ring

$$R = \begin{bmatrix} F & 0 & 0 \\ F & F & 0 \\ F[x]\dfrac{1}{x} & F[x] & F[x] \end{bmatrix}$$

has a semiprimary classical left quotient ring

$$\begin{bmatrix} F & 0 & 0 \\ F & F & 0 \\ F(x) & F(x) & F(x) \end{bmatrix}.$$

However, by Theorem 3.11, R is not left hereditary since $N(R)e_1$ is not the set of matrices over Re_2. Note that the ring

$$\begin{bmatrix} F & 0 & 0 \\ F & F & 0 \\ F[x]\frac{1}{x} & F[x]\frac{1}{x} & F[x] \end{bmatrix}$$

is left hereditary. $\square$

We now shift our direction towards two-sided classical quotient rings. In case a semihereditary ring possesses a semiprimary classical two-sided quotient ring, then every flat left Q-module is also R-flat. Thus Theorem 3.7 reduces to

THEOREM 3.11. *Let R be an indecomposable semihereditary ring with semiprimary classical two-sided quotient ring Q. Then each P_i is a semihereditary two-sided Goldie prime ring with two-sided classical quotient ring K_i and $P_{ij} = K_{ij}$. Conversely, any such ring is semihereditary.*

REMARK. Theorem 3.11 gives a surprisingly complete characterization of a class of semihereditary rings including the class of two-sided finite dimensional rings; it basically says that such a ring is a hereditary semiprimary ring except that prime rings are sprinkled along the diagonal.

Now let R be a two-sided hereditary ring which is a two-sided order in a semiprimary ring. Then by Lemma 3.6, K_i cannot be a projective left or right P_i-module unless $P_i = K_i$. Hence it follows that $K_{ij} = 0$ whenever $P_i \neq K_i$ or $P_j \neq K_j$. Thus we can order the P_i's such that $R \simeq \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}$ where A is a semiprimary ring and S is a direct sum of prime rings. Putting all of this together, we have

THEOREM 3.12. *A (two-sided) hereditary ring which is a two-sided order in a semiprimary ring is a direct sum of a semiprimary ring and a semiprime ring.* $\square$

As a corollary we have a new proof of Chatters' result [9] which does not use the restricted minimum condition.

COROLLARY 3.13. *A hereditary Noetherian ring is a direct sum of an Artinian ring and a direct sum of prime rings.* $\square$

Finally we consider the situation when a semihereditary ring with semiprimary classical left quotient ring has a two-sided maximal quotient ring. Part (i) of the following theorem has already been proved by Goodearl [13]; however, our proof is somewhat shorter than his.

THEOREM 3.14. *Let R be an indecomposable semihereditary ring with semiprimary classical left quotient ring Q. If R has a two-sided maximal quotient ring T, then*

(i) *Q is Morita equivalent to a full block lower triangular matrix ring over a division ring (and hence is a hereditary serial ring); and*

(ii) *each P_i is a two-sided order in K_i and $K_{ij} = P_{ij}$.*

PROOF. First we show (i). As in §2, T will be a direct sum of full linear rings over division rings. Since T is the two-sided maximal quotient ring of R, T will be left and right self-injective. Since a full linear ring is two-sided self-injective if and only if it is semisimple (Osofsky [26]) and Q is semiprimary, we have that Q is right and left Artinian by Sandomierski [28]. Since by Cateforis [6], a semisimple maximal left quotient ring is two-sided if and only if injective hulls of finitely generated projective modules are flat, it will be sufficient to assume that Q is basic. In that case the structure of Q is given by Theorem 2.1. In terms of the notation of Theorem 2.1, by the left-hand version of Theorem 2.1, $\cdots l_1 \geqslant l_2 \geqslant \cdots \geqslant l_n$ and by the right-hand version of Theorem 2.1, $l_n \geqslant l_{n-1} \geqslant \cdots \geqslant l_1$. Thus $l_1 = l_2 = \cdots = l_n$. By Proposition 2.7, the ranks for a family of generators for Ne_i must be l_i; hence it must be that Qe_n is the only projective simple left module and each Qe_i has simple socle. Thus it follows that Q is a lower block matrix ring over some division ring.

To prove (ii) note that since the maximal quotient ring T of R is two-sided, T must be the two-sided maximal quotient ring of Q also. Hence T is semisimple and Q is a hereditary serial ring by the first part of the proof. It follows that R_R must have finite Goldie dimension and R is a two-sided order in Q. Then the desired structure follows from Theorem 3.11. □

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