TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 212, 1975

THE WEDDERBURN PRINCIPAL THEOREM FOR GENERALIZED ALTERNATIVE ALGEBRAS I

BY

HARRY F. SMITH

ABSTRACT. A generalized alternative ring I is a nonassociative ring R in which the identities $(wx, y, z) + (w, x, (y, z)) - w(x, y, z) - (w, y, z)x$; $((w, x), y, z) + (w, x, yz) - y(w, x, z) - (w, x, y)z$; and (x, x, x) are identically zero. Let A be a finite-dimensional algebra of this type over a field F of characteristic $\neq 2, 3$. Then it is established that (1) A cannot be a nodal algebra, and (2) the standard Wedderburn principal theorem is valid for A.

1. **Preliminaries.** Let R be a nonassociative ring. For x, y, z in R we denote by (x, y, z) the associator $(x, y, z) = (xy)z - x(yz)$ and by (x, y) the commutator $(x, y) = xy - yx$.

In [3] Kleinfeld has defined a generalized alternative ring I to be a nonassociative ring R such that for all w, x, y, z in R the following identities are satisfied:

$$(wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x, \tag{1}$$

$$((w, x), y, z) + (w, x, yz) = y(w, x, z) + (w, x, y)z, \tag{2}$$

$$(x, x, x) = 0. \tag{3}$$

In particular, these identities are satisfied by any alternative ring, that is any ring which satisfies the identities $(x, x, y) = 0 = (y, x, x)$. Conversely, from [3] and [8] it is known that if R is a ring of this type with characteristic $\neq 2, 3$, then R is alternative if R is prime and contains an idempotent $e \neq 1$. Also, from [3] we have that R is alternative if whenever x, y, z are contained in a subring of R generated by two elements and $(x, y, z)^2 = 0$, then $(x, y, z) = 0$.

Throughout this work we shall assume A to be a finite-dimensional generalized alternative algebra I over a field F of characteristic $\neq 2, 3$. We note that from [9] it is then known that if A is a nilalgebra, A is nilpotent.

In addition to the above defining identities, we shall also need to make use of the following:

$$(w, xy, z) - (w, x, zy) + (w, x, y)z - (w, y, z)x = 0, \tag{4}$$

$$(w, xy, z) - (xw, y, z) + w(x, y, z) - y(w, x, z) = 0, \tag{5}$$

Received by the editors June 6, 1974.

AMS (MOS) subject classifications (1970). Primary 17A30.

Key words and phrases. Generalized alternative ring I, Jordan algebra, nodal algebra, Penico solvable, semisimple, Wedderburn principal theorem.

$$(x, x, yz) = y(x, x, z) + (x, x, y)z, \tag{6}$$

$$(x^2, x, y) = (x, x^2, y) = 2(x, x, yx). \tag{7}$$

Identities (4), (5) and (6) are established in [8], while (7) can be found in [9].

2. **Nodal algebras.** If A is an algebra over a field F of characteristic $\neq 2$, we can construct a new algebra A^+ over F, where the vector space operations are the same as those in A but multiplication is defined by the (commutative) product $x \circ y = \frac{1}{2}(xy + yx)$. A Jordan algebra is a commutative algebra which satisfies the identity $(x, y, x^2) = 0$.

LEMMA 1. *If A is a generalized alternative algebra I over a field F of characteristic $\neq 2, 3$, then A^+ is a Jordan algebra.*

PROOF. From [3] we have $(x, y, x^2) = 2(x, y, x)x$, $2x(x, y, x) = (x^2, y, x)$, and $(x, y, x)x = x(x, y, x)$, whence

$$(x, y, x^2) = (x^2, y, x). \tag{8}$$

Next letting $z = y = x$ in (4) we obtain $(w, x^2, x) - (w, x, x^2) = 0$, whence

$$(y, x^2, x) = (y, x, x^2). \tag{9}$$

Now using (7), (8), and (9) we have

$$\begin{aligned} 0 &= (x, x^2, y) - (x^2, x, y) + (x, y, x^2) - (x^2, y, x) + (y, x, x^2) - (y, x^2, x) \\ &= (xy)x^2 + (yx)x^2 + x^2(xy) + x^2(yx) - x(yx^2) - x(x^2y) - (yx^2)x - (x^2y)x \\ &= 4((x \circ y) \circ x^2 - x \circ (y \circ x^2)). \end{aligned}$$

Thus $(x, y, x^2) = 0$ in A^+, and so A^+ is a Jordan algebra.

Let A be a finite-dimensional power-associative algebra with unity element over a field F. If every x in A is of the form $x = \alpha 1 + n$ with α in F and n nilpotent, and if the set N of nilpotent elements of A does not form a subalgebra of A, then A is called a nodal algebra.

Let A be a nodal generalized alternative algebra I. Since the Jordan algebra A^+ cannot be a nodal algebra [2], $A^+ = F1 + N^+$ where N^+ is a nilideal of A^+; that is $A = F1 + N$ where N is a subspace of A consisting of all nilpotent elements of A, and $x \circ y$ is in N for all x in N, y in A. We denote by $N \circ N$ the subspace of N generated by all elements of the form $x \circ y$ with x, y in N.

LEMMA 2. *If A is a nodal generalized alternative algebra I over a field F of characteristic $\neq 2, 3$, then $(N \circ N)N \subseteq N$ and $N(N \circ N) \subseteq N$.*

PROOF. From [3] we have $(x, y, x)^2 = 0$, whence (x, y, x) is in N for all x, y in A. Continuing now as in the proof of Lemma 2 in [1], we have $(x \circ y)x = \frac{1}{2}(x, y, x) + (yx) \circ x$ is in N for x in N. Linearization of $(x \circ y)x$

then gives $(x \circ y)z + (z \circ y)x$ is in N for x, z in N. Taking $z = y$, this in turn yields $(x \circ y)y + y^2x$ is in N, whence y^2x is in N for x, y in N. Linearization of y^2x now gives $2(y \circ z)x$, hence $(y \circ z)x$ is in N for x, y, z in N. Since this implies $x(y \circ z) = 2((y \circ z) \circ x) - (y \circ z)x$ is also in N, we have shown $(N \circ N)N$ and $N(N \circ N)$ are contained in N.

LEMMA 3. *If A is a nodal generalized alternative algebra I over a field F of characteristic $\neq 2, 3$, then (x, x, y) and (y, x, x) are in N for all x, y in A.*

PROOF. It is clear we may assume that x, y are in N. Let $xy = \alpha 1 + n$. Then (7) gives $2(x, x, yx) = (x^2, x, y) = x^3y - x^2(xy) = x^3y - \alpha x^2 - x^2n$ is in $N \circ N + (N \circ N)N$, which by Lemmas 1 and 2 is contained in N. Thus (x, x, yx) is in N. Next from (6) and (3) we have $(x, x, y \circ x) = x \circ (x, x, y)$ is in N, whence (x, x, xy) is also in N. If we take $y = x$ in (6) and apply (3), we obtain

$$(x, x, xy) = x(x, x, y). \tag{10}$$

Then using (10) we have $x(x(x,x,y)) = x(x, x, xy) = x(x, x, \alpha 1 + n) = x(x, x, n) = (x, x, xn)$ is in N. Hence $(x, x, (x, x, y)) = x^2(x, x, y) - x(x(x, x, y))$ is in N, since by Lemmas 1 and 2 $x^2(x, x, y)$ is in N. Set $(x, x, y) = u$. Then using (6) we have N contains

$$\begin{aligned}(x, x, (x, x, y^2)) &= 2(x, x, y \circ u) \\ &= 2(y \circ (x, x, u) + u \circ (x, x, y)) = 2(y \circ (x, x, u) + u^2).\end{aligned}$$

This implies $2u^2$, hence u^2, is in N, since $2y \circ (x, x, u)$ is in N by Lemma 1. Thus $(x, x, y) = u$ is itself in N. Lastly, linearization of (3) gives $(y, x, x) = -(x, y, x) - (x, x, y)$ is in N, since as in the proof of Lemma 2 we know (x, y, x) to be in N.

LEMMA 4. *If A is a nodal generalized alternative algebra I over a field F of characteristic $\neq 2, 3$, then $((N \circ N)N)N \subseteq N$ and $N((N \circ N)N) \subseteq N$.*

PROOF. Let x, y, z be in N. Then from (1) we have $(x^2, y, z) = -(x, x, (y, z)) + 2x \circ (x, y, z)$, whence (x^2, y, z) is in N by Lemmas 1 and 3. Since $x^2(yz)$ is in N by Lemmas 1 and 2, this in turn implies $(x^2y)z = (x^2, y, z) + x^2(yz)$ is in N. Linearization of $(x^2y)z$ now yields $2((w \circ x)y)z$, hence $((w \circ x)y)z$, is in N for w, x, y, z in N. Since this implies $z((w \circ x)y) = 2((w \circ x)y) \circ z - ((w \circ x)y)z$ is also in N, we have proven $((N \circ N)N)N$ and $N((N \circ N)N)$ to be contained in N.

LEMMA 5. *If A is a nodal generalized alternative algebra I over a field F of characteristic $\neq 2, 3$, then $K = N \circ N + (N \circ N)N + ((N \circ N)N)N$ is an ideal of A contained in N.*

PROOF. That K is contained in N follows directly from Lemmas 1, 2, and 4. Take $x = \alpha 1 + n$ in A, k in K. Then $kx = \alpha k + kn$ and $xk = \alpha k + nk = \alpha k + 2n \circ k - kn$. Thus AK and KA are both contained in $K + KN$, and so to show K is an ideal of A it is sufficient to show $(((N \circ N)N)N)N$, hence KN, is contained in K.

Let u, v, w, x, y, z be in N. Then taking $w = u \circ v$, from (2) we obtain

$$\begin{aligned}
&((x(u \circ v))y)z + y(((u \circ v)x)z) \\
&\quad = (x(u \circ v))(yz) - (u \circ v)(x(yz)) + y((u \circ v)(xz)) + ((u \circ v)(xy))z \\
&\quad = (2(x \circ (u \circ v))(yz) - ((u \circ v)x)(yz)) - (u \circ v)(x(yz)) \\
&\qquad + (2y \circ ((u \circ v)(xz)) - ((u \circ v)(xz))y) + ((u \circ v)(xy))z
\end{aligned}$$

is in K. Since

$$\begin{aligned}
((x(u \circ v))y)z + y(((u \circ v)x)z) = {} & (2((x \circ (u \circ v))y)z - (((u \circ v)x)y)z) \\
& + (2y \circ (((u \circ v)x)z) - (((u \circ v)x)z)y),
\end{aligned}$$

this gives

(i) $(((u \circ v)x)y)z \equiv -(((u \circ v)x)z)y \bmod K$.

Now from (1) we obtain $(((u \circ v)x)y)z - (((u \circ v)y)z)x = ((u \circ v)x)(zy) - (u \circ v)(x(zy)) + (u \circ v)((xy)z) - ((u \circ v)(yz))x$ is in K. Using (i) this gives

(ii) $(((u \circ v)x)y)z \equiv -(((u \circ v)y)x)z \bmod K$.

Next taking $y = u \circ v$, from (1) we obtain

$$\begin{aligned}
&w(x(z(u \circ v))) - w((x(u \circ v))z) - ((w(u \circ v))z)x + (w((u \circ v)z))x \\
&\quad = -((wx)(u \circ v))z + (wx)(z(u \circ v)) \\
&\quad = -(2((wx) \circ (u \circ v))z - ((u \circ v)(wx))z) \\
&\qquad + (2(2(wx) \circ (z \circ (u \circ v))) - 2(z \circ (u \circ v))(wx) \\
&\qquad\qquad - 2(wx) \circ ((u \circ v)z) + ((u \circ v)z)(wx))
\end{aligned}$$

is in K. Letting $w = y$ this gives

(iii) $y(x(z(u \circ v))) - y((x(u \circ v))z) - ((y(u \circ v))z)x + (y((u \circ v)z))x \equiv 0 \bmod K$.

Noting that $nk + kn = 2n \circ k$ implies $nk \equiv -kn \bmod N \circ N$ and so that also $N(N \circ N), N(N(N \circ N)), (N(N \circ N))N, N((N \circ N)N)$ are in K, from (iii) we now have

$$\begin{aligned}
0 &\equiv y(x(z(u \circ v))) - y((x(u \circ v))z) - ((y(u \circ v))z)x + (y((u \circ v)z))x \\
&\equiv -(x(z(u \circ v)))y + ((x(u \circ v))z)y + (((u \circ v)y)z)x - (((u \circ v)z)y)x \\
&\equiv (x((u \circ v)z))y - (((u \circ v)x)z)y + (((u \circ v)y)z)x - (((u \circ v)z)y)x \\
&\equiv -(((u \circ v)z)x)y - (((u \circ v)x)z)y + (((u \circ v)y)z)x - (((u \circ v)z)y)x \bmod K.
\end{aligned}$$

That is

(iv) $-(((u \circ v)z)x)y - (((u \circ v)x)z)y + (((u \circ v)y)z)x - (((u \circ v)z)y)x \equiv 0 \bmod K$.

Finally, applying (ii) to (iv) we have $0 \equiv (((u \circ v)x)z)y - (((u \circ v)x)z)y + (((u \circ v)y)z)x + (((u \circ v)y)z)x = 2(((u \circ v)y)z)x \bmod K$. Thus $(((N \circ N)N)N)N$ is contained in K, and so it follows that K is an ideal of A contained in N.

LEMMA 6. *There are no nodal generalized alternative algebras I over fields of characteristic* $\neq 2, 3$ *such that* $n^2 = 0$ *for each n in N.*

PROOF. Suppose that A is a nodal generalized alternative algebra I over a field F of characteristic $\neq 2, 3$ such that $n^2 = 0$ for each n in N. We first note that for x, y in N we have $0 = (x + y)^2 = xy + yx$ implies $xy = -yx$. Let $xy = \alpha 1 + n = -yx$. Then taking $w = x$ and $z = y$ in (1) we have

$$\begin{aligned} 0 &= (x^2y)y - x^2y^2 - x((xy)y) + x(xy^2) - ((xy)y)x + (xy^2)x \\ &= -x((xy)y) - ((xy)y)x = -x((\alpha 1 + n)y) - ((\alpha 1 + n)y)x \\ &= -\alpha xy - x(ny) - \alpha yx - (ny)x = -2x \circ (ny). \end{aligned}$$

Thus

(v) $x \circ (ny) = 0$.

Next taking w and y as x, x and z as y in (1) we have

$$\begin{aligned} 0 &= ((xy)x)y - (xy)(yx) + x(y(yx)) - x((yx)y) - (x^2y)y + (x(xy))y \\ &= ((xy)x)y + (xy)(xy) - x(y(xy)) + x((xy)y) + (x(xy))y \\ &= ((\alpha 1 + n)x)y + (\alpha 1 + n)^2 - x(y(\alpha 1 + n)) + x((\alpha 1 + n)y) + (x(\alpha 1 + n))y \\ &= \alpha xy + (nx)y + \alpha^2 1 + 2\alpha n + n^2 - \alpha xy - x(yn) + \alpha xy + x(ny) + \alpha xy + (xn)y \\ &= \alpha^2 1 + 2\alpha n + 2\alpha xy + 2x(ny) = 3\alpha^2 1 + 4\alpha n + 2x(ny). \end{aligned}$$

Thus

(vi) $3\alpha^2 1 + 4\alpha n + 2x(ny) = 0$.

Now taking $w = y$ and $z = x$ in (2) we have

$$\begin{aligned} 0 &= ((xy)y)x - (xy)(yx) + y(x(yx)) + y((yx)x) - y(yx^2) - (y(xy))x \\ &= ((xy)y)x + (xy)(xy) - y(x(xy)) - y((xy)x) - (y(xy))x \\ &= ((\alpha 1 + n)y)x + (\alpha 1 + n)^2 - y(x(\alpha 1 + n)) - y((\alpha 1 + n)x) - (y(\alpha 1 + n))x \\ &= \alpha yx + (ny)x + \alpha^2 1 + 2\alpha n + n^2 - \alpha yx - y(xn) - \alpha yx - y(nx) - \alpha yx - (yn)x \\ &= \alpha^2 1 + 2\alpha n + 2\alpha xy + 2(ny)x = 3\alpha^2 1 + 4\alpha n + 2(ny)x. \end{aligned}$$

Thus

(vii) $3\alpha^2 1 + 4\alpha n + 2(ny)x = 0$.

Finally, adding (vi) and (vii) and using (v) we obtain $0 = 6\alpha^2 1 + 8\alpha n + 4x \circ (ny) = 6\alpha^2 1 + 8\alpha n$. But then $6\alpha^2 = 0$ implies $\alpha = 0$, that is xy is in N for every x,y

in N. Since this means the set N of nilpotent elements of A is a subalgebra of A, A cannot be a nodal algebra.

THEOREM 1. *There are no nodal generalized alternative algebras I over fields of characteristic* $\neq 2, 3$.

PROOF. Suppose that A is a nodal generalized alternative algebra I over a field F of characteristic $\neq 2, 3$. Then A has a homomorphic image which is a simple nodal algebra, and so we can assume A itself to be simple. Now by Lemma 5, since the ideal $K = N \circ N + (N \circ N)N + ((N \circ N)N)N$ of A is contained in N, it must be zero. In particular, $N \circ N = 0$, and so $n^2 = 0$ for each n in N. But then, by Lemma 6, A cannot be a nodal algebra.

3. **Wedderburn principal theorem.**

LEMMA 7. *Let A be a generalized alternative algebra I. If B is an ideal of A, then $AB^2 + B^2A + B^2$ and $B^* = B^3 + A(BB^2) + (B^2B)A$ are ideals of A with $B^* \subseteq B^2$.*

PROOF. Take a_i in A, b_j in B for $i = 1, 2$; $j = 1, 2, 3$. Then from (1) we have $a_1(b_1, b_2, a_2) + (a_1, b_2, a_2)b_1 = (a_1b_1, b_2, a_2) + (a_1, b_1, (b_2, a_2))$, whence $A(B^2A) \subseteq AB^2 + B^2A + B^2$. Also from (1) we have $(b_1, b_2, a_1)a_2 + b_1(a_2, b_2, a_1) = (b_1a_2, b_2, a_1) + (b_1, a_2, (b_2, a_1))$, whence $(B^2A)A \subseteq B^2A + B^2$. Now using (2), in symmetric fashion one obtains that $(AB^2)A \subseteq AB^2 + B^2A + B^2$ and $A(AB^2) \subseteq AB^2 + B^2$. Thus $AB^2 + B^2A + B^2$ is an ideal of A.

To show B^* is an ideal of A, we first note that from (1) we have $(b_1b_2, a_1, b_3) + (b_1, b_2, (a_1, b_3)) = b_1(b_2, a_1, b_3) + (b_1, a_1, b_3)b_2$ or $((b_1b_2)a_1)b_3$ is in B^3. Symmetrically from (2) one also has $b_1(a_1(b_2b_3))$ is in B^3. Hence

(viii) $(B^2A)B, B(AB^2) \subseteq B^3$.

Now (5) gives $a_1(b_1, b_2, b_3) = (b_1, b_2a_1, b_3) - (b_2b_1, a_1, b_3) + b_1(b_2, a_1, b_3)$, and using (viii) this implies $A(B^2B) \subseteq B^3 + A(BB^2) \subseteq B^*$. Symmetrically (4) and (viii) imply $(BB^2)A \subseteq B^3 + (B^2B)A \subseteq B^*$. Thus we have shown $AB^3, B^3A \subseteq B^*$.

Next, letting $z = b_2b_3$, (2) yields $a_1(a_2, b_1, b_2b_3) + (a_2, b_1, a_1)(b_2b_3) = ((a_2, b_1), a_1, b_2b_3) + (a_2, b_1, a_1(b_2b_3))$, whence using (viii) and that $AB^3 \subseteq B^*$ we have $A(A(BB^2)) \subseteq B^*$. Then using our earlier calculations that $A(B^2B) \subseteq B^3 + A(BB^2)$ and $AB^3 \subseteq B^*$, this in turn gives $A(A(B^2B)) \subseteq AB^3 + A(A(BB^2)) \subseteq B^*$. Still letting $z = b_2b_3$, (4) now yields $(a_1, b_1, b_2b_3)a_2 = (a_1, a_2b_1, b_2b_3) - (a_1, a_2, (b_2b_3)b_1) + (a_1, a_2, b_1)(b_2b_3)$, whence using $B^3A, AB^3, A(A(B^2B)) \subseteq B^*$ we have $(A(BB^2))A \subseteq B^*$. Thus we have shown $A(A(BB^2)), (A(BB^2))A \subseteq B^*$. In symmetric fashion using (1) and (5) one also has $((B^2B)A)A, A((B^2B)A) \subseteq B^*$; and this completes the proof that B^* is an ideal of A.

Finally, from (1) we have $(b_1b_2, b_3, a_1) + (b_1, b_2, (b_3, a_1)) = b_1(b_2, b_3, a_1) + (b_1, b_3, a_1)b_2$ or $(B^2B)A \subseteq B^2$. Symmetrically from (2) we have $A(BB^2) \subseteq B^2$, and thus $B^* \subseteq B^2$.

For any nonassociative algebra A one can obtain a descending chain of subalgebras $A^{(0)} \supseteq A^{(1)} \supseteq \cdots \supseteq A^{(n)} \supseteq \cdots$ by defining inductively $A^{(0)} = A$, $A^{(i+1)} = (A^{(i)})^2$. The algebra A is called solvable in case $A^{(n)} = 0$ for some n.

Let A be a generalized alternative algebra I. If B is any ideal of A, we define $B^{\langle i \rangle}$ inductively by $B^{\langle 0 \rangle} = B, B^{\langle i+1 \rangle} = A(B^{\langle i \rangle})^2 + (B^{\langle i \rangle})^2A + (B^{\langle i \rangle})^2$. Then by Lemma 7 we obtain a descending chain $B^{\langle 0 \rangle} \supseteq B^{\langle 1 \rangle} \supseteq \cdots \supseteq B^{\langle m \rangle} \supseteq \cdots$ of ideals of A which we call a Penico sequence. As in [7], we shall call B Penico solvable in case there is some m for which $B^{\langle m \rangle} = 0$.

LEMMA 8. *Let A be a generalized alternative algebra I. If B is an ideal of A, then B is solvable if and only if B is Penico solvable.*

PROOF. If B is Penico solvable, then B is solvable since $B^{(i)} \subseteq B^{\langle i \rangle}$. To see that B solvable implies B is Penico solvable, suppose $B^{\langle 2 \rangle} \subseteq B^* \subseteq B^2 \subseteq B^{(1)}$. Then, as in the proof of Theorem 3 in [7], by induction one has $B^{\langle 2k \rangle} \subseteq B^{(k)}$, since $B^{\langle 2(k+1) \rangle} = (B^{\langle 2k \rangle})^{\langle 2 \rangle} \subseteq (B^{\langle 2k \rangle})^{(1)} \subseteq (B^{(k)})^{(1)} = B^{(k+1)}$. Hence if B is solvable, then $B^{\langle 2k \rangle} \subseteq B^{(k)} = 0$ for some k, and B is Penico solvable. Thus to prove B solvable implies B is Penico solvable, it is sufficient to prove

$$A((AB^2 + B^2A + B^2)^2) + ((AB^2 + B^2A + B^2)^2)A + (AB^2 + B^2A + B^2)^2 = B^{\langle 2 \rangle} \subseteq B^*.$$

To do this, since by Lemma 7 B^* is itself an ideal of A, it is in turn sufficient to show $(AB^2 + B^2A + B^2)^2 \subseteq B^*$. Now B an ideal of A gives $B^2B^2, B^2(B^2A)$, $(B^2A)B^2, B^2(AB^2), (AB^2)B^2 \subseteq B^3 \subseteq B^*$. Also, using (viii) from the proof of Lemma 7, we have $(B^2A)(B^2A), (B^2A)(AB^2) \subseteq (B^2A)B \subseteq B^3 \subseteq B^*$ and $(AB^2)(AB^2) \subseteq B(AB^2) \subseteq B^3 \subseteq B^*$. Lastly, taking a in A, b_i in B, from (1) we obtain $b_1(b_2, b_3, a) + (b_1, b_3, a)b_2 = (b_1b_2, b_3, a) + (b_1, b_2, (b_3, a))$, whence again using (viii) we have $B(B^2A) \subseteq B^3 + (B^2A)B + (B^2B)A \subseteq B^3 + (B^2B)A \subseteq B^*$. But then $(AB^2)(B^2A) \subseteq B(B^2A) \subseteq B^*$. We now have shown $(AB^2 + B^2A + B^2)^2 \subseteq B^*$, and so our proof is complete.

Let A be a finite-dimensional generalized alternative algebra I over a field F of characteristic $\neq 2, 3$. We define the radical N of A to be the maximal nilideal (= solvable = nilpotent [9]) of A, and we call A semisimple in case $N = 0$. If, in addition, A is semisimple over every scalar extension of the base field F, then A is said to be separable. We note too that A/N, as is the case for any power-associative algebra, is semisimple.

THEOREM 2. *Let A be a finite-dimensional generalized alternative algebra I over a field F of characteristic $\neq 2, 3$. If A is semisimple, then A has a unity* *element and is the direct sum of simple algebras.*

PROOF. The proof is the same as that of Theorem 9 in [4].

THEOREM 3 (WEDDERBURN PRINCIPAL THEOREM). *Let A be a finite-dimensional generalized alternative algebra I over a field F of characteristic $\neq 2, 3$. If A/N is separable, then $A = S + N$ (vector space direct sum) where S is a subalgebra of A such that $S \cong A/N$.*

PROOF. If A has dimension one, then since either $N = 0$ or $N = A$ the theorem is clearly true. We make an induction on the dimension of A and assume the theorem to be true for all algebras of lesser dimension.

Now, as in the proof of Theorem 2.4 in [9], it is possible to make the following standard reductions. First one may assume N not to properly contain any ideals of A. Then using Theorem 1.3 in [9] and our Lemma 8, one can reduce to the case $N^2 = 0$, and hence to the case F is an algebraically closed field. Next, using our Theorem 2 and the fact from [3] that A_1 and A_0 are subalgebras in the Albert decomposition for A relative to an idempotent e, we can use Theorem 2.1 in [5] to assume A has a unity element and that A/N is simple. As a final reduction we note, as in the proof of Theorem 2.2 of [5], that if there exists a primitive idempotent e such that our theorem is true for the ideal of A generated by the subspace $A_{1/2}$ in the Albert decomposition of A, then it is valid for A as well.

We suppose first that 1 is the only idempotent in A/N. Then since we are assuming the field F to be algebraically closed, and since by Theorem 1 there are no nodal generalized alternative algebras I over fields of characteristic $\neq 2, 3$, we must have $A/N = F1$. Now by Lemma 2.1 in [5], 1 lifts to an idempotent e in A. But then we have Fe a subalgebra of A such that $Fe \cong A/N$, and our theorem is proven. Thus we may assume that A/N, hence A, contains an idempotent $e \neq 1$. Furthermore, since A is finite-dimensional, we can take e to be primitive. Now by Theorem 1 in [8], $I = (e, e, A)$ is an ideal of A such that $I^2 = 0$. Hence, since we are assuming N not to properly contain any ideals of A, either $I = 0$ or $I = N$.

Suppose that $I = 0$. Then, as in the proof of the corollary in [8], A has a Peirce decomposition relative to e. Let $H = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$. As in the proof of Theorem 2 in [8], H is an ideal of A. In particular, H must be the ideal of A generated by $A_{1/2} = A_{10} + A_{01}$. Now if H is a proper ideal of A, then our induction assumption implies that the theorem is true for H. But then our final reduction applies, and so we may conclude that the theorem is true for A itself. On the other hand, if $H = A$, then $A_{11} = A_{10}A_{01}$ and $A_{00} = A_{01}A_{10}$. Take $w_{ij}, x_{ij}, y_{ij}, z_{ij}$ in A_{ij} for $i, j = 0, 1$. Then using the fact established in [3] that the Peirce subspaces of a generalized alternative algebra I multiply the same as for an alternative algebra, from (4) we obtain

$$(w_{11}, x_{11}, z_{10}y_{01}) = (w_{11}, x_{11}y_{01}, z_{10}) + (w_{11}, x_{11}, y_{01})z_{10} - (w_{11}, y_{01}, z_{10})x_{11} = 0,$$

and

$$(w_{00}, x_{00}, z_{01}y_{10}) = (w_{00}, x_{00}y_{10}, z_{01}) + (w_{00}, x_{00}, y_{10})z_{01} - (w_{00}, y_{10}, z_{01})x_{00} = 0.$$

Hence A_{11} and A_{00} are associative subalgebras of A. But then it follows from the proof of Theorem 2 in [8] and the proof of Theorem 3 in [3] that A is an alternative algebra. Since in this case the theorem is known from [6] to be valid for A, our induction is complete.

We consider now the other possibility, namely $I = N$, and take $k = (e, e, x) \neq 0$. For the Albert decomposition of A, we have from [3] that $A_{1/2}A_i$, $A_iA_{1/2} \subseteq A_{1/2}$ for $i = 0, 1$. In particular, this says that $N = (e, e, A) \subseteq A_{1/2}$. Now if H is the ideal in A generated by the subspace $A_{1/2}$, then $H = A_{1/2} + (A_{1/2})^2$. To see this, take x_i, y_i, z_i in A_i for $i = 0, 1/2, 1$. Then for $i = 0, 1$ we have

$$\begin{aligned}(x_{1/2}y_{1/2})z_i &= (x_{1/2}, y_{1/2}, z_i) + x_{1/2}(y_{1/2}z_i)\\ &= (x_{1/2}, y_{1/2} + z_i, y_{1/2} + z_i) - (x_{1/2}, y_{1/2}, y_{1/2})\\ &\quad - (x_{1/2}, z_i, z_i) - (x_{1/2}, z_i, y_{1/2}) + x_{1/2}(y_{1/2}z_i).\end{aligned}$$

But by Theorem 3 in [3] our assumption that A/N is simple implies A/N is alternative, that is (a, a, b) and (b, a, a) are in N for all a, b in A, so we have shown that $(x_{1/2}y_{1/2})z_i$ is in $N + (A_{1/2})^2 \subseteq A_{1/2} + (A_{1/2})^2$. Similarly one has $z_i(x_{1/2}y_{1/2})$ is in $A_{1/2} + (A_{1/2})^2$ for $i = 0, 1$. Since the cases for $i = 1/2$ are immediate if one writes $x_{1/2}y_{1/2} = a_1 + a_{1/2} + a_0$ with a_i in A_i, we have established $H = A_{1/2} + (A_{1/2})^2$ as claimed. Now by Theorem 1 in [8] $Hk = 0$, but from the proof of that same Theorem 1 $ek = \frac{1}{2}k \neq 0$. Hence e is not in H. But then H is a proper ideal of A, and so by the induction assumption the theorem is true for H. Our final reduction now applies to complete the induction and the proof of the theorem.

BIBLIOGRAPHY

1. J. D. Arrison and M. Rich, *On nearly commutative degree one algebras*, Pacific J. Math. 35 (1970), 533–536. MR 43 #298.

2. N. Jacobson, *A theorem on the structure of Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 140–147. MR 17, 822.

3. E. Kleinfeld, *Generalization of alternative rings*. I, J. Algebra 18 (1971), 304–325. MR 43 #308.

4. E. Kleinfeld, F. Kosier, J. M. Osborn and D. Rodabaugh, *The structure of associator dependent rings*, Trans. Amer. Math. Soc. 110 (1964), 473–483. MR 28 #1221.

5. D. J. Rodabaugh, *On the Wedderburn principal theorem,* Trans. Amer. Math. Soc. 138 (1969), 343–361.

6. R. D. Schafer, *The Wedderburn principal theorem for alternative algebras,* Bull. Amer. Math. Soc. 55 (1949), 604–614. MR 10, 676.

7. ———, *Standard algebras,* Pacific J. Math. 29 (1969), 203–223. MR 39 #5647.

8. H. F. Smith, *Prime generalized alternative rings I with nontrivial idempotent,* Proc. Amer. Math. Soc. 39 (1973), 242–246. MR 47 #1903.

9. ———, *The Wedderburn principal theorem for a generalization of alternative algebras,* Trans. Amer. Math. Soc. 198 (1974), 139–154.

DEPARTMENT OF MATHEMATICS, MADISON COLLEGE, HARRISONBURG, VIRGINIA 22801