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ON SOME REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

BY

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ABSTRACT. A principal circle bundle over a real hypersurface of a complex projective space CP^n can be regarded as a hypersurface of an odd-dimensional sphere. From this standpoint we can establish a method to translate conditions imposed on a hypersurface of CP^n into those imposed on a hypersurface of S^{2n+1}. Some fundamental relations between the second fundamental tensor of a hypersurface of CP^n and that of a hypersurface of S^{2n+1} are given.

Introduction. As is well known a sphere S^{2n+1} of dimension $2n + 1$ is a principal circle bundle over a complex projective space CP^n and the Riemannian structure on CP^n is given by the submersion π: $S^{2n+1} \longrightarrow CP^n$ [4], [7]. This suggests that fundamental properties of a submersion would be applied to the study of real submanifolds of a complex projective space. In fact, H. B. Lawson [2] has made one step in this direction. His idea is to construct a principal circle bundle $\overline{M}^{2n}$ over a real hypersurface M^{2n-1} of CP^n in such a way that $\overline{M}^{2n}$ is a hypersurface of S^{2n+1} and then to compare the length of the second fundamental tensors of M^{2n-1} and $\overline{M}^{2n}$. Thus we can apply theorems on hypersurfaces of S^{2n+1}.

In this paper, using Lawson's method, we prove a theorem which characterizes some remarkable classes of real hypersurfaces of CP^n. First of all, in §1, we state a lemma for a hypersurface of a Riemannian manifold of constant curvature for the later use. In §2, we recall fundamental formulas of a submersion which are obtained in [4], [7] and those established between the second fundamental tensors of M and $\overline{M}$. In §3, we give some identities which are valid in a real hypersurface of CP^n. After these preparations, we show, in §4, a geometric meaning of the commutativity of the second fundamental tensor of M in CP^n and a fundamental tensor of the submersion π: $\overline{M} \longrightarrow M$.

1. Hypersurfaces of a Riemannian manifold of constant curvature. Let $\widetilde{M}$ be an $(m + 1)$-dimensional Riemannian manifold with a Riemannian metric $\overline{G}$ and i: $\overline{M} \longrightarrow \widetilde{M}$ be an isometric immersion of an m-dimensional differentiable

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manifold $\overline{M}$ into $\widetilde{M}$. The Riemannian metric $\bar{g}$ of $\overline{M}$ is naturally induced from $\bar{G}$ in such a way that $\bar{g}(\overline{X}, \overline{Y}) = \bar{G}(i(\overline{X}), i(\overline{Y}))$, where $\overline{X}$, $\overline{Y}$ are vector fields on $\overline{M}$ and we denote by the same letter i the differential of the immersion. For an arbitrary point $\bar{x} \in \overline{M}$, we choose a unit normal vector and extend it to a field $\overline{N}$. The Riemannian connections $\bar{D}$ in $\widetilde{M}$ and $\overline{\nabla}$ in $\overline{M}$ are related by the following formulas:

$$D_{i(\overline{X})} i(\overline{Y}) = i(\overline{\nabla}_{\overline{X}} \overline{Y}) + \bar{g}(\overline{HX}, \overline{Y})\overline{N}, \tag{1.1}$$

$$\bar{D}_{i(\overline{X})} \overline{N} = -\, i(\overline{HX}), \tag{1.2}$$

where $\overline{H}$ is the second fundamental tensor of $\overline{M}$ in $\widetilde{M}$.

The mean curvature $\bar{\mu}$ of $\overline{M}$ in $\widetilde{M}$ is defined by

$$m\bar{\mu} = \text{trace } \overline{H}. \tag{1.3}$$

Let $\widetilde{R}$ and $\overline{R}$ be curvature tensors of $\widetilde{M}$ and of $\overline{M}$ respectively, then we have the following Gauss and Mainardi-Codazzi equations:

$$\begin{aligned} \bar{G}(\widetilde{R}(i(\overline{X}), i(\overline{Y}))i(\overline{Z}), i(\overline{W})) = \bar{g}(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W}) &- \bar{g}(\overline{HY}, \overline{Z})\bar{g}(\overline{HX}, \overline{W}) \\ &+ \bar{g}(\overline{HX}, \overline{Z})\bar{g}(\overline{HY}, \overline{W}), \end{aligned} \tag{1.4}$$

$$\bar{G}(\widetilde{R}(i(\overline{X}), i(\overline{Y}))i(\overline{Z}), \overline{N}) = \bar{g}((\overline{\nabla}_{\overline{X}} \overline{H})\overline{Y}, \overline{Z}) - \bar{g}((\overline{\nabla}_{\overline{Y}} \overline{H})\overline{X}, \overline{Z}), \tag{1.5}$$

where $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and $\overline{W}$ are vector fields on $\overline{M}$.

If the ambient manifold is of constant curvature k, the curvature tensor $\widetilde{R}$ has the form

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = k\{\bar{G}(\widetilde{Y}, \widetilde{Z})\widetilde{X} - \bar{G}(\widetilde{X}, \widetilde{Z})\widetilde{Y}\} \tag{1.6}$$

for vector fields $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ on $\widetilde{M}$. Consequently we have

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = k\{\bar{g}(\overline{Y}, \overline{Z})\overline{X} - \bar{g}(\overline{X}, \overline{Z})\overline{Y}\} + \bar{g}(\overline{HY}, \overline{Z})\overline{HX} - \bar{g}(\overline{HX}, \overline{Z})\overline{HY}, \tag{1.7}$$

$$(\overline{\nabla}_{\overline{X}} \overline{H})\overline{Y} = (\overline{\nabla}_{\overline{Y}} \overline{H})\overline{X}. \tag{1.8}$$

We assume that $\overline{M}$ has constant mean curvature, that is, trace $\overline{H}$ = const.

Let $\{\overline{E}_1, \ldots, \overline{E}_m\}$ be an orthonormal basis in $T_{\bar{x}}(\overline{M})$ and extend them to vector fields in a normal neighborhood of $\bar{x}$ by parallel translation along geodesics with respect to the Riemannian connection of $\overline{M}$. Then we have $\overline{\nabla}\overline{E}_i = 0$ $(i = 1, \ldots, m)$ at $\bar{x}$. Since $\overline{H}$ and $\overline{\nabla}_{\overline{E}_i}\overline{H}$ are both symmetric linear transformations on $T(\overline{M})$, we get, by using (1.8)

$$\begin{aligned} \bar{g}\left(\sum_{i=1}^m (\overline{\nabla}_{\overline{E}_i}\overline{H})\overline{E}_i, \overline{X}\right) = \sum_{i=1}^m \bar{g}(\overline{E}_i, (\overline{\nabla}_{\overline{E}_i}\overline{H})\overline{X}) &= \sum_{i=1}^m \bar{g}(\overline{E}_i, (\overline{\nabla}_{\overline{X}}\overline{H})\overline{E}_i) \\ &= \text{trace}\,(\overline{\nabla}_{\overline{X}}\overline{H}) = \overline{X}(\text{trace } \overline{H}) = 0, \end{aligned}$$

which implies that

$$\sum_{i=1}^{m} (\overline{\nabla}_{\overline{E}_i}\overline{H})\overline{E}_i = 0. \tag{1.9}$$

Thus we have

$$\sum_{i=1}^{m} (\overline{\nabla}_{\overline{X}}(\overline{\nabla}_{\overline{E}_i}\overline{H}))\overline{E}_i = 0 \quad \text{at } \overline{x}. \tag{1.10}$$

Now we prove the

LEMMA 1.1. *Let $\overline{M}$ be a hypersurface of a Riemannian manifold of constant curvature k. If the second fundamental tensor $\overline{H}$ satisfies for a constant α,*

$$\overline{H}^2\overline{X} = \alpha\overline{HX} + k\overline{X}, \quad \overline{X} \in \overline{T}(\overline{M}) \tag{1.11}$$

then we have $\overline{\nabla H} = 0$.

PROOF. Since $\overline{H}$ is a symmetric operator and (1.7), (1.8) are valid, we have

$$\begin{aligned}
(\overline{\nabla}_{\overline{X}}(\overline{\nabla}_{\overline{Y}}\overline{H}) - \overline{\nabla}_{\overline{Y}}(\overline{\nabla}_{\overline{X}}\overline{H}) - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{H})\overline{Z} &= \overline{R}(\overline{X},\overline{Y})\overline{HZ} - \overline{H}(\overline{R}(\overline{X},\overline{Y})\overline{Z}) \\
&= k\{\overline{g}(\overline{Y},\overline{HZ})\overline{X} - \overline{g}(\overline{X},\overline{HZ})\overline{Y}\} + \overline{g}(\overline{HY},\overline{HZ})\overline{HX} - \overline{g}(\overline{HX},\overline{HZ})\overline{HY} \\
&\quad - k\{\overline{g}(\overline{Y},\overline{Z})\overline{HX} - \overline{g}(\overline{X},\overline{Z})\overline{HY}\} - \overline{g}(\overline{HY},\overline{Z})\overline{H}^2\overline{X} + \overline{g}(\overline{HX},\overline{Z})\overline{H}^2\overline{Y} = 0.
\end{aligned}$$

Let $\{\overline{E}_1, \ldots, \overline{E}_m\}$ be an orthonormal basis which is chosen as above and $\overline{X}$ be a tangent vector at $\overline{x}$. Extend $\overline{X}$ to a vector field in a normal neighborhood of $\overline{x}$ by parallel translation along geodesics, then $\overline{\nabla X} = 0$ at $\overline{x}$. In the last equation we replace $\overline{Y}$ and $\overline{Z}$ by $\overline{E}_i$ and sum over i. Then we have, from (1.8) and (1.10),

$$\sum_{i=1}^{m} (\overline{\nabla}_{\overline{E}_i}(\overline{\nabla}_{\overline{X}}\overline{H}))\overline{E}_i = \sum_{i=1}^{m} (\overline{\nabla}_{\overline{E}_i}(\overline{\nabla}_{\overline{E}_i}\overline{H}))\overline{X} = 0 \quad \text{at } \overline{x}, \tag{1.12}$$

because from (1.11) we know that $\overline{M}$ has constant mean curvature. Furthermore (1.11) implies that trace $\overline{H}^2 = \alpha$ trace $\overline{H} + mk$ = const. Differentiating this covariantly, we have

$$\tfrac{1}{2}\overline{YX}(\text{trace } \overline{H}^2) = \text{trace}(\overline{\nabla}_{\overline{Y}}(\overline{\nabla}_{\overline{X}}\overline{H}))\overline{H} + \text{trace}(\overline{\nabla}_{\overline{Y}}\overline{H})(\overline{\nabla}_{\overline{X}}\overline{H}) = 0,$$

from which, at $\overline{x}$,

$$\text{trace}(\overline{\nabla}_{\overline{X}}\overline{H})^2 = -\,\text{trace}(\overline{\nabla}_{\overline{X}}(\overline{\nabla}_{\overline{X}}\overline{H}))\overline{H} = -\sum_{i=1}^{m} \overline{g}((\overline{\nabla}_{\overline{X}}(\overline{\nabla}_{\overline{X}}\overline{H}))\overline{E}_i, \overline{HE}_i).$$

Thus we have

$$\overline{g}(\overline{\nabla H}, \overline{\nabla H}) = \sum_{i=1}^{m} \text{trace}(\overline{\nabla}_{\overline{E}_i}\overline{H})^2 = -\sum_{i,j=1}^{m} \overline{g}((\overline{\nabla}_{\overline{E}_j}(\overline{\nabla}_{\overline{E}_j}\overline{H}))\overline{E}_i, \overline{HE}_i) = 0,$$

because of (1.12). This completes the proof.

2. **Submersion and immersion.** Let $\overline{M}$ and M be differentiable manifolds of dimension $n+1$ and n respectively and assume that there exists a differentiable mapping π of $\overline{M}$ onto M which has maximum rank, that is, each differential map π_* of π is onto. Hence, for each $x \in M$, $\pi^{-1}(x)$ is a 1-dimensional submanifold of $\overline{M}$, which is called the fibre over x. We suppose that every fibre is connected. A vector field on $\overline{M}$ is called vertical if it is always tangent to fibres, horizontal if always orthogonal to fibres; we use corresponding terminology for individual vectors. Thus $\overline{X} \in T_{\overline{x}}(\overline{M})$ decomposes as $\overline{X}^V + \overline{X}^H$, where $\overline{X}^V$ and $\overline{X}^H$ denote respectively vertical part and horizontal part of $\overline{X}$.

We assume that the mapping π is a Riemannian submersion, that is, there are given in $\overline{M}$ a vertical vector field $\overline{V}$ and a Riemannian metric $\overline{g}$ of $\overline{M}$ satisfying the condition that $\overline{V}$ is a unit Killing vector field with respect to the Riemannian metric $\overline{g}$. Then a Riemannian metric g can be defined on M by

$$g(X, Y)(x) = \overline{g}(X^L, Y^L)(\pi(\overline{x})), \tag{2.1}$$

where $\overline{x}$ is an arbitrary point of $\overline{M}$ such that $\pi(\overline{x}) = x$ and X^L, Y^L are the lifts of X, $Y \in T_x(M)$ respectively. Hence we have

$$g(X, Y)^L = \overline{g}(X^L, Y^L). \tag{2.2}$$

The fundamental tensor F of the submersion π is a skew-symmetric tensor of type (1.1) on M and is related to covariant differentiation $\overline{\nabla}$ and ∇ in $\overline{M}$ and M, respectively, by the following formulas:

$$\overline{\nabla}_{Y^L} X^L = (\nabla_Y X)^L + \overline{g}(F^L Y^L, X^L)\overline{V} = (\nabla_Y X)^L + g(FY, X)^L \overline{V}, \tag{2.3}$$

$$\overline{\nabla}_{\overline{V}} X^L = \overline{\nabla}_{X^L} \overline{V} = -F^L X^L. \tag{2.4}$$

Now we consider two Riemannian submersions $\widetilde{\pi}: \widetilde{M} \longrightarrow M'$ and $\pi: \overline{M} \longrightarrow M$ with 1-dimensional fibres and suppose that $\overline{M}$ is a hypersurface of $\widetilde{M}$ which respects the submersion $\widetilde{\pi}$. That is, suppose that there are immersions $\widetilde{i}: \overline{M} \longrightarrow \widetilde{M}$ and $i: M \longrightarrow M'$ such that the diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\ \widetilde{i}\ } & \widetilde{M} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}} \\ M & \xrightarrow[\ i\ ]{} & M' \end{array}$$

commutes and the immersion $\widetilde{i}$ is a diffeomorphism on the fibres. The commutativity implies that for the unit vertical vector field $\overline{V}$ of $\overline{M}$, $\widetilde{i}(\overline{V})$ is also the unit vertical vector field of $\widetilde{M}$ and that for any tangent vector field X to M, $i(X)^L = \widetilde{i}(X^L)$. Furthermore, for a field of unit normal vector N to M defined in a neighborhood of $x \in M$, the lift N^L is a field of unit normal vectors to $\overline{M}$ defined in a tubular neighborhood of $\bar{x}$, where $\bar{x}$ is an arbitrary point on a fibre over x.

We denote by $\overline{D}$, $\overline{\nabla}$, D and ∇ the Riemannian connections of $\widetilde{M}$, $\overline{M}$, M' and M respectively. By means of (1.1), (2.3) and (2.4), we have

$$\overline{D}_{\tilde{\imath}(X^L)} \tilde{\imath}(Y^L) = \tilde{\imath}(\overline{\nabla}_{X^L} Y^L) + \bar{g}(\overline{H}X^L, Y^L)N^L$$
$$= \tilde{\imath}((\nabla_X Y)^L + \bar{g}(F^L X^L, Y^L)\overline{V}) + \bar{g}(\overline{H}X^L, Y^L)N^L,$$

$$\overline{D}_{\tilde{\imath}(X^L)} \tilde{\imath}(\overline{V}) = \tilde{\imath}(\overline{\nabla}_{X^L} \overline{V}) + g(\overline{H\overline{V}}, X^L)N^L.$$

Using the above two equations and Gauss equation (1.1) and comparing the vertical parts and horizontal parts, we have

$$\bar{g}(\overline{H}X^L, Y^L) = g(HX, Y)^L, \tag{2.5}$$

$$('Fi(X))^L = \tilde{\imath}(FX)^L - \bar{g}(\overline{H\overline{V}}, X^L)N^L, \tag{2.6}$$

where $'F$ is the fundamental tensor of the submersion $\tilde{\pi}$. Thus the transforms $'Fi(X)$ and $'FN$ of $i(X)$ and N by $'F$ can be written in the form:

$$'Fi(X) = i(FX) + u(X)N, \tag{2.7}$$

$$'FN = -i(U), \tag{2.8}$$

$u(X) = g(U, X)$. Moreover the following identities are known [1].

$$\bar{g}(\overline{H\overline{V}}, X^L) = -g(U, X)^L, \tag{2.9}$$

$$\bar{g}(\overline{H\overline{V}}, \overline{V}) = 0, \tag{2.10}$$

$$\text{trace } \overline{H} = (\text{trace } H)^L. \tag{2.11}$$

LEMMA 2.1. *If the second fundamental tensor $\overline{H}$ of the hypersurface $\overline{M}$ is parallel, the second fundamental tensor H of M and the fundamental tensor F of the submersion π commutes.*

PROOF. Differentiating (2.5) covariantly in the direction of $\overline{V}$ and making use of the fact that $g(HX, Y) \circ \pi$ is invariant along the fibre, we get

$$\overline{V}(g(HX, Y) \circ \pi) = \overline{V}(\bar{g}(\overline{H}X^L, Y^L)) = \bar{g}(\overline{H}\overline{\nabla}_{\overline{V}} X^L, Y^L) + \bar{g}(\overline{H}X^L, \overline{\nabla}_{\overline{V}} Y^L)$$
$$= -\bar{g}(\overline{H}F^L X^L, Y^L) - \bar{g}(\overline{H}X^L, F^L Y^L)$$
$$= -g(HFX, Y)^L + g(FHX, Y)^L = 0,$$

where we have used (2.4) and the skew-symmetric property of F. This completes the proof.

3. **Real hypersurfaces of a complex projective space.** Let S^{n+2} be an odd-dimensional unit sphere in an $(n + 3)$-dimensional Euclidean space $E^{n+3} = C^{(n+3)/2}$ and $\tilde{J}$ the natural almost complex structure on $C^{(n+3)/2}$. The image $\tilde{V} = \tilde{J}\tilde{N}$ of the outward unit normal vector $\tilde{N}$ to S^{n+2} by $\tilde{J}$ defines a tangent vector field on S^{n+2} and the integral curves of $\tilde{V}$ are great circles S^1 in S^{n+2} which are the fibres of the standard fibration $\tilde{\pi}$,

$$S^1 \longrightarrow S^{n+2} \xrightarrow{\tilde{\pi}} CP^{(n+1)/2} \tag{3.1}$$

onto complex projective space. The usual Riemannian structure on $CP^{(n+1)/2}$ is characterized by the fact that $\tilde{\pi}$ is a submersion.

Let M^n be a real hypersurface of a complex projective space $CP^{(n+1)/2}$. Then the principal circle bundle $\bar{M}^{n+1}$ over M^n is a hypersurface of S^{n+2} and tha natural immersion $\bar{M}^{n+1}$ into S^{n+2} respects the submersion $\tilde{\pi}$. Thus S^{n+2} and $CP^{(n+1)/2}$ are in the same situations as $\tilde{M}$ and M' respectively, so we continue to use the same notations as those in §2. In the sequel, we always assume that the hypersurface is connected.

In S^{n+2} we have the family of products $M_{p,q} = S^p \times S^q$, where $p + q = n + 1$. By choosing the spheres to lie in complex subspaces we get fibrations

$$S^1 \longrightarrow M_{2p+1,\, 2q+1} \longrightarrow M^c_{p,q},$$

compatible with (3.1), where $p + q = (n - 1)/2$. In the special case $p = 0$, the hypersurface is a homogeneous, positively curved manifold diffeomorphic to the sphere.

The almost complex structure J of $CP^{(n+1)/2}$ is nothing but the fundamental tensor of the submersion $\tilde{\pi}$, that is,

$$J^L\tilde{X} = -\bar{D}_{\tilde{X}}\tilde{V}, \quad \tilde{X} \in T(S^{n+2}). \tag{3.2}$$

From the discussions of §2, the transform $Ji(X)$ of $i(X)$ by J, can be put

$$Ji(X) = i(FX) + g(U, X)N \tag{3.3}$$

and we know that F, U and g define the induced almost contact metric structure on M. Hence we have, for any $X \in T(M)$,

$$F^2X = -X + g(U, X)U, \tag{3.4}$$

$$g(U, U) = 1, \tag{3.5}$$

$$FU = 0. \tag{3.6}$$

Differentiating (3.3) covariantly and making use of the fact that the almost complex structure J of $CP^{(n+1)/2}$ is covariant constant, we have easily

(3.7) $$(\nabla_Y F)X = u(X)HY - g(HX, Y)U,$$

(3.8) $$\nabla_Y U = FHY.$$

LEMMA 3.1. $\bar{g}(\overline{HV}, \overline{HV}) = 1$.

PROOF. Let $\bar{x}$ be an arbitrary point of M and $\{E_1, \cdots, E_n\}$ be an orthonormal basis at $T_{\pi(\bar{x})}(M)$. We choose an orthonormal basis $\{\bar{E}_1, \ldots, \bar{E}_{n+1}\}$ at $T_{\bar{x}}(\bar{M})$ in such a way that $\bar{E}_i = E_i^L (i = 1, 2, \ldots, n)$ and $\bar{E}_{n+1} = \bar{V}$. Then, we have

$$\bar{g}(\overline{HV}, \overline{HV}) = \sum_{\alpha=1}^{n+1} \bar{g}(\overline{HV}, \bar{E}_\alpha)\bar{g}(\overline{HV}, \bar{E}_\alpha) = \sum_{i=1}^{n} \bar{g}(\overline{HV}, E_i^L)\bar{g}(\overline{HV}, E_i^L)$$

$$= \sum_{i=1}^{n} g(U, E_i)g(U, E_i) = g(U, U) = 1,$$

because of (2.9), (2.10) and (3.5).

4. Real hypersurface satisfying a certain commutative condition. In the following we assume that a real hypersurface M^n of a complex projective space $CP^{(n+1)/2}$ satisfies the commutative condition

(4.1) $$FH = HF.$$

By virtue of Lemma 2.1 if, as a hypersurface of S^{n+2}, the principal circle bundle $\bar{M}^{n+1}$ over M^n has the parallel second fundamental tensor, then M satisfies (4.1) and $M_{p,q}^c$ is an example. In this section we discuss the converse problem, that is, we want to prove that $M_{p,q}^c$ is the only hypersurface of $CP^{(n+1)/2}$ which satisfies (4.1).

We recall the structure equations of a hypersurface of a complex projective space $CP^{(n+1)/2}$ of the maximal sectional curvature 4:

(4.2) $$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY \\ & - 2g(FX, Y)FZ + g(HY, Z)HX - g(HX, Z)HY, \end{aligned}$$

(4.3) $$(\nabla_X H)Y - (\nabla_Y H)X = g(U, X)FY - g(U, Y)FX - 2g(FX, Y)U,$$

where R denotes the curvature tensor of the hypersurface. So we have

(4.4) $$g((\nabla_X H)Y, U) - g((\nabla_Y H)X, U) = -2g(FX, Y),$$

because of (3.5) and (3.6). From (4.1) we easily see that U is an eigenvector of H, that is,

(4.5) $$HU = \alpha U, \qquad \alpha = g(HU, U).$$

Differentiating (4.5) covariantly and making use of (3.8) and (4.1), we have

$$g((\nabla_X H)Y, U) + g(H^2FX, Y) = (X\alpha)g(U, Y) + \alpha g(HFX, Y).$$

Forming a similar equation by interchanging X and Y in the last equation and using (4.4), we get

$$(4.6) \quad -2g(FX, Y) + 2g(H^2FX, Y) = (X\alpha)g(U, Y) - (Y\alpha)g(U, X) + 2\alpha g(HFX, Y).$$

In (4.6) if we replace X by U, we obtain $Y\alpha = (U\alpha)g(U, Y)$ and substituting this into (4.6) yields $FH^2X - \alpha FHX - FX = 0$. Transforming this by F and making use of (3.4), we have

$$(4.7) \qquad H^2X - \alpha HX - X + g(U, X)U = 0.$$

We prove the

LEMMA 4.1. *If a hypersurface M^n of $CP^{(n+1)/2}$ satisfies* (4.1), *the eigenvalue α is constant.*

PROOF. From the above discussions we have grad $\alpha = \beta U$, $\beta = U\alpha$. Differentiating this covariantly, we get ∇_X grad $\alpha = (X\beta)U + \beta FHX$, from which

$$(4.8) \qquad (Y\beta)g(U, X) - (X\beta)g(U, Y) = 2\beta g(FHX, Y),$$

because of the fact that $g(\nabla_X \operatorname{grad} \alpha, Y) = g(\nabla_Y \operatorname{grad} \alpha, X)$.

Replacing X by U and making use of (3.5), (3.6), we get $Y\beta = (U\beta)g(U, Y)$. Substituting this into (4.8), we get $\beta g(FHX, Y) = 0$. Now let x be a point of M^n where $\beta(x) \neq 0$. Then the last equation shows that $FH = 0$ at x. Hence, from (4.6), $FX = 0$. But F has the maximal rank; this is a contradiction. Thus we know that at every point of M^n, $\beta = 0$. Hence α is constant.

LEMMA 4.2. *If the second fundamental tensor H of the hypersurface M^n in $CP^{(n+1)/2}$ satisfies* (4.7), *the second fundamental tensor $\overline{H}$ of $\overline{M}^{n+1}$ in S^{n+2} satisfies*

$$(4.9) \qquad \overline{H}^2\overline{X} = \alpha\overline{H}\overline{X} + \overline{X},$$

for any $\overline{X} \in T(M^{n+1})$.

PROOF. Let X be a tangent vector of M^n and first compute $\overline{H}^2X^L - \alpha\overline{H}X^L - X^L$ at $\overline{x} \in \overline{M}^{n+1}$. Since any tangent vector $\overline{Y}$ of $\overline{M}^{n+1}$ can be written in the form $\overline{Y} = \overline{Y}^H + \overline{Y}^V = Y^L + \overline{g}(\overline{Y}, \overline{V})\overline{V}$, at $\overline{x}$, where Y is a tangent vector of M^n at $\pi(\overline{x})$, we have

$$(4.10) \qquad \begin{aligned} \overline{g}(\overline{H}^2X^L - \alpha\overline{H}X^L - X^L, \overline{Y}) &= \overline{g}(\overline{H}^2X^L - \alpha\overline{H}X^L - X^L, Y^L) \\ &\quad + \overline{g}(\overline{H}^2X^L - \alpha\overline{H}X^L, \overline{V})\overline{g}(\overline{Y}, \overline{V}). \end{aligned}$$

Since (4.5) implies that $g(HX, U) = \alpha g(U, X)$, it follows from (2.9) that $\bar{g}(\bar{H}(HX)^L, \bar{V}) = -\alpha g(U, X)^L$.

On the other hand, (2.5) and the relation $g(HX, Y)^L = \bar{g}((HX)^L, Y^L)$ show that

$$\bar{H}X^L = (HX)^L + \bar{g}(\bar{H}X^L, \bar{V})\bar{V} = (HX)^L - g(X, U)^L\bar{V}. \tag{4.11}$$

Hence

$$\bar{H}^2X^L = (H^2X)^L - \alpha g(X, U)^L\bar{V} - g(X, U)^L\overline{HV}. \tag{4.12}$$

Thus we have

$$\bar{H}^2X^L - \alpha\bar{H}X^L - X^L = (H^2X - \alpha HX - X)^L - g(X, U)^L\overline{HV}, \tag{4.13}$$

and consequently

$$\begin{aligned}&\bar{g}(\bar{H}^2X^L - \alpha\bar{H}X^L - X^L, \bar{Y})\\ &\qquad = g(H^2X - \alpha HX - X + g(X, U)U, Y)^L = 0,\end{aligned} \tag{4.14}$$

because of (2.10) and (4.7).

Next we consider $\bar{H}^2\bar{V} - \alpha\overline{HV} - \bar{V}$. For any $\bar{Y} \in T_{\bar{x}}(\bar{M}^{n+1})$, we get

$$\begin{aligned}\bar{g}(\bar{H}^2 V - \alpha\overline{HV} - \bar{V}, \bar{Y}) &= \bar{g}(\bar{H}^2\bar{V} - \alpha\overline{HV} - \bar{V}, Y^L + \bar{g}(\bar{V}, \bar{Y})\bar{V})\\ &= \bar{g}(\bar{H}^2\bar{V}, Y^L) - \alpha\bar{g}(\overline{HV}, Y^L),\end{aligned}$$

because of (2.10) and Lemma 3.1.

Making use of (4.12), we have

$$\bar{g}(\bar{H}^2\bar{V} - \alpha\overline{HV} - \bar{V}, \bar{Y}) = -\alpha g(U, Y)^L + \alpha g(U, Y)^L = 0. \tag{4.15}$$

Combining (4.14) and (4.15), we have (4.9). This completes the proof.

As a consequence of Lemmas 1.1, 2.1 and 4.2, we have

THEOREM 4.3. *Let M^n be a hypersurface of a complex projective space $CP^{(n+1)/2}$ and π: $\bar{M}^{n+1} \longrightarrow M^n$ the submersion which is compatible with the fibration $S^1 \longrightarrow S^{n+2} \longrightarrow CP^{(n+1)/2}$. In order that the second fundamental tensor H of M^n commute with the fundamental tensor F of the submersion π, it is necessary and sufficient that the second fundamental tensor $\bar{H}$ of $\bar{M}^{n+1}$ is parallel.*

From this theorem and theorems in Ryan's papers [5], [6], we have

THEOREM 4.4. *$M^c_{p,q}$ are the only complete hypersurfaces of a complex projective space in which the second fundamental tensor H commutes with the fundamental tensor F of the submersion π.*

Since in [3] we proved that the induced almost contact structure of a hypersurface of a Kaehlerian manifold is normal if and only if H commutes with F, we have

COROLLARY 4.5. *$M^c_{p,q}$ is the only normal almost contact hypersurface of a complex projective space.*

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