ON SOME REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

BY

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ABSTRACT. A principal circle bundle over a real hypersurface of a complex projective space $\mathbb{C}P^n$ can be regarded as a hypersurface of an odd-dimensional sphere. From this standpoint we can establish a method to translate conditions imposed on a hypersurface of $\mathbb{C}P^n$ into those imposed on a hypersurface of $S^{2n+1}$. Some fundamental relations between the second fundamental tensor of a hypersurface of $\mathbb{C}P^n$ and that of a hypersurface of $S^{2n+1}$ are given.

Introduction. As is well known a sphere $S^{2n+1}$ of dimension $2n + 1$ is a principal circle bundle over a complex projective space $\mathbb{C}P^n$ and the Riemannian structure on $\mathbb{C}P^n$ is given by the submersion $\pi: S^{2n+1} \rightarrow \mathbb{C}P^n$ [4], [7]. This suggests that fundamental properties of a submersion would be applied to the study of real submanifolds of a complex projective space. In fact, H. B. Lawson [2] has made one step in this direction. His idea is to construct a principal circle bundle $\tilde{M}^{2n}$ over a real hypersurface $M^{2n-1}$ of $\mathbb{C}P^n$ in such a way that $\tilde{M}^{2n}$ is a hypersurface of $S^{2n+1}$ and then to compare the length of the second fundamental tensors of $M^{2n-1}$ and $\tilde{M}^{2n}$. Thus we can apply theorems on hypersurfaces of $S^{2n+1}$.

In this paper, using Lawson's method, we prove a theorem which characterizes some remarkable classes of real hypersurfaces of $\mathbb{C}P^n$. First of all, in §1, we state a lemma for a hypersurface of a Riemannian manifold of constant curvature for the later use. In §2, we recall fundamental formulas of a submersion which are obtained in [4], [7] and those established between the second fundamental tensors of $M$ and $\tilde{M}$. In §3, we give some identities which are valid in a real hypersurface of $\mathbb{C}P^n$. After these preparations, we show, in §4, a geometric meaning of the commutativity of the second fundamental tensor of $M$ in $\mathbb{C}P^n$ and a fundamental tensor of the submersion $\pi: \tilde{M} \rightarrow M$. 

1. Hypersurfaces of a Riemannian manifold of constant curvature. Let $\tilde{M}$ be an $(m + 1)$-dimensional Riemannian manifold with a Riemannian metric $\tilde{G}$ and $i: \tilde{M} \rightarrow \tilde{M}$ be an isometric immersion of an $m$-dimensional differentiable
manifold $M$ into $\tilde{M}$. The Riemannian metric $\bar{g}$ of $\tilde{M}$ is naturally induced from $\bar{g}$ in such a way that $\bar{g}(\bar{X}, \bar{Y}) = \bar{g}(i(\bar{X}), i(\bar{Y}))$, where $\bar{X}, \bar{Y}$ are vector fields on $\tilde{M}$ and we denote by the same letter $i$ the differential of the immersion. For an arbitrary point $\bar{x} \in \tilde{M}$, we choose a unit normal vector and extend it to a field $\tilde{N}$. The Riemannian connections $\bar{D}$ in $\tilde{M}$ and $\bar{\nabla}$ in $\tilde{M}$ are related by the following formulas:

\begin{align}
\bar{D}i(\bar{X})i(\bar{Y}) &= i(\bar{\nabla}_X \bar{Y}) + \bar{g}(H_X, \bar{Y})\bar{N}, \\
\bar{D}i(\bar{X})\bar{N} &= -i(H_X),
\end{align}

where $H$ is the second fundamental tensor of $M$ in $\tilde{M}$.

The mean curvature $\bar{m}$ of $\tilde{M}$ in $\tilde{M}$ is defined by

\begin{equation}
\bar{m} = \text{trace } H.
\end{equation}

Let $\tilde{R}$ and $\bar{R}$ be curvature tensors of $\tilde{M}$ and of $\tilde{M}$ respectively, then we have the following Gauss and Mainardi-Codazzi equations:

\begin{align}
\bar{g}(\tilde{R}(i(\bar{X}), i(\bar{Y}))i(\bar{Z}), i(\bar{W})) &= \bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) - \bar{g}(H_{\bar{Y}}, \bar{Z})\bar{g}(H_{\bar{X}}, \bar{W}) \\
&\quad + \bar{g}(H_{\bar{X}}, \bar{Z})\bar{g}(H_{\bar{Y}}, \bar{W}), \\
\bar{g}(\tilde{R}(i(\bar{X}), i(\bar{Y}))i(\bar{Z}), \bar{N}) &= \bar{g}((\bar{\nabla}_X H)\bar{Y}, \bar{Z}) - \bar{g}((\bar{\nabla}_Y H)\bar{X}, \bar{Z}),
\end{align}

where $\bar{X}, \bar{Y}, \bar{Z}$ and $\bar{W}$ are vector fields on $\tilde{M}$.

If the ambient manifold is of constant curvature $k$, the curvature tensor $\tilde{R}$ has the form

\begin{equation}
\tilde{R}(\bar{X}, \bar{Y})\bar{Z} = k \{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\}
\end{equation}

for vector fields $\bar{X}, \bar{Y}$ and $\bar{Z}$ on $\tilde{M}$. Consequently we have

\begin{align}
\tilde{R}(\bar{X}, \bar{Y})\bar{Z} &= k \{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\} + \bar{g}(H_{\bar{Y}}, \bar{Z})H_{\bar{X}} - \bar{g}(H_{\bar{X}}, \bar{Z})H_{\bar{Y}}, \\
(\bar{\nabla}_X H)\bar{Y} &= (\bar{\nabla}_X H)\bar{X}.
\end{align}

We assume that $M$ has constant mean curvature, that is, $\text{trace } H = \text{const}$. Let $\{\bar{E}_1, \ldots, \bar{E}_m\}$ be an orthonormal basis in $T_{\bar{x}}(\bar{M})$ and extend them to vector fields in a normal neighborhood of $\bar{x}$ by parallel translation along geodesics with respect to the Riemannian connection of $\bar{M}$. Then we have $\bar{\nabla}_{\bar{E}_i} = 0$ ($i = 1, \ldots, m$) at $\bar{x}$. Since $\bar{H}$ and $\bar{\nabla}_{\bar{E}_i} H$ are both symmetric linear transformations on $T(\bar{M})$, we get, by using (1.8)

\begin{align}
\bar{g}\left(\sum_{i=1}^{m} (\bar{\nabla}_{\bar{E}_i} H)\bar{E}_i, \bar{X}\right) &= \sum_{i=1}^{m} \bar{g}(\bar{E}_i, (\bar{\nabla}_{\bar{E}_i} H)\bar{X}) = \sum_{i=1}^{m} \bar{g}(\bar{E}_i, (\bar{\nabla}_{\bar{X}} H)\bar{E}_i) \\
&= \text{trace } (\bar{\nabla}_{\bar{X}} H) = \bar{X}(\text{trace } H) = 0,
\end{align}
which implies that
\[(1.9) \quad \sum_{i=1}^{m} (\nabla_{E_i} \bar{H}) E_i = 0.\]

Thus we have
\[(1.10) \quad \sum_{i=1}^{m} (\nabla_{\bar{X}} (\nabla_{E_i} \bar{H})) E_i = 0 \quad \text{at } \bar{x}.\]

Now we prove the

**Lemma 1.1.** Let $\bar{M}$ be a hypersurface of a Riemannian manifold of constant curvature $k$. If the second fundamental tensor $\bar{H}$ satisfies for a constant $\alpha$,
\[(1.11) \quad \bar{H}^2 \bar{X} = \alpha \bar{H} \bar{X} + k \bar{X}, \quad \bar{X} \in \bar{T}(\bar{M})\]
then we have $\nabla \bar{H} = 0$.

**Proof.** Since $\bar{H}$ is a symmetric operator and (1.7), (1.8) are valid, we have
\[
(\nabla_{\bar{X}}(\nabla_{\bar{Y}} \bar{H}) - \nabla_{\bar{Y}}(\nabla_{\bar{X}} \bar{H}) - \nabla_{\{\bar{X}, \bar{Y}\}} \bar{H}) \bar{Z} = \bar{R}(\bar{X}, \bar{Y}) \bar{H} \bar{Z} - \bar{H}(\bar{R}(\bar{X}, \bar{Y}) \bar{Z})
\]
\[= k(g(\bar{Y}, \bar{H} \bar{Z})\bar{X} - g(\bar{X}, \bar{H} \bar{Z})\bar{Y}) + g(\bar{H} \bar{Y}, \bar{H} \bar{Z}) \bar{H} \bar{X} - g(\bar{H} \bar{X}, \bar{H} \bar{Z}) \bar{H} \bar{Y}
\]
\[- k(g(\bar{Y}, \bar{Z}) \bar{H} \bar{X} - g(\bar{X}, \bar{Z}) \bar{H} \bar{Y}) - g(\bar{H} \bar{Y}, \bar{Z}) \bar{H}^2 \bar{X} + g(\bar{H} \bar{X}, \bar{Z}) \bar{H}^2 \bar{Y} = 0.
\]

Let $\{\bar{E}_1, \ldots, \bar{E}_m\}$ be an orthonormal basis which is chosen as above and $\bar{X}$ be a tangent vector at $\bar{x}$. Extend $\bar{X}$ to a vector field in a normal neighborhood of $\bar{x}$ by parallel translation along geodesics, then $\nabla \bar{X} = 0$ at $\bar{x}$. In the last equation we replace $\bar{Y}$ and $\bar{Z}$ by $\bar{E}_i$ and sum over $i$. Then we have, from (1.8) and (1.10),
\[(1.12) \quad \sum_{i=1}^{m} (\nabla_{\bar{E}_i} (\nabla_{\bar{X}} \bar{H})) \bar{E}_i = \sum_{i=1}^{m} (\nabla_{\bar{E}_i} (\nabla_{\bar{E}_i} \bar{H})) \bar{X} = 0 \quad \text{at } \bar{x},\]

because from (1.11) we know that $\bar{M}$ has constant mean curvature. Furthermore (1.11) implies that trace $\bar{H}^2 = \alpha$ trace $\bar{H} + mk = \text{const}$. Differentiating this covariantly, we have
\[
\frac{1}{2} \bar{X} \bar{X} \text{trace } \bar{H}^2 = \text{trace} (\nabla_{\bar{X}}(\nabla_{\bar{X}} \bar{H})) \bar{H} + \text{trace} (\nabla_{\bar{X}} \bar{H})(\nabla_{\bar{X}} \bar{H}) = 0,
\]
from which, at $\bar{x}$,
\[
\text{trace} (\nabla_{\bar{X}} \bar{H})^2 = - \text{trace} (\nabla_{\bar{X}}(\nabla_{\bar{X}} \bar{H})) \bar{H} = - \sum_{i=1}^{m} g((\nabla_{\bar{X}}(\nabla_{\bar{X}} \bar{H})) \bar{E}_i, \bar{H} \bar{E}_i).
\]

Thus we have
\[
\bar{g}(\nabla \bar{H}, \nabla \bar{H}) = \sum_{i=1}^{m} \text{trace} (\nabla_{\bar{E}_i} \bar{H})^2 = - \sum_{i,j=1}^{m} g((\nabla_{\bar{E}_i}(\nabla_{\bar{E}_j} \bar{H})) \bar{E}_i, \bar{H} \bar{E}_j) = 0,
\]
because of (1.12). This completes the proof.
2. Submersion and immersion. Let \( \overline{M} \) and \( M \) be differentiable manifolds of dimension \( n + 1 \) and \( n \) respectively and assume that there exists a differentiable mapping \( \pi \) of \( \overline{M} \) onto \( M \) which has maximum rank, that is, each differential map \( \pi_* \) of \( \pi \) is onto. Hence, for each \( x \in M \), \( \pi^{-1}(x) \) is a 1-dimensional submanifold of \( \overline{M} \), which is called the fibre over \( x \). We suppose that every fibre is connected.

A vector field on \( \overline{M} \) is called vertical if it is always tangent to fibres, horizontal if always orthogonal to fibres; we use corresponding terminology for individual vectors. Thus \( \overline{X} \in T_x(\overline{M}) \) decomposes as \( \overline{X}^V + \overline{X}^H \), where \( \overline{X}^V \) and \( \overline{X}^H \) denote respectively vertical part and horizontal part of \( \overline{X} \).

We assume that the mapping \( \pi \) is a Riemannian submersion, that is, there are given in \( \overline{M} \) a vertical vector field \( \overline{V} \) and a Riemannian metric \( \overline{g} \) of \( \overline{M} \) satisfying the condition that \( \overline{V} \) is a unit Killing vector field with respect to the Riemannian metric \( \overline{g} \). Then a Riemannian metric \( g \) can be defined on \( M \) by

\[
g(X, Y)(x) = \overline{g}(X^L, Y^L)(\pi(x)),
\]

where \( \overline{x} \) is an arbitrary point of \( \overline{M} \) such that \( \pi(\overline{x}) = x \) and \( X^L, Y^L \) are the lifts of \( X, Y \in T_x(M) \) respectively. Hence we have

\[
g(X, Y)^L = \overline{g}(X^L, Y^L).
\]

The fundamental tensor \( F \) of the submersion \( \pi \) is a skew-symmetric tensor of type \( (1, 1) \) on \( M \) and is related to covariant differentiation \( \nabla \) and \( \nabla \) in \( \overline{M} \) and \( M \), respectively, by the following formulas:

\[
\nabla_Y X^L = (\nabla_Y X)^L + \overline{g}(F^L X^L, X^L)\overline{V} = (\nabla_Y X)^L + \overline{g}(FY, X^L)\overline{V},
\]

\[
\nabla_X Y^L = \nabla_X Y^L \overline{V} = -F^L X^L.
\]

Now we consider two Riemannian submersions \( \overline{\pi}: \overline{M} \to M' \) and \( \pi: \overline{M} \to M \)

with 1-dimensional fibres and suppose that \( \overline{M} \) is a hypersurface of \( \overline{M} \) which respects the submersion \( \overline{\pi} \). That is, suppose that there are immersions \( \overline{i}: \overline{M} \to \overline{M} \) and \( i: M \to M' \) such that the diagram

\[
\begin{array}{ccc}
\overline{M} & \xrightarrow{\overline{i}} & \overline{M} \\
\pi \downarrow & & \downarrow \overline{\pi} \\
M & \xrightarrow{i} & M'
\end{array}
\]

commutes and the immersion \( \overline{i} \) is a diffeomorphism on the fibres. The commutativity implies that for the unit vertical vector field \( \overline{V} \) of \( \overline{M} \), \( \overline{i}(\overline{V}) \) is also the unit vertical vector field of \( \overline{M} \) and that for any tangent vector field \( X \) to \( M \), \( i(X)^L = \overline{i}(X^L) \). Furthermore, for a field of unit normal vector \( N \) to \( M \) defined in a neigh-

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borhood of \( x \in M \), the lift \( N^L \) is a field of unit normal vectors to \( \bar{M} \) defined in a tubular neighborhood of \( \bar{x} \), where \( \bar{x} \) is an arbitrary point on a fibre over \( x \).

We denote by \( \bar{D}, \bar{V}, D \) and \( \nabla \) the Riemannian connections of \( \bar{M}, \bar{M}, M' \) and \( M \) respectively. By means of (1.1), (2.3) and (2.4), we have

\[
\bar{D}_{\mathbf{T}(X^L)} \bar{\nabla}(Y^L) = \bar{\nabla}(\nabla_X Y^L) + \bar{g}(\bar{H}X^L, Y^L)N^L,
\]

\[
= \bar{\nabla}((\nabla_X Y)^L + \bar{g}(F^L X^L, Y^L)\bar{V}) + \bar{g}(\bar{H}X^L, Y^L)N^L,
\]

\[
\bar{D}_{\mathbf{T}(X^L)} \bar{\nabla}(\bar{V}) = \bar{\nabla}(\nabla_X \bar{V}) + \bar{g}(\bar{H}\bar{V}, X^L)N^L.
\]

Using the above two equations and Gauss equation (1.1) and comparing the vertical parts and horizontal parts, we have

\[
\bar{g}(\bar{H}X^L, Y^L) = g(HX, Y)^L,
\]

\[
('F)(X)^L = \bar{\nabla}(FX)^L - \bar{g}(\bar{H}\bar{V}, X^L)N^L,
\]

where \('F' \) is the fundamental tensor of the submersion \( \bar{\pi} \). Thus the transforms \('F')(X) \) and \('F'N \) of \( i(X) \) and \( N \) by \('F' \) can be written in the form:

\[
('F')(X) = i(FX) + u(X)N,
\]

\[
('F'N = -i(U),
\]

\( u(X) = g(U, X) \). Moreover the following identities are known [1].

\[
\bar{g}(\bar{H}\bar{V}, X^L) = -g(U, X)^L,
\]

\[
\bar{g}(\bar{H}\bar{V}, \bar{V}) = 0,
\]

\[
\text{trace } \bar{H} = \text{(trace } H)^L.
\]

**Lemma 2.1.** If the second fundamental tensor \( H \) of the hypersurface \( M \) is parallel, the second fundamental tensor \( \bar{H} \) of \( \bar{M} \) and the fundamental tensor \( F \) of the submersion \( \pi \) commutes.

**Proof.** Differentiating (2.5) covariantly in the direction of \( \bar{V} \) and making use of the fact that \( g(HX, Y)^ \circ \pi \) is invariant along the fibre, we get

\[
\bar{V}(g(HX, Y)^ \circ \pi) = \nabla(g(\bar{H}X^L, Y^L)) = \bar{g}(\bar{H}\nabla_{\bar{V}} X^L, Y^L) + \bar{g}(\bar{H}X^L, \nabla_{\bar{V}} Y^L)
\]

\[
= -\bar{g}(\bar{H}F^L X^L, Y^L) - \bar{g}(\bar{H}X^L, F^L Y^L)
\]

\[
= -g(HFX, Y)^L + g(FHX, Y)^L = 0,
\]

where we have used (2.4) and the skew-symmetric property of \( F \). This completes the proof.
3. Real hypersurfaces of a complex projective space. Let $S^{n+2}$ be an odd-dimensional unit sphere in an $(n + 3)$-dimensional Euclidean space $E^{n+3} = C^{(n+3)/2}$ and $\mathcal{J}$ the natural almost complex structure on $C^{(n+3)/2}$. The image $\mathcal{V} = \mathcal{J}\mathcal{N}$ of the outward unit normal vector $\mathcal{N}$ to $S^{n+2}$ by $\mathcal{J}$ defines a tangent vector field on $S^{n+2}$ and the integral curves of $\mathcal{V}$ are great circles $S^1$ in $S^{n+2}$ which are the fibres of the standard fibration $\mathcal{F}$,

$$S^1 \rightarrow S^{n+2} \xrightarrow{\mathcal{F}} C\mathbb{P}(n+1)/2$$

onto complex projective space. The usual Riemannian structure on $C\mathbb{P}(n+1)/2$ is characterized by the fact that $\mathcal{F}$ is a submersion.

Let $M^n$ be a real hypersurface of a complex projective space $C\mathbb{P}(n+1)/2$. Then the principal circle bundle $\tilde{M}^{n+1}$ over $M^n$ is a hypersurface of $S^{n+2}$ and the natural immersion $\tilde{M}^{n+1}$ into $S^{n+2}$ respects the submersion $\mathcal{F}$. Thus $S^{n+2}$ and $C\mathbb{P}(n+1)/2$ are in the same situations as $\tilde{M}$ and $M'$ respectively, so we continue to use the same notations as those in §2. In the sequel, we always assume that the hypersurface is connected.

In $S^{n+2}$ we have the family of products $M_{p,q} = S^p \times S^q$, where $p + q = n + 1$. By choosing the spheres to lie in complex subspaces we get fibrations

$$S^1 \rightarrow M_{2p+1, 2q+1} \rightarrow M_{p,q},$$

compatible with (3.1), where $p + q = (n - 1)/2$. In the special case $p = 0$, the hypersurface is a homogeneous, positively curved manifold diffeomorphic to the sphere.

The almost complex structure $J$ of $C\mathbb{P}(n+1)/2$ is nothing but the fundamental tensor of the submersion $\mathcal{F}$, that is,

$$J^L\mathcal{X} = -\bar{D}_X\mathcal{V}, \quad \mathcal{X} \in \mathcal{T}(S^{n+2}).$$

From the discussions of §2, the transform $J_i(X)$ of $i(X)$ by $J$, can be put

$$J_i(X) = i(FX) + g(U, X)N$$

and we know that $F$, $U$ and $g$ define the induced almost contact metric structure on $M$. Hence we have, for any $X \in T(M)$,

$$F^2 X = -X + g(U, X)U,$$

$$g(U, U) = 1,$$

$$FU = 0.$$

Differentiating (3.3) covariantly and making use of the fact that the almost complex structure $J$ of $C\mathbb{P}(n+1)/2$ is covariant constant, we have easily
(3.7) \( (\nabla_Y F)X = u(X)HY - g(HX, Y)U \),
(3.8) \( \nabla_Y U = FHY \).

**Lemma 3.1.** \( g(\overline{HV}, \overline{HV}) = 1 \).

**Proof.** Let \( \bar{x} \) be an arbitrary point of \( M \) and \( \{E_1, \ldots , E_n\} \) be an orthonormal basis at \( T_{\pi(\bar{x})}(M) \). We choose an orthonormal basis \( \{\overline{E}_1, \ldots , \overline{E}_{n+1}\} \) at \( T_{\bar{x}}(M) \) in such a way that \( \overline{E}_i = E_i^L \) (\( i = 1, 2, \ldots , n \)) and \( \overline{E}_{n+1} = \overline{V} \). Then, we have

\[
\sum_{a=1}^{n+1} \overline{g}(\overline{HV}, \overline{E}_a)\overline{g}(\overline{HV}, \overline{E}_a) = \sum_{i=1}^{n} \overline{g}(\overline{HV}, E_i^L)\overline{g}(\overline{HV}, E_i^L)
\]

because of (2.9), (2.10) and (3.5).

4. **Real hypersurface satisfying a certain commutative condition.** In the following we assume that a real hypersurface \( M^n \) of a complex projective space \( \mathbb{C}P^{(n+1)/2} \) satisfies the commutative condition

(4.1) \( FH = HF \).

By virtue of Lemma 2.1 if, as a hypersurface of \( S^{n+2} \), the principal circle bundle \( M^{n+1} \) over \( M^n \) has the parallel second fundamental tensor, then \( M \) satisfies (4.1) and \( M^{p,q} \) is an example. In this section we discuss the converse problem, that is, we want to prove that \( M^{p,q} \) is the only hypersurface of \( \mathbb{C}P^{(n+1)/2} \) which satisfies (4.1).

We recall the structure equations of a hypersurface of a complex projective space \( \mathbb{C}P^{(n+1)/2} \) of the maximal sectional curvature 4:

(4.2) \( R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY \)

(4.3) \( (\nabla_Y^H)Y = (\nabla_Y X)Y = g(U, X)FY - g(U, Y)FX - 2g(FX, Y)U \),

where \( R \) denotes the curvature tensor of the hypersurface. So we have

(4.4) \( g((\nabla_X^H)Y, U) - g((\nabla_Y^H)X, U) = -2g(FX, Y) \),

because of (3.5) and (3.6). From (4.1) we easily see that \( U \) is an eigenvector of \( H \), that is,

(4.5) \( HU = \alpha U, \quad \alpha = g(HU, U) \).
Differentiating (4.5) covariantly and making use of (3.8) and (4.1), we have
\[ g((\nabla_X H)Y, U) + g(H^2FX, Y) = (X\alpha)g(U, Y) + \alpha g(HFX, Y). \]

Forming a similar equation by interchanging X and Y in the last equation and using (4.4), we get
\[ (4.6) \quad -2g(FX, Y) + 2g(H^2FX, Y) = (X\alpha)g(U, Y) - (Y\alpha)g(U, X) + 2\alpha g(HFX, Y). \]

In (4.6) if we replace X by U, we obtain \( Y\alpha = (U\alpha)g(U, Y) \) and substituting this into (4.6) yields \( FH^2X - \alpha FHX - FX = 0 \). Transforming this by F and making use of (3.4), we have
\[ (4.7) \quad H^2X - \alpha HX - X + g(U, X)U = 0. \]

We prove the

**Lemma 4.1.** If a hypersurface \( M^n \) of \( CP^{(n+1)/2} \) satisfies (4.1), the eigenvalue \( \alpha \) is constant.

**Proof.** From the above discussions we have grad \( \alpha = \beta U, \beta = \nabla^a \alpha \). Differentiating this covariantly, we get \( \nabla_X \text{grad} \alpha = (X\beta)U + \beta FHX \), from which
\[ (4.8) \quad (Y\beta)g(U, X) - (X\beta)g(U, Y) = 2\beta g(FHX, Y), \]
because of the fact that \( g(\nabla_X \text{grad} \alpha, Y) = g(\nabla_Y \text{grad} \alpha, X) \).

Replacing X by U and making use of (3.5), (3.6), we get \( Y\beta = (U\beta)g(U, Y) \). Substituting this into (4.8), we get \( \beta g(FHX, Y) = 0 \). Now let \( x \) be a point of \( M^n \) where \( \beta(x) \neq 0 \). Then the last equation shows that \( FH = 0 \) at \( x \). Hence, from (4.6), \( FX = 0 \). But F has the maximal rank; this is a contradiction. Thus we know that at every point of \( M^n \), \( \beta = 0 \). Hence \( \alpha \) is constant.

**Lemma 4.2.** If the second fundamental tensor \( H \) of the hypersurface \( M^n \) in \( CP^{(n+1)/2} \) satisfies (4.7), the second fundamental tensor \( \bar{H} \) of \( \bar{M}^{n+1} \) in \( S^{n+2} \) satisfies
\[ (4.9) \quad \bar{H}^2\bar{X} = \alpha \bar{H}\bar{X} + \bar{X}, \]
for any \( \bar{X} \in T(\bar{M}^{n+1}) \).

**Proof.** Let \( X \) be a tangent vector of \( M^n \) and first compute \( \bar{H}^2X^L - \alpha \bar{H}X^L - X^L \) at \( \bar{x} \in \bar{M}^{n+1} \). Since any tangent vector \( \bar{Y} \) of \( \bar{M}^{n+1} \) can be written in the form \( \bar{Y} = \bar{Y}^H + \bar{Y}^V = Y^L + \bar{g}(\bar{Y}, \bar{V})\bar{V} \), at \( \bar{x} \), where \( Y \) is a tangent vector of \( M^n \) at \( \pi(\bar{x}) \), we have
\[ \bar{g}(\bar{H}^2X^L - \alpha \bar{H}X^L - X^L, \bar{Y}) = \bar{g}(\bar{H}^2X^L - \alpha \bar{H}X^L - X^L, Y^L) \]
\[ + \bar{g}(\bar{H}^2X^L - \alpha \bar{H}X^L, \bar{V})\bar{g}(\bar{Y}, \bar{V}). \]
Since (4.5) implies that \( g(HX, U) = \alpha g(U, X) \), it follows from (2.9) that

\[ \bar{g}(\bar{H}(HX)^L, \bar{V}) = -\alpha g(U, X)^L. \]

On the other hand, (2.5) and the relation \( g(HX, Y)^L = \bar{g}((HX)^L, Y^L) \) show that

\[ HX^L = (HX)^L + \bar{g}(\bar{H}X^L, \bar{V})\bar{V} = (HX)^L - g(X, U)^L\bar{V}. \]

Hence

\[ (4.11) \quad HX^L = (HX)^L + \bar{g}(\bar{H}X^L, \bar{V})\bar{V} = (HX)^L - g(X, U)^L\bar{V}. \]

Thus we have

\[ (4.12) \quad HX^L = (H^2 X)^L - \alpha g(X, U)^L\bar{V} - g(X, U)^L\bar{H}\bar{V}. \]

and consequently

\[ (4.13) \quad HX^L = (H^2 X)^L - \alpha HX^L - X^L = (H^2 X - \alpha HX - X)^L - g(X, U)^L\bar{H}\bar{V}, \]

because of (2.10) and (4.7).

Next we consider \( HX^L - \alpha HX^L - V \). For any \( \bar{V} \in T_{\bar{X}}(\bar{M}^{n+1}) \), we get

\[ \bar{g}(H^2 \bar{V} - \alpha H\bar{V} - \bar{V}, \bar{Y}) = \bar{g}(H^2 \bar{V} - \alpha H\bar{V} - \bar{V}, \bar{V}) + \bar{g}(\bar{V}, \bar{Y})\bar{V} \]

\[ = \bar{g}(H^2 \bar{V}, \bar{V}) - \alpha \bar{g}(H\bar{V}, \bar{V}^L), \]

because of (2.10) and Lemma 3.1.

Making use of (4.12), we have

\[ (4.15) \quad \bar{g}(H^2 \bar{V} - \alpha H\bar{V} - \bar{V}, \bar{Y}) = -\alpha g(U, Y)^L + \alpha g(U, Y)^L = 0. \]

Combining (4.14) and (4.15), we have (4.9). This completes the proof.

As a consequence of Lemmas 1.1, 2.1 and 4.2, we have

**Theorem 4.3.** Let \( M^n \) be a hypersurface of a complex projective space \( CP(n+1)/2 \) and \( \pi: M^{n+1} \rightarrow M^n \) the submersion which is compatible with the fibration \( S^1 \rightarrow S^{n+2} \rightarrow CP(n+1)/2 \). In order that the second fundamental tensor \( H \) of \( M^n \) commute with the fundamental tensor \( F \) of the submersion \( \pi \), it is necessary and sufficient that the second fundamental tensor \( \bar{H} \) of \( M^{n+1} \) is parallel.

From this theorem and theorems in Ryan's papers [5], [6], we have

**Theorem 4.4.** \( M^c_{p,q} \) are the only complete hypersurfaces of a complex projective space in which the second fundamental tensor \( H \) commutes with the fundamental tensor \( F \) of the submersion \( \pi \).
Since in [3] we proved that the induced almost contact structure of a hypersurface of a Kaehlerian manifold is normal if and only if $H$ commutes with $F$, we have

**Corollary 4.5.** $M^c_{p,q}$ is the only normal almost contact hypersurface of a complex projective space.

**BIBLIOGRAPHY**


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