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ON COMMUTATORS OF SINGULAR INTEGRALS AND BILINEAR SINGULAR INTEGRALS

BY

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ABSTRACT. L^p estimates for multilinear singular integrals generalizing Calderón's commutator integral are obtained. The methods introduced involve Fourier and Mellin analysis.

1. In this paper we introduce new methods to obtain estimates for commutators of singular integrals as well as other related operators.

Let $A(f) = A(x)f(x)$ and $H(f) = pv\int f(t)\,dt/(x-t)$. It has been shown by A. P. Calderón [3] that

$$\|[A, dH/dx]\, f\|_r \leqslant C\|dA/dx\|_{p_1}\|f\|_{p_2}, \tag{1.1}$$

where $[A, H] = A \cdot H - H \cdot A$, $1/r = 1/p_1 + 1/p_2$ and $1 < p_1 \leqslant \infty$, $1 < p_2 < \infty$, $1 < r < \infty$.

Calderón's theorem was proved using a characterization of the Hardy space $H^1(R)$ in terms of the Lusin area function. It is the special form of the kernel which permits analysis using complex variables. It will be shown here that these L^p estimates (as well as others) can also be obtained by making use of the Fourier transform followed by the Mellin transform.

These ideas are applied to obtain estimates of the following type

$$\|[[A, d^2H/dx^2], B]\, f\|_r \leqslant C\|dA/dx\|_{p_1}\|dB/dx\|_{p_2}\|f\|_{p_3}, \tag{1.2}$$

where $1/r = 1/p_1 + 1/p_2 + 1/p_3$, $1 < p_1 \leqslant \infty$, $1 < p_2 \leqslant \infty$, $1 < p_3 < \infty$.

Actually, results improving both (1.1) (1.2) can be obtained. In fact let

$$C(a, b, f)(x) = pv\int_{-\infty}^{\infty} \frac{[A(x) - A(y)]\,[B(x) - B(y)]}{(x-y)^3}\, f(y)\,dy,$$

where $a = dA/dx$, $b = dB/dx$. Then for $\infty > p > 0$ we have

$$\int |C(a, b, f)(x)|^p\,dx \leqslant C_p \int (a^*(x)b^*(x)f^*(x))^p\,dx, \tag{1.3}$$

where a^* is the Hardy-Littlewood maximal function of a.

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We should point out that estimates of type (1.1) are reducible to results involving bilinear singular integrals. Let $S(f, g)$ be a bilinear operator commuting with simultaneous translations and simultaneous dilations of both functions f, g in $L^{p_1} \times L^{p_2}$. More specifically, assume that

$$S(g, f)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(t-x, t-y) f(x) g(y)\, dx\, dy,$$

where k is an odd Calderón-Zygmund kernel (i.e., k is homogeneous of degree -2 and the restriction of k to the unit circle is of bounded variation), then one obtains the result

$$\|S(g, f)\|_{L^r} \leqslant C\|f\|_{p_1}\|g\|_{p_2}$$

for $1/r = 1/p_1 + 1/p_2$, $1 < r < \infty$, $1 < p_1 < \infty$, $1 < p_2 < \infty$. Calderón's estimate (1.1) is obtained by using

$$(1.4) \qquad k(x, y) = \frac{e(x) - e(y)}{(x-y)^2} \quad \text{where } e(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

The estimates (1.2) are obtained by using a similar analysis on a kernel related to the double commutator occurring in that expression.

The paper is split into two main parts. The first deals with some basic estimates obtained by Fourier transforms for special L^p-spaces. The second part extends these results to the full range of spaces by means of real variable techniques. There the main idea is to prove the so-called "good λ distribution function inequalities" (4.11), which permit this extrapolation.

To conclude we would like to thank Guido Weiss with whom we had many helpful discussions and Bogdan Baishansky with whom some of the real variable ideas were developed.

2. We illustrate the ideas developed later by examining first the special case of the commutator singular integral of Calderón.

$$C(a, f)(x) = \int \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy, \qquad \text{where } A' = a.$$

We want to estimate

$$I = \int C(a, f)(x) g(x)\, dx$$
$$= \iiint f(y) g(x) \frac{a(x + t(y-x))}{(x-y)} \chi(t)\, dt\, dx\, dy,$$

where χ is the characteristic function of $[0, 1]$ and a, f, g are real-valued test functions. We now write

$$a(x+t(y-x)) = \int e^{i(xu+t(y-x)u)} a^\wedge(u)\,du$$

which gives us

$$\begin{aligned} I &= \iiint f(y)g(x)\frac{e^{ixu}}{x-y} a^\wedge(u)\chi^\wedge(u(x-y))\,dx\,dy\,du \\ &= \int\left[\iiint f(x-s)g(x)e^{ixu}\chi^\wedge(us)\frac{ds}{s}\,dx\right] a^\wedge(u)\,du \\ &= \int a^\wedge(u)\left[\int f^{\bar\wedge}(v)g^{\bar\wedge}(u-v)\left(\int e^{isv}\chi^\wedge(us)\frac{ds}{s}\right)dv\right]du \\ &= \int\left(\int f^{\bar\wedge}(v)g^{\bar\wedge}(u-v)\operatorname{sgn} u\,\Phi\left(\frac{v}{u}\right)dv\right)a^\wedge(u)\,du \end{aligned}$$

where

$$\text{(2.1)}\qquad \Phi(t) = C\begin{Bmatrix} -1, & t<0, \\ 2t-1, & 0<t<1, \\ 1, & t>1, \end{Bmatrix} = C\left[2\int_{-\infty}^{t}\chi(s)\,ds - 1\right].$$

Thus we are led to study the function $F(x)$ defined by

$$F^\wedge(u) = \operatorname{sgn} u \int_{-\infty}^{\infty} f^\wedge(v)g^\wedge(u-v)\Phi\left(\frac{v}{u}\right)dv.$$

We write $f = f_+ + f_-$ and $g = g_+ + g_-$, where

$$f_+^\wedge(v) = \begin{cases} f^\wedge(v), & v>0, \\ 0, & v<0, \end{cases}$$

and denote the corresponding integrals by $F_{++}^\wedge, F_{+-}^\wedge, F_{-+}^\wedge$ and $F_{--}^\wedge$.

$F_{+-}^\wedge$ and $F_{-+}^\wedge$ are essentially trivial to study, since then we have either $v>0$, $u<v$ and $\operatorname{sgn} u\Phi(v/u) = 1$ or $v<0$, $u>v$ and $\operatorname{sgn} u\Phi(v/u) = -1$. Thus,

$$F_{+-}(x) = f_+(x)g_-(x) \quad\text{and}\quad F_{-+}(x) = -f_-(x)g_+(x).$$

For F_{++} we have $u>v>0$ and

$$F_{++}^\wedge(u) = 2\int_0^u f_+^\wedge(v)g_+^\wedge(u-v)\frac{v}{u}\,dv - (f_+(x)g_+(x))^\wedge(u);$$

because

$$\frac{v}{u} = C\int_{-\infty}^{\infty}\left(\frac{v}{u}\right)^{i\gamma}\frac{d\gamma}{1+\gamma^2} \quad\text{for } 0<v<u.$$

The first term then becomes

$$\text{(2.2)}\quad \int_{-\infty}^{\infty}\frac{d\gamma}{1+\gamma^2}u^{-i\gamma}\int_{-\infty}^{\infty} v^{i\gamma}f_+^\wedge(v)g_+^\wedge(u-v)\,dv = \int_{-\infty}^{\infty}\frac{M_{-\gamma}(g_+M_\gamma(f_+))}{1+\gamma^2}\,d\gamma$$

where

$$M_\gamma(f)(x) = (u^{i\gamma} f^\wedge(u))^\vee(x).$$

We now claim that for all $\epsilon > 0$ and $1 < p < \infty$,

$$\|M_\gamma(f)\|_p \leq C_\epsilon(1 + |\gamma|^{|1/2-1/p|+\epsilon})\|f\|_p,$$

the result being valid for $p = 1$ if f is in $H^1(\mathbf{R})$ (see Lemma (2.4)). If we now take $f \in L^p$, $g \in L^q$ so that $1/r = 1/p + 1/q \leq 1$, $\infty > p > 1$, $\infty > q > 1$, we obtain

$$\|M_{-\gamma}(g_+ M_\gamma(f_+))\|_{H^r} \leq C(1 + |\gamma|^{|1/2-1/r|+|1/2-1/p|+\epsilon})\|g\|_q\|f\|_p.$$

Thus, applying Minkowski's integral inequality in (2.2) we get

$$\|F_{++}(x)\|_{L^r} \leq C\|g\|_q\|f\|_p.$$

F_{--} can obviously be treated by the same method; taking $a \in L^{r'}$ we obtain the following inequality of Calderón:

$$\|C(a, f)\|_{q'} \leq C\|a\|_{r'}\|f\|_p. \tag{2.3}$$

Lemma (2.4). *Let*

$$M_\gamma(f) = (|t|^{i\gamma} f^\wedge(t))^\vee;$$

then, for $1 < p < \infty$,

$$\|M_\gamma(f)\|_{L^p} \leq C_p(1 + |\gamma|^{|1/p-1/2|}\ln(1 + |\gamma|))\|f\|_p;$$

for $p = 1$ *the result remains valid provided* $f \in H^1$.

Proof. We have $M_\gamma(f) = k_\gamma * f$, where

$$k_\gamma(x) = \frac{\Gamma((1 + i\gamma)/2)}{\Gamma(-i\gamma/2)\pi^{i\gamma+1/2}} \frac{1}{|x|^{1+i\gamma}} = C_\gamma \frac{1}{|x|^{1+i\gamma}}.$$

Moreover, $C_\gamma = O(\sqrt{|\gamma|})$ as $|\gamma| \to \infty$. It is clear that

$$\int_{|x|>2|y|} \left| \frac{1}{|x-y|^{1+i\gamma}} - \frac{1}{|x|^{1+i\gamma}} \right| dx = O(\ln|\gamma|) \quad \text{as } |\gamma| \to \infty.$$

Now using results of Fefferman and Stein [8] we obtain

$$\|M_\gamma(f)\|_{H^1} \leq O(\sqrt{|\gamma|}\log(|\gamma|))\,\|f\|_{H^1}, \qquad |\gamma| \to \infty.$$

The result for L^p follows by convexity and interpolation. We should note that the estimate

$$\|M_\gamma(f)\|_p \leq O(1 + |\gamma|^{|1/p-1/2|+\epsilon})\|f\|_p, \qquad 1 < p < \infty,$$

can be obtained directly from the Calderón-Zygmund theory. It can also be obtained from the multiplier theorem stating that $(m(t)f^\wedge)^\vee$ is a bounded operator on L^p provided $m(t) \in L^\infty$,

$$|m(t+h) - m(t)| \leqslant |h/t|^\alpha \quad \text{for } 2|h| < |t| \quad \text{and} \quad \alpha > |1/p - 1/2|$$

and the observation that

$$\big|\,|t+h|^{i\gamma} - |t|^{i\gamma}\big| \leqslant (1 + |\gamma|^\alpha)\,|h/t|^\alpha$$

for $0 < \alpha < 1$; see [7].

The previous result (2.3) is a particular case of the study of bilinear singular integrals commuting with translations and dilations. In fact, we have

THEOREM I. *Let*

$$S(f, g)(t) = \iint k(x-t, y-t) f(x) g(y)\, dx\, dy,$$

where $k(x, y)$ is odd, homogeneous of degree -2 and the restriction $\Omega(\theta)$ of k to the unit circle is of bounded variation; then

$$\|S(f, g)(t)\|_r \leqslant C\|f\|_p \|g\|_q$$

for $\infty > p > 1$, $\infty > q > 1$, $\infty > r > 1$ and $1/r = 1/p + 1/q$; moreover S is in weak L^1 for $r = 1$.

REMARKS. 1°. The result remains valid for $r = 1$ if $k(x, y) = 0$ whenever $xy > 0$, which is exactly the case for Calderón's operator where (1.4) holds.

2°. The theorem can be read as a restriction theorem of the singular integral transformation in $\mathbf{R}^2$ to the line (t, t); it follows immediately that one can restrict to any line through 0 which is not horizontal or vertical. The restrictions to the coordinate axis are possible if we add the assumption that $\int_0^{\pi/2} \Omega(\theta)\, d\theta = \int_{\pi/2}^{\pi} \Omega(\theta)\, d\theta = 0$ where $\Omega(\theta) = k(\cos\theta, \sin\theta)$.

3°. In view of the preceding remark it is natural to conjecture that convolution with k in $\mathbf{R}^2$ is a bounded operator on the tensor product

$$L^p \otimes L^q = \left\{F(x, y)\colon F(x, y) = \sum f_i(x) g_i(y) \text{ with } \sum \|f_i\|_p \|g_i\|_q < \infty\right\}.$$

This, however, is false even for Calderón's kernel. It is quite easy to obtain sufficient conditions on k for boundedness on $L^p \otimes L^q$.

PROOF OF THEOREM 1. As before, we calculate the Fourier transform of $h(t) = S(f, g)(t)$ by observing that

$$\begin{aligned} h(t) &= \iint k(x-t, y-t) f(x) g(y)\, dx\, dy \\ &= \iint \overline{k^\wedge(u, v)} e^{it(u+v)} f^\wedge(u) g^\wedge(v)\, du\, dv \\ &= \int e^{its} \left(\int \overline{k^\wedge(u, s-u)} f^\wedge(u) g^\wedge(s-u)\, du \right) ds. \end{aligned}$$

Thus,

$$h^\wedge(s) = \int \overline{k^\wedge(u, s-u)} f^\wedge(u) g^\wedge(s-u)\, du.$$

We know from the Calderón and Zygmund theory [9] that

$$k^\wedge(\cos\theta, \sin\theta) = C\int_0^{2\pi} \Omega(\theta - \phi)\operatorname{sgn}(\cos\phi)\, d\phi = \omega(\theta);$$

thus $\omega'(\theta)$ is of bounded variation.

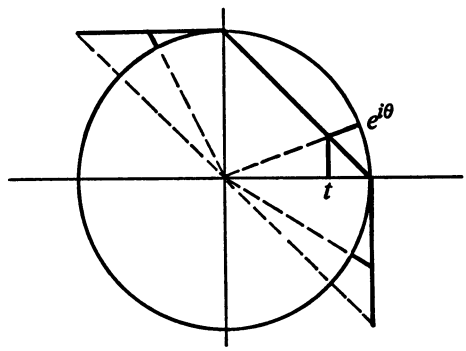

FIGURE 1

As before, put $f = f_+ + f_-$, $g = g_+ + g_-$. We first consider

$$h^\wedge_{++}(s) = \int_0^s \overline{k^\wedge(u, s-u)}\, f^\wedge_+(u) g^\wedge_+(s-u)\, du$$

$$= \int_0^s \overline{k^\wedge\left(\frac{u}{s}, 1 - \frac{u}{s}\right)}\, f^\wedge_+(u) g^\wedge_+(s-u).$$

We let

$$\phi(t) = \overline{k^\wedge(t, 1-t)} - \overline{k^\wedge(0, 1)}, \qquad 0 \leq t \leq 1.$$

As can easily be seen from Figure 1, the transformation linking t to θ is in C^∞, and thus $\phi'(t)$ is of bounded variation on $(0, 1)$; thus, the function $\phi(e^{-s})$ can be extended to $(-\infty, \infty)$ in such a way as to have a Fourier transform $m(\gamma)$ bounded by $C/(1 + \gamma^2)$.

We thus obtain, for $0 \leq t \leq 1$,

$$\phi(t) = \int_{-\infty}^{\infty} t^{+i\gamma} m(\gamma)\, d\gamma;$$

hence,

$$h^\wedge_{++}(s) = \int_{-\infty}^{\infty} m(\gamma) s^{-i\gamma} \int f^\wedge_+(u) u^{+i\gamma} g^\wedge_+(s-u)\, du$$

$$+ \overline{k^\wedge(0, 1)}\, (f_+(x) g_+(x))^\wedge(s)$$

which is estimated as above.

We now study $h^{\wedge}_{+-}(s)$. Here we have to distinguish $s > 0$, $s < 0$, and proceed as above.

If $s > 0$ we must have $u > 0$, $s > 0$, $s - u < 0$; thus, $0 < (u - s)/u < 1$. We now introduce

$$\phi_1(t) = \overline{k^{\wedge}(1, -t)} - \overline{k^{\wedge}(1, 0)}.$$

It is again clear from Figure 1 that $\phi_1'(t)$ is of bounded variation. As before we write, for $0 < t < 1$, $\phi_1(t) = \int_{-\infty}^{\infty} t^{i\gamma} m_1(\gamma)\, d\gamma$ for some m_1 satisfying $|m_1(\gamma)| \leqslant C/(1 + \gamma^2)$. Thus,

$$e(s)h^{\wedge}_{+-}(s) = \left\{ \int m_1(\gamma) \left(\int u^{-i\gamma} f^{\wedge}_{+}(u) |s - u|^{i\gamma} g^{\wedge}(s - u)\, du \right) d\gamma + \overline{k^{\wedge}}(1, 0)(f_+ g_-)^{\wedge}(s) \right\} e(s).$$

If $s < 0$ we must have $s < 0 < u$, thus $u < u/(u - s) < 1$. Introduce

$$\phi_2(t) = \overline{k^{\wedge}(-t, 1)} - \overline{k^{\wedge}(0, 1)}.$$

As above, we represent $\phi_2(t) = \int_{-\infty}^{\infty} t^{i\gamma} m_2(\gamma)\, d\gamma$, with $|m_2(\gamma)| \leqslant C/(1 + \gamma^2)$ and $0 \leqslant t \leqslant 1$. Hence,

$$e(-s)h^{\wedge}_{+-}(s) = \left\{ \int m_2(\gamma) \left(\int u^{i\gamma} f^{\wedge}_{+}(u)(s - u)^{-i\gamma} g^{\wedge}_{-}(s - u)\, du \right) d\gamma + \overline{k^{\wedge}}(0, 1)(f_+ g_-)^{\wedge}(s) \right\} e(-s).$$

Estimating as before we deduce that the functions in curly brackets are Fourier transforms of functions in L^r ($r \geqslant 1$) if $1/p + 1/q = 1/r$, $1 < p < \infty$, $1 < q < \infty$. Using the boundedness on L^r of the operators $f_{\pm}(x) = (e(\pm s) f^{\wedge})^{\vee}(x)$ we deduce that h_{+-} is in L^r if $r > 1$ and in weak L^1 for $r = 1$.

3. As an application of the ideas illustrated above we would like to study the double commutator integral

$$C(a, b, f)(x) = \int_{-\infty}^{\infty} \frac{[A(x) - A(y)]\,[B(x) - B(y)]}{(x - y)^3} f(y)\, dy$$

where $a = A'$, $b = B'$.

THEOREM II. *Let ϕ denote an absolutely continuous nonnegative increasing function satisfying $\phi(2x) \leqslant C\phi(x)$ and let*

$$f^*(x) = \sup_{\epsilon} \frac{1}{\epsilon} \int_{|t| < \epsilon} |f(x + t)|\, dt;$$

we then have the following inequality:

$$\int \phi(C(a, b, f)(x))\,dx \leqslant C_\phi \int \phi(a^*(x)b^*(x)f^*(x))\,dx.$$

In particular, if $a \in L^\infty$, $b \in L^\infty$ *and* $\phi(t) = t^p$ *then*

$$\|C(a, b, f)\|_p \leqslant C_p \|f\|_p, \qquad 1 < p < \infty.$$

The proof is split into two main parts. In the first we use the previous methods to show that

$$\|C(a, b, f)\|_{4/3} \leqslant C\|a\|_4 \|b\|_4 \|f\|_4.$$

(The Fourier method yields a wider range of indices (see Lemma (3.2)) but does not include the case $a \in L^\infty$, $b \in L^\infty$, $f \in L^p$ for which we need real variable ideas.) In the second part we use this estimate to analyse C further by real variable methods, proving "good λ" inequalities.

PART I. *The basic estimate.* As before we study

$$\begin{aligned} I &= \int C(a, b, f)(y)g(y)\,dy \\ &= \iiiint f(x)g(y)(x-y)^{-1}\chi(s)\chi(t)a(x + t(y-x))b(x + s(y-x))\,dx\,dy\,ds\,dt \\ &= \iint \left(\iint f(x)g(x-s)e^{ix(u+v)}\, s^{-1}\chi\hat{}(us)\chi\hat{}(vs)\,ds\,dx\right) \overline{a\hat{}(u)}\; \overline{b\hat{}(v)}\,du\,dv \\ &= \iint \left[\int f\hat{}(u+v-w)g\hat{}(w)k(w, u, v)\,dw\right] \overline{a\hat{}(u)}\; \overline{b\hat{}(v)}\,du\,dv, \end{aligned}$$

where

$$k(w, u, v) = \int_{-\infty}^{\infty} e^{iws}\chi\hat{}(us)\chi\hat{}(vs)\frac{ds}{s}. \tag{3.1}$$

We first study k by observing that

$$\begin{aligned} k_0(w, u) = k(w, u, 0) &= \operatorname{sgn} u\,\Phi\left(\frac{w}{u}\right) \\ &= \int_{-\infty}^{\infty} e^{iws}\chi\hat{}(us)\frac{ds}{s} \quad \text{(see (2.1))}. \end{aligned}$$

Thus

$$k(w, u, v) = \frac{1}{v}\int \chi\left(\frac{s}{v}\right)k_0(w-s, u)\,ds = \operatorname{sgn} u\,\frac{u}{v}\left[\Omega\left(\frac{w}{u}\right) - \Omega\left(\frac{w-v}{u}\right)\right]$$

where

$$\Omega(t) = \begin{cases} -t, & t < 0, \\ t^2 - t, & 0 < t < 1, \\ t - 1, & t > 1. \end{cases}$$

It is also clear from (3.1) that $k(w, u, v)$ is homogeneous of degree 0, odd in (w, u, v), symmetric in u, v and

$$k(w, u, v) = k(w + u + v, -u, -v)$$

(we have used the fact that $\chi^\wedge(us) = (e^{ius} - 1)/us$).

We will study I by splitting a, b, f, g into a_+, a_-, f_+, f_-, etc. and denote the corresponding integrals by I_{++--}, etc. As in Calderón's case the hardest case involves I_{++++}, I_{----} while the cases where we have some mixture of signs exhibit some degeneracy.

The case of I_{++++}. We write

$$k_{++}(w, u, v) = k(w, u, v) - k(w, u + v, 0) + k(w, u + v, 0).$$

$$k(w, u, v) = \frac{1}{v}\int_0^v k_0(w - s, u)\, ds$$

equals $-\operatorname{sgn} u$ for $w < 0$ and is $\operatorname{sgn} u$ for $w > u + v$; thus, $k(w, u, v) - k_0(w, u + v)$ is supported in $[0, u + v]$. We now define

$$K(x, y) = k(x, y, 1 - y) - k_0(x, 1)$$

where

$$x = \frac{w}{u + v}, \quad y = \frac{u}{u + v}, \quad 1 - y = \frac{v}{u + v}, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1.$$

We claim

LEMMA (3.2).

$$K(x, y) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{i\gamma_1} y^{i\gamma_2} m(\gamma_1, \gamma_2)\, d\gamma_1\, d\gamma_2,$$

where

$$\iint |m(\gamma_1, \gamma_2)|(1 + |\gamma_1|^\alpha)(1 + |\gamma_2|^\beta)(1 + |\gamma_1 + \gamma_2|^\gamma) < \infty \tag{3.3}$$

as long as $0 < \alpha < ½, 0 < \beta < ½, 0 < \gamma < ½, 0 < \alpha + \beta + \gamma < 1$; *moreover,*

$$m(\gamma_1, \gamma_2) = \frac{2}{i\gamma_1(1 + i\gamma_1)(2 + i\gamma_2)} \cdot \left[\frac{\Gamma(i\gamma_1 + 2)\Gamma(i\gamma_2 - 1)}{\Gamma(i(\gamma_1 + \gamma_2) + 1)} + \frac{\Gamma'}{\Gamma}(i\gamma_2 - 1) - \frac{\Gamma'}{\Gamma}(i(\gamma_1 + \gamma_2) + 1)\right] + \frac{2}{i\gamma_1(1 + i\gamma_1)i\gamma_2}.$$

The Mellin transform of K can be computed directly. For our purpose, however, it is sufficient to obtain (3.3) for $\alpha < ½, \beta < ½$ and $\gamma = 0$. This can be achieved by showing that the function

$$h(s, t) = \begin{cases} 0, & t < 0 \text{ or } s < 0, \\ K(e^{-s}, e^{-t}), & s > 0, t > 0, \end{cases}$$

has a Fourier transform $m(\gamma_1, \gamma_2)$ which is integrable with respect to $(1 + |\gamma_1|^\alpha)\cdot(1 + |\gamma_2|^\beta)\,d\gamma_1\,d\gamma_2$. This follows from the Hausdorff-Young inequality whenever h, $\partial h/\partial t$, $\partial h/\partial s$, $\partial^2 h/\partial s\partial t$ all belong to $L^{2-\epsilon}(\mathbf{R}^2)$, $\epsilon > 0$. This can readily be checked using the following expression for K:

$$K(x, y) = \begin{cases} x^2/y(1-y) - 2x, & 0 \leqslant x \leqslant \inf(y, 1-y), \\ \inf\left(\dfrac{y}{1-y}, \dfrac{1-y}{y}\right)(2x-1), & \inf(y, 1-y) \leqslant x \leqslant \sup(y, 1-y), \\ (1-x)\left(2 - \dfrac{1-x}{y(1-y)}\right), & \sup(y, 1-y) \leqslant x \leqslant 1. \end{cases}$$

We now consider $f, g, a, b \in L^4$ and we estimate

$$I_{++++} = \iint \overline{\hat{a}_+(u)}\,\overline{\hat{b}_+(v)}\left(\int \hat{f}_+(u+v-w)\hat{g}_+(w)k(w, u, v)\,dw\right)du\,dv.$$

In view of the fact that $0 < w < u + v$, we can use Lemma (3.2) to represent k as

$$k(w, u, v) = \iint \left(\frac{w}{u+v}\right)^{i\gamma_1}\left(\frac{u}{u+v}\right)^{i\gamma_2} m(\gamma_1\gamma_2)\,d\gamma_1\,d\gamma_2$$

$$+ C\int \left(\frac{w}{u+v}\right)^{i\gamma}\frac{d\gamma}{1+\gamma^2} - 1$$

(where the second term comes from $k_0(w, u+v) = 2w/(u+v) - 1$). Thus

$$I_{++++} = \iint m(\gamma_1, \gamma_2)\left[\iint \overline{\hat{a}_+(u)u^{i\gamma_2}}\,\overline{\hat{b}_+(v)}\,(u+v)^{-i(\gamma_1+\gamma_2)}\right.$$

$$\cdot \int \hat{f}_+(u+v-w)\hat{g}_+(w)w^{i\gamma_1}\,dw$$

$$+ C\int \frac{d\gamma}{1+\gamma^2}\iint \overline{\hat{a}_+(u)}\,\overline{\hat{b}_+(v)}\,(u+v)^{-i\gamma}$$

$$\cdot\int \hat{f}_+(u+v-w)\hat{g}_+(w)w^{i\gamma}\,dw\,du\,dv$$

$$- \iint \overline{\hat{a}_+(u)}\,\overline{\hat{b}_+(v)}\left(\int \hat{f}_+(u+v-w)\hat{g}_+(w)\,dw\right)du\,dv.$$

Observe first that the last integral equals

$$\int f_+(x) g_+(x) \bar{b}_+(x) \bar{a}_+(x)\, dx$$

while the preceding one is

$$\int \frac{d\gamma}{1+\gamma^2} \int M_{-\gamma}(f_+ M_\gamma(g_+))(x) \overline{b_+(x)}\, \overline{a_+(x)}\, dx$$

and the first becomes

$$\int m(\gamma_1 \gamma_2) \left\{ \int \overline{M_{-\gamma_2}(a_+)}(x) \overline{b_+(x)} M_{-\gamma_1-\gamma_2}(f_+ M_{\gamma_1}(g))(x)\, dx \right\} d\gamma_1\, d\gamma_2.$$

Since

$$\|M_\gamma f\|_4 \leqslant C(1 + |\gamma|^{1/4+\epsilon}) \|f\|_4,$$

we obtain the estimate

$$I_{++++} \leqslant C \|a\|_4 \|b\|_4 \|f\|_4 \|g\|_4.$$

The cases. I_{+++-}: Here $u + v > w$, $w < 0$, $u > 0$, $v > 0$ so that $k(w, u, v) = -1$ in this range and the estimate is trivial. Similarly for the next two cases:

I_{++-+}: Here $u > 0$, $v > 0$, $u + v < w$, $w > 0$ so that $k(w, u, v) = +1$.

I_{++--}: $u > 0$, $v > 0$, $u + v \leqslant w \leqslant 0$ domain of integration is null so that $I_{++--} = 0$.

I_{+-++}: $u > 0$, $v < 0$, $u + v > w > 0$, $0 < w/u < 1$, $0 < -v/u < 1$ and

$$k(w, u, v) = 2\frac{w}{u} - \frac{v}{u} - 1 = C \int \left\{ 2\left(\frac{w}{u}\right)^{i\gamma} + \left(-\frac{v}{u}\right)^{i\gamma} \right\} \frac{d\gamma}{1+\gamma^2} - 1.$$

The estimate follows along the same line as before.

I_{+---}: $u > 0$, $v < 0$, $w > u + v$, $w < 0$ and $k(w, u, v) = 2w/v - u/v - 1$ is estimated as above.

I_{+--+}: $u > 0$, $v < 0$, $w > 0$, $w > u + v$,

$$k(w, u, v) = \begin{cases} 1 + (w-u)^2/uv, & w < u, \\ 1, & w > u. \end{cases}$$

This case is slightly different. We introduce $s = w - u$ and let

$$k_0(s, u, v) = \begin{cases} 1 + s^2/uv, & \max(-u, v) < s < 0, \\ 1, & 0 \leqslant s; \end{cases}$$

then

$$I_{+--+} = \iint a_+^\wedge(u) b_-^\wedge(v) \int k(w, u, v) f_-^\wedge(u + v - w) g_+^\wedge(w)\, dw\, du\, dv$$

$$= \int_0^\infty \left[\iint \overline{a_+^\wedge(u)} \overline{b_-^\wedge(v)} f_-^\wedge(v - s) g_+^\wedge(u + s)\, dv\, du\right] ds$$

$$- \int_{-\infty}^0 \left[\iint \overline{a_+^\wedge(u)} \overline{b_-^\wedge(v)} \frac{s}{-u} \frac{s}{v} f_-^\wedge(v - s) g_+^\wedge(u + s)\, du\, dv\right] ds.$$

Observing that $0 < s/-u < 1$, $0 < s/v < 1$, the second integral can be rewritten as

$$C \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{d\alpha\, d\beta}{(1 + \alpha^2)(1 + \beta)^2} \int_{s<0} |s|^{i(\alpha+\beta)} \left(\int u^{-i\alpha}\, \overline{a_+^\wedge(u)} g_+^\wedge(u + s)\, du\right)$$

$$\cdot \left(\int |v|^{-i\beta}\, \overline{b_-^\wedge(v)} f_-^\wedge(v - s)\, dv\right) ds$$

$$= \iint \frac{d\alpha\, d\beta}{(1 + \alpha^2)(1 + \beta^2)} \int G_\alpha(x) \overline{F_\beta(x)}\, dx$$

$$\leqslant \iint \frac{(1 + |\alpha|^{1/4+\epsilon})(|\beta|^{1/4+\epsilon} + 1)}{(1 + \alpha^2)(1 + \beta)^2} \|a\|_4 \|b\|_4 \|f\|_4 \|g\|_4$$

where

$$[G_\alpha(x)]^\wedge(s) = |s|^{i\alpha} \left[\int u^{-i\alpha} a_+^\wedge(u) g_+^\wedge(u + s)\, du\right] e(-s)$$

and

$$[F_\beta(x)]^\wedge(s) = |s|^{-i\beta} \left[\int |v|^{+i\beta} b_-^\wedge(v) f_-^\wedge(v - s)\, dv\right] e(-s).$$

I_{+-+-}: $u > 0$, $v < 0$, $0 > w$, $u + v > w$ and

$$k(w, u, v) = \begin{cases} -(1 + (w - v)^2/uv), & v < w, \\ -1, & w < v, \end{cases}$$

is treated as above.

Combining all the cases above we have just estimated the integral

$$\int a_+^\wedge(u) b^\wedge(v) \int k(w, u, v) f^\wedge(u + v - w) g^\wedge(w)\, dw$$

for all $b, f, g \in L^4$. The case with $a_-^\wedge(u)$ can be reduced to the case above by changing $b(x)$ into $b(-x)$, $f(x)$ into $f(-x)$, $g(x)$ into $g(-x)$ and using the fact that k is odd.

4. **Real variable methods.** Our purpose now is to extend the preceding estimate to a wider range of spaces. In particular, we need first to obtain

$$\|C(a, b, f)\|_p \leqslant C\|a\|_\infty \|b\|_\infty \|f\|_p, \qquad 1 < p < \infty.$$

The idea is to control the local behavior of the operator and use the fact that if $a \in L^\infty$, $b \in L^\infty$, they are locally in L^4. This is achieved by proving "good λ inequalities". For technical reasons we need to estimate the operator

$$C_*(a, b, f)(x) = \sup_\epsilon \left| \int_{|x-y|>\epsilon} \frac{[A(x) - A(y)]\,[B(x) - B(y)]}{(x-y)^3} f(y)\,dy \right|,$$

since it is only for this operator that the good λ inequalities can be proved. We proceed as in [1], [2] using the method of Cotlar.

We let

$$k(x, y) = \frac{(A(x) - A(y))(B(x) - B(y))}{(x-y)^3}.$$

LEMMA (4.1). *Let* $|x - x_1| < \epsilon/4$ *and* $|x - x_2| < \epsilon/4$; *then*

$$\left| \int_{|y-x_1|>\epsilon} [k(x, y) - k(x_1, y)]\, f(y)\, dy \right| \leq Cf^*(x_2)a^*(x_2)b^*(x_2),$$

where f^* *is the Hardy-Littlewood maximal operator.*

PROOF. The inequality

$$|k(x, y) - k(x_1, y)| \leq C \frac{|x - x_1|}{|y - x_1|^2} a^*(x_2)b^*(x_2)$$

can be verified directly (just use $|(A(x) - A(y))/(x-y)| \leq a^*(x)$, etc). Moreover,

$$\int_{|y-x_1|>\epsilon} \frac{|x - x_1|}{|y - x_1|^2} f(y)\, dy \leq Cf^*(x_2)$$

by a standard argument.

LEMMA (4.2). *Let* $p_1 \geq 1, p_2 \geq 1, p_3 \geq 1$ *be such that*

(4.3) $$|C(a, b, f)(x) > \lambda| \leq C(\|a\|_{p_1}\|b\|_{p_2}\|f\|_{p_3}/\lambda)^q,$$

where $1/q = 1/p_1 + 1/p_2 + 1/p_3$; *then for* $0 < \delta < q < \infty$

(4.4) $$C_*(a, b, f)(x) \leq C_1 [\Lambda_\delta(C(a, b, f))(x) + \Lambda_{p_1}(a)\Lambda_{p_2}(b)\Lambda_{p_3}(f)],$$

where $\Lambda_p(f) = (|f|^s)^{*1/s}$ *and* C_* *satisfies the same weak type inequality* (4.3).

PROOF. Let $\chi_\epsilon(t)$ be the characteristic function of $[x - \epsilon, x + \epsilon]$ and consider, for $|x_1 - x| < \epsilon/4$,

$$C_\epsilon(f)(x) + C(\chi_\epsilon f)(x_1) - C(f)(x_1) = \int_{|y-x|>\epsilon} [k(x, y) - k(x_1, y)]\, f(y)\, dy.$$

Using Lemma (4.1) we obtain

$$|C_\epsilon(f)(x)| \leq [|C(\chi_\epsilon f)(x_1)| + |C(f)(x_1)| + ca^*(x)b^*(x)f^*(x)].$$

Forming the power δ of both sides, averaging in x_1 on $(x - \epsilon/4, x + \epsilon/4)$ and, finally, taking the power $1/\delta$, we obtain

$$|C_\epsilon(f)(x)| \leqslant c[\Lambda_\delta(C(f))(x) + a^*(x)b^*(x)f^*(x) + M_\delta(f)(x)],$$

where

$$M_\delta(f)(x) = \sup_{\epsilon>0}\left(\frac{2}{\epsilon}\int_{x-\epsilon/4}^{x+\epsilon/4} |C(\chi_\epsilon f)(x_1)|^\delta\, dx_1\right)^{1/\delta}.$$

We first observe that on $(x - \epsilon/4, x + \epsilon/4)$,

$$C(\chi_\epsilon f)(x_1) = C(\chi_\epsilon a, \chi_\epsilon b, \chi_\epsilon f)(x_1).$$

Now, using Kolmogorov's inequality, we obtain, for $0 < \delta < q$,

$$\begin{aligned}\frac{2}{\epsilon}\int_{x-\epsilon/4}^{x+\epsilon/4} |C(\chi_\epsilon f)|^\delta\, dx &\leqslant C_\delta \frac{2}{\epsilon}\left(\frac{2}{\epsilon}\right)^{1-\delta/q} \|\chi_\epsilon a\|_{p_1}^\delta \|\chi_\epsilon b\|_{p_2}^\delta \|\chi_\epsilon f\|_{p_3}^\delta \\ &= c\left(\frac{1}{\epsilon}\int_{|x-t|<\epsilon} |a(t)|^{p_1}\right)^{\delta/p_1}\left(\frac{1}{\epsilon}\int_{|x-t|<\epsilon} |b(t)|^{p_2}\right)^{\delta/p_2} \\ &\quad\cdot\left(\frac{1}{\epsilon}\int_{|x-t|<\epsilon} f^{p_3}\right)^{\delta/p_3};\end{aligned}$$

thus,

$$M_\delta(f)(x) \leqslant c\Lambda_{p_1}(a)(x)\Lambda_{p_2}(b)(x)\Lambda_{p_3}(f)(x).$$

Obviously, M_δ satisfies inequality (4.3).

To see that the first term of the right-hand side of (4.4) satisfies the weak type inequality (4.3), we use the fact that for the noncentered maximal operator

$$|E_\lambda| \leqslant \frac{2}{\lambda}\int_{E_\lambda} |f|\, dx \quad \text{where } E_\lambda = \{f^*(x) > \lambda\}.$$

Thus, using Kolmogorov's inequality again, we have

$$\begin{aligned}|E_\lambda| &= |\Lambda_\delta(C(f)) > \lambda| = |(C(f))^{\delta\,*}(x) > \lambda^\delta| \\ &\leqslant \frac{2}{\lambda^\delta}\int_{E_\lambda} |C(f)|^\delta\, dx \leqslant C_\delta \frac{|E_\lambda|^{1-\delta/q}}{\lambda^\delta} \|a\|_{p_1}^\delta \|b\|_{p_2}^\delta \|f\|_{p_3}^\delta;\end{aligned}$$

simplifying and taking power $1/\delta$ we obtain the desired inequality.

LEMMA (4.5). *Under the assumption* (4.3) *of Lemma* (4.2) *we have*

$$|C_*(a, b, f)(x) > 2\lambda, \Lambda_{p_1}(a)\Lambda_{p_2}(b)\Lambda_{p_3}(f) \leqslant \gamma\lambda| \leqslant C\gamma^q |C_*(a, b, f) > \lambda|,$$

where $1/q = 1/p_1 + 1/p_2 + 1/p_3$.

PROOF. $\{C_*(a, b, f)(x) > \lambda\} = UI_i$, where the intervals $I_i = (\alpha_i, \alpha_i + \delta_i)$ are disjoint and $C_*(a, b, f)(\alpha_i) \leqslant \lambda$. It is enough to show that

$$|x \in I_i\colon C_*(a, b, f)(x) > 2\lambda, \Lambda_{p_1}(a)\Lambda_{p_2}(b)\Lambda_{p_3}(f) \leq \gamma\lambda| \leq C\gamma^q |I_i|.$$

Now, let us fix the index i and put

$$\bar{I}_i = (\alpha_i - 2\delta_i, \alpha_i + 2\delta_i);$$

we write $f = f_1 + f_2$, where $f_1 = \chi_{\bar{I}_i} f$. We can also assume the existence of $\xi \in I_i$ such that $\Lambda_{p_1}(a)\Lambda_{p_2}(b)\Lambda_{p_3}(f)(\xi) \leq \gamma\lambda$. We first observe that, for $x \in I_i$,

$$C_*(a, b, f_1)(x) = C_*(\chi_{\bar{I}_i} a, \chi_{\bar{I}_i} b, \chi_{\bar{I}_i} f)(x)$$

and that, by (4.3) of Lemma (4.2),

$$|x \in I_i\colon C_*(a, b, f_1) > \beta\lambda| \leq c(\|\chi_{\bar{I}_i} a\|_{p_1} \|\chi_{\bar{I}_i} b\|_{p_2} \|\chi_{\bar{I}_i} f\|_{p_3} / \beta\lambda)^q$$

$$\leq c|I_i|(\Lambda_{p_1}(a)(\xi)\Lambda_{p_2}(b)(\xi)\Lambda_{p_3}(f)(\xi)/\beta\lambda)^q \leq c|I_i|(\gamma/\beta)^q.$$

For the function f_2 we have

$$C_\epsilon(f_2(x)) = C_\epsilon(f_2)(x) - C_\epsilon(f_2)(\alpha_i) + C_\epsilon(f_2)(\alpha_i),$$

$$|C_\epsilon(f_2)(x) - C_\epsilon(f_2)(\alpha_i)| \leq \left| \int_{|y-\alpha_i|>\epsilon} [k(x, y) - k(\alpha_i, y)]\, f_2(y)\, dy \right|$$

$$+ \left| \int_{\alpha_i - \epsilon}^{x-\epsilon} |k(x, y)| f_2(y)\, dy \right| + \left| \int_{\alpha_i+\epsilon}^{x+\epsilon} k(x, y) f_2(y)\, dy \right|$$

$$\leq Cf^*(\xi)a^*(\xi)b^*(\xi) \leq C\gamma\lambda$$

where the 1st term is estimated by using Lemma (4.1), while for the second and third we observe that

$$|k(x, y)| \leq Ca^*(\xi)b^*(\xi)/(|I_i| + \epsilon),$$

which implies the inequality. By construction, $C_\epsilon(f_2)(\alpha_i) \leq \lambda$. Thus

$$C_*(f_2)(x) \leq \lambda + C\gamma\lambda \quad \text{for all } x \in I_i.$$

Combining this with the estimate for $C_*(f_1)$ we obtain

$$|x \in I_i\colon C_*(a, b, f) > 2\lambda, \Lambda_{p_1}(a)\Lambda_{p_2}(b)\Lambda_{p_3}(f) \leq \gamma\lambda|$$

$$\leq |x \in I_i\colon C_*(a, b, f_1) > (1 - C\gamma)\lambda| + |x \in I_i\colon C_*(a, b, f_2) > (1 + C\gamma)\lambda|$$

$$\leq C(\gamma/(1 - C\gamma))^q |I_i| + 0. \quad \text{Q.E.D.}$$

LEMMA (4.6). *For all $p > 0$ and p_1, p_2, p_3, for which* (4.3) *is true we have*

$$\int |C(a, b, f)(x)|^p \leqslant C_p \int (\Lambda_{p_1}(a)\Lambda_{p_2}(b)\Lambda_{p_3}(f))^p \, dx.$$

This follows immediately from Lemma (4.5) by a standard argument (see [6]).

COROLLARY (4.7). *If* $a \in L^\infty$, $b \in L^\infty$, $f \in L^p$, $\infty > p > 4$, *then* $\|C(a, b, f)\|_p \leqslant C\|a\|_\infty \|b\|_\infty \|f\|_p$.

In fact we know that for $p_1 = p_2 = p_3 = 4$, condition (4.3) in Lemma (4.2) is verified. Since Λ_4 is a bounded operator on L^p for $p > 4$, the conclusion follows from Lemma (4.6).

LEMMA (4.8). *If* $p_1 = p_2 = p_3 = 1$ *then* (4.3) *in Lemma* (4.2) *is satisfied*:

$$|C(a, b, f)(x) > \lambda| \leqslant C(\|a\|_1 \|b\|_1 \|f\|_1 / \lambda)^{1/3}.$$

PROOF. We use the Calderón-Zygmund decomposition of f, b, a in order to extend the range of the estimate in Corollary (4.7). We first observe that if $a \in L^\infty$, $b \in L^\infty$, then the estimate in Lemma (4.1) gives

$$|k(x, y) - k(x, y_0)| \leqslant \frac{|y - y_0|}{|x - y_0|^2} \|a\|_\infty \|b\|_\infty$$

for $|x - y_0| > 2|y - y_0|$. This permits us to see immediately by the Calderón-Zygmund method that

$$|C(a, b, f)(x) > \lambda| < \|a\|_\infty \|b\|_\infty \|f\|_1 / \lambda. \tag{4.9}$$

We now leave $a \in L^\infty$ and take $b \in L^1$, $\|b\|_1 = 1$, $\|f\|_1 = 1$ and we decompose (as in [5]) $b = b_1 + b_2$, where $b_1 \leqslant \lambda^{1/2}$, $b_2 = \Sigma b_k$ with b_k supported in I_k, having mean 0,

$$\frac{1}{|I_k|} \int |b_k| \, dx \leqslant \lambda^{1/2} \quad \text{and} \quad \sum |I_k| \leqslant C/\lambda^{1/2}.$$

We obtain from (4.9)

$$|C(a, b_1, f)(x) > \lambda| \leqslant C/\lambda^{1/2}.$$

Observe that for $x \notin 2I_k$, $B_k(x) = 0$ and $|B_k(y)| \leqslant \lambda^{1/2}|I_k|$. Thus,

$$|C(a, b_k, f)| = \left| - \int_{I_k} \frac{B_k(y)}{(x - y)^2} \frac{A(x) - A(y)}{x - y} f(y) \, dy \right|$$

$$\leqslant \lambda^{1/2} \frac{|I_k|}{(x - y_k)^2 + |I_k|^2} \int_{I_k} |f(y)| \, dy.$$

Now

$$\left\| \sum \frac{|I_k|}{(x - y_k)^2 + |I_k|^2} \int_{I_k} |f(y)|\, dy \right\|_1 \leq C\|f\|_1 \leq C.$$

Thus,

$$\left| x \notin U(2I_k) \sum C(a, b_k, f)(x) > \lambda \right| \leq 1/\lambda^{1/2}.$$

We obtained, by combining the estimates above,

$$|\dot{C}(a, b, f) > \lambda| \leq (\|a\|_\infty \|b\|_1 \|f\|_1/\lambda)^{1/2}. \tag{4.10}$$

Repeating this argument for $a \in L^1$, $\|b\|_1 = \|f\|_1 = \|a\|_1 = 1$, we write

$$a = a_1 + a_2, \quad |a_1| \leq C\lambda^{1/3}; \quad a_2 = \sum a_k$$

and we proceed as above to obtain for $x \notin 2I_k$,

$$|C(a_k, b, f)(x)| \leq \lambda^{1/3} \frac{|I_k|}{(x - y_k)^k + I_k^2} \left(\int_{I_k} |f|\, dy \right) b^*(x).$$

Again combining the estimates, using (4.10) and the fact that b^* is in weak L^1,

$$|C(a, b, f) > \lambda| \leq (\|a\|_1 \|b\|_1 \|f\|_1/\lambda)^{1/3}.$$

Lemma (4.5) now gives the good λ inequality

$$|C_*(a, b, f)(x) > 2\lambda,\ a^*(x)b^*(x)f^*(x) \leq \gamma\lambda| \leq C\gamma^{1/3}|C_*(a, b, f) > \lambda| \tag{4.11}$$

which, of course, implies Theorem II.

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