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INTEGRABILITY OF INFINITE SUMS OF INDEPENDENT VECTOR-VALUED RANDOM VARIABLES[1]

BY

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ABSTRACT. Let B be a normed vector space (possibly a Banach space, but it could be more general) and $\{X_n\}$ a sequence of B-valued independent random variables on some probability space. Let $S_n = \Sigma_{j=1}^n X_j$, $M = \sup_n |S_n|$ and $S = \lim_n S_n$ is norm, whenever it exists. Assuming that S exists or $M < \infty$ a.s. and given certain nondecreasing functions φ, we find conditions in terms of the distributions of $\|X_n\|$ such that $E(\varphi(M))$ or $E(\varphi(\|S\|))$ is finite.

Let $\{u_n\}$ be a sequence of elements in B and $\{\epsilon_n\}$ a sequence of independent, identically distributed random variables such that $P\{\epsilon_1 = 1\} = P\{\epsilon_1 = -1\} = ½$. We prove some comparison theorems which generalize the following *contraction principle* of Kahane: If $\{\lambda_n\}$ is a bounded sequence of scalars, then $\Sigma\epsilon_n u_n$ converges in norm a.s. (or is bounded a.s.) implies the corresponding conclusion for the series $\Sigma\lambda_n\epsilon_n u_n$. Some generalizations of this contraction principle have already been carried out by Hoffmann-Jørgensen. All these earlier results are subsumed by ours.

Applications of our results are made to Gaussian processes, random Fourier series and other random series of functions.

1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B, \mathcal{B})$ a measurable space where B is a linear space over the real field R^1 and $\mathcal{B}$ a σ-algebra of subsets of B which is compatible with the linear structure of B, i.e.

(1.1) $(x, y) \longrightarrow x + y$ is a $\mathcal{B}|\mathcal{B} \times \mathcal{B}$ measurable map from $B \times B$ into B, and

(1.2) for $\lambda \in R^1$, $x \in B$, the map $(\lambda, x) \longrightarrow \lambda x$ from $R^1 \times B$ into B is a $\mathcal{B}|\mathcal{B}_1 \times \mathcal{B}$ measurable mapping, where $\mathcal{B}_1$ denotes the Borel sets of R^1.

DEFINITION 1.1. A function X from Ω into B, which is $\mathcal{B}|\mathcal{F}$ measurable, will be called a *B-valued random variable.* If $X_1, \ldots, X_n$ are B-valued random variables, then we call them *independent* if, given $A_1, \ldots, A_n \in \mathcal{B}$,

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = \prod_{i=1}^n P(X_i \in A_i).$$

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An infinite sequence of B-valued random variables on $(\Omega, \mathcal{F}, P)$ is said to be a sequence of independent B-valued random variables if any finite subset of the sequence is independent.

DEFINITION 1.2. A *measurable norm* on $(B, \mathcal{B})$ is a mapping $x \longrightarrow \|x\|$ from B to $[0, \infty)$ such that

(1.3) $\|x\| = 0$ if and only if $x = 0$,

(1.4) $\|x + y\| \leqslant \|x\| + \|y\|$,

(1.5) $\|\lambda x\| = |\lambda| \, \|x\|$, λ real, and

(1.6) $\{x \in B \colon \|x\| \leqslant a\} \in \mathcal{B}$ for all $a \in R^1$.

If B is a Banach space, then its norm is a measurable norm in the sense of Definition 1.2, $\mathcal{B}$ is taken to consist of the Borel sets.

If X is a B-valued random variable on $(\Omega, \mathcal{F}, P)$ and $\|\cdot\|$ is a measurable norm on B, then $\|X\|$ is a real-valued random variable on $(\Omega, \mathcal{F}, P)$ in the usual sense. From now on we will assume that B has a measurable norm $\|\cdot\|$.

Let $\{X_n, n \geqslant 1\}$ be a sequence of B-valued independent random variables on $(\Omega, \mathcal{F}, P)$. We will always write

(1.7) $S_n = \Sigma_{j=1}^n X_j$,

(1.8) $M_n = \sup_{1 \leqslant j \leqslant n} \|S_j\|$,

(1.9) $N_n = \sup_{1 \leqslant j \leqslant n} \|X_j\|$,

(1.10) $M = \sup_{j \geqslant 1} \|S_j\|$,

(1.11) $N = \sup_{j \geqslant 1} \|X_j\|$,

and if S_n converges in norm, then

(1.12) $S = \Sigma_{j=1}^\infty X_j$.

When $B = R^1$ the question of convergence of S_n has been very satisfactorily answered by Kolmogorov's three-series theorem. However, there are no comprehensive results when, e.g., X_n are independent Banach space-valued random variables. Extensive results exist in special cases, like the study of random trigonometric series or Gaussian processes (for references see [4], [5]), but even in these cases results are not complete.

In this article we will assume that S exists, or that $M < \infty$ a.s. Then given certain nondecreasing functions φ we find conditions in terms of φ and the distributions of $\|X_n\|$ such that $E(\varphi(M))$ or $E(\varphi(\|S\|))$ is finite, where E denotes expectation. Some of our results appear to be new even when $B = R^1$ (see, for example, Theorem 3.6).

For earlier work in this direction, we mention the result of Landau and Shepp [6], who showed that if $\{X_n\}$ is a sequence of real-valued Gaussian random variables such that $Z = \sup_n |X_n| < \infty$ a.s., then $E(\exp(\epsilon Z^2)) < \infty$ for some $\epsilon > 0$. Fernique [2] obtained a similar result. Kahane [5] proves results of this nature for random trigonometric series involving Rademacher sequences, and Hoffmann-Jørgensen [3], whose paper stimulated our present research, considers the same question we do when φ is a power. Most of these results, often in stronger form, will be rederived. These references will be discussed in greater detail in appropriate context later on.

In §2 we give some more notation and preliminary lemmas. §3 contains the main results. An important special case is discussed in §4, where X_n is of the form $v_n Y_n$, v_n a nonrandom element of B and Y_n a real-valued random variable. A counterexample given in this section shows how drastically the situation differs from the usual real-valued case.

In §5 we also consider the case where X_n is of the form $v_n Y_n$. What we show is essentially this. If $\sup_n \|\Sigma_{k=1}^n v_k Y_k\| < \infty$ a.s. and if $\{\xi_n\}$ is a sequence of independent symmetric random variables that are smaller than the $\{Y_n\}$ in some appropriate sense then $\sup_n \|\Sigma_{k=1}^n v_k \xi_k\| < \infty$ a.s. also. A similar result holds if the first series converges. Furthermore, results of the type

$$E\left[\varphi\left(\sup_n \left\|\sum_{k=1}^n v_k \xi_k\right\|\right)\right] \leqslant E\left[\varphi\left(\sup_n \left\|\sum_{k=1}^n v_k Y_k\right\|\right)\right],$$

for certain convex functions φ, also hold. These results are actually extensions of the *contraction principle* of Kahane [5].

Applications of the results of §§3, 4 and 5 to Gaussian processes, random Fourier series and other random series of functions are given in §6.

Note. After these results were submitted for publication, [3] appeared in the form of [3a] with certain results in improved form over those of [3]. Consequently some special cases of our results are now equivalent to the results in [3a].

2. Preliminaries and notation. We will add to the definitions and notation given in the previous section. Some well-known results, that will be needed later, will also be given here.

DEFINITION 2.0. A B-valued random variable X on $(\Omega, \mathcal{F}, P)$ is said to be symmetric if

$$P(X \in A) = P(-X \in A), \qquad A \in \mathcal{B}. \tag{2.1}$$

For symmetric random variables we have the following inequality of P. Lévy (see [5, p. 12] for a proof).

LEMMA 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of symmetric, B-valued, independent random variables such that $S = \Sigma_{j=1}^\infty X_j$ exists, then, for all $\lambda > 0$,*

$$P(M > \lambda) \leqslant 2P(\|S\| > \lambda). \tag{2.2}$$

Note that one can take $X_j = 0$ for $j > n$, then (2.2) applies to M_n and S_n in place of M and S, respectively.

The distribution function F of a real-valued random variable X on $(\Omega, \mathcal{F}, P)$ is defined by

$$F(x) = P(X \leqslant x). \tag{2.3}$$

For a real-valued measurable function g on R^1, G a right-continuous function with bounded variation on compact sets, $\int g\, dG$ will always denote the integral of g with respect to the measure μ determined by G on R^1 by setting $\mu(a, b] = G(b) - G(a)$.

If φ is a nondecreasing, continuous function on R^1, F a distribution function (right-continuous, nonnegative, nondecreasing, $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$) and $Q = 1 - F$, then for $-\infty < a \leqslant b < \infty$ we have the well-known integration by parts formula

$$\int \varphi I_{(a,b]}\, dF = \varphi(a)Q(a) - \varphi(b)Q(b) + \int_a^b Q\, d\varphi, \tag{2.4}$$

where $I_{(a,b]}(x) = 1$ or 0 according as $x \in (a, b]$ or $x \notin (a, b]$. If the limit as $b \longrightarrow \infty$ exists for the left side in (2.4), then $\varphi(b)Q(b) \longrightarrow 0$, and consequently $\lim_{b\to\infty} \int_a^b Q\, d\varphi$ exists. In this case we also have

$$\int_{(a,\infty)} \varphi\, dF = \varphi(a)Q(a) + \int_a^\infty Q\, d\varphi. \tag{2.5}$$

If $\varphi \geqslant 0$, $\varphi(0) = 0$, and X is a nonnegative finite random variable with distribution function F, then

$$E(\varphi(X)) = \int_{(0,\infty)} \varphi\, dF = \int_0^\infty Q\, d\varphi, \tag{2.6}$$

by (2.5).

Notation. We will write $I[A]$ to mean the indicator of the set A. Φ will denote the class of all finite-valued nonnegative, nondecreasing, continuous, functions on $[0, \infty)$, which are not identically zero. Let $\Phi_0 = \{\varphi \in \Phi: \varphi(0) = 0\}$.

For $\varphi \in \Phi$, we shall write

$$\varphi_a(x) = \varphi(x) I[x > a]. \tag{2.7}$$

The abbreviation "a.s." stands for "almost surely". $(\Omega, \mathcal{F}, P)$ will be used generically for a probability space. E will denote the integration operator on $L^1(\Omega, \mathcal{F}, P)$.

3. **Integrability of $\varphi(M)$ and $\varphi(\|S\|)$.** Let $\{X_n, n \geqslant 1\}$ be a sequence of independent B-valued random variables on $(\Omega, \mathcal{F}, P)$. Let $\varphi \in \Phi$. The main results of this section, Theorems 3.3, 3.6, 3.8 and 3.11, give conditions in terms of φ and the distributions of $\|X_n\|$ under which $\varphi(M)$ and $\varphi(\|S\|)$ are integrable.

The following lemma is part of Theorem 3.3, except that more restrictive conditions on φ are imposed there.

LEMMA 3.1. *Let* $\{Y_n, n \geqslant 1\}$ *be a sequence of independent nonnegative random variables on* (Ω, F, P). *Let* $\varphi \in \Phi$. *If* $N = \sup_n Y_n$, *then*

$$\text{(3.1)}\quad N < \infty \text{ a.s. and } E(\varphi(N)) < \infty \Leftrightarrow \; a > 0 \text{ such that } \sum_{n=1}^{\infty} E(\varphi_a(Y_n)) < \infty.$$

REMARK 3.2. If $\varphi \equiv 1$, then (3.1) reads as

$$\text{(3.2)}\qquad N < \infty \text{ a.s.} \Leftrightarrow \; a > 0, \text{ such that } \sum_{n=1}^{\infty} P(Y_n > a) < \infty.$$

PROOF OF LEMMA 3.1. If φ is bounded, then, using the facts that φ is nondecreasing and not identically zero, it is easily seen that (3.1) reduces to (3.2). We prove (3.2) first. Using independence,

$$\text{(3.3)}\qquad \begin{aligned} P(N > u) &= \sum_{j=1}^{\infty} P(Y_1 \leqslant u, \ldots, Y_{j-1} \leqslant u, Y_j > u) \\ &= \sum_{j=1}^{\infty} (1 - \alpha_j(u))P(Y_j > u), \end{aligned}$$

where $\alpha_j(u) = P(\bigcup_{k=1}^{j-1} [Y_k > u]) \uparrow P(N > u)$. If $N < \infty$ a.s., then $\exists u_0$ such that $P(N > u_0) \leqslant \frac{1}{2}$. Therefore from (3.3) we have $\Sigma P(Y_j > u_0) < \infty$. Conversely if the right side in (3.2) holds, then $\Sigma_j P(Y_j > u) \longrightarrow 0$ as $u \longrightarrow \infty$, hence from (3.3) we then have $\lim_{u \to \infty} P(N > u) = 0$. Hence $N < \infty$ a.s.

For the general case, let $N_k = \max_{1 < j < k} Y_j$. Then using independence, we have

$$P(N_k > u) = 1 - \prod_{n=1}^{k} (1 - P(Y_n > u)) \geqslant 1 - \exp\left(-\sum_{n=1}^{k} P(Y_n > u)\right),$$

hence letting $k \uparrow \infty$, we get

$$\text{(3.4)}\qquad P(N > u) \geqslant 1 - \exp\left(-\sum_{n=1}^{\infty} P(Y_n > u)\right).$$

Assuming the left side in (3.1), we have $\Sigma_{n=1}^{\infty} P(Y_n > a) < \infty$ for some $a > 0$ by (3.2). Hence

$$\text{(3.5)}\qquad \lim_{u \to \infty} \sum_{n=1}^{\infty} P(Y_n > u) = 0,$$

and there exists $a > 0$ such that $\Sigma_{n=1}^{\infty} P(Y_n > a) \leqslant \frac{1}{2}$, we have

$$\text{(3.6)}\qquad \begin{aligned} &\frac{1}{2} \int_a^{\infty} \sum_{n=1}^{\infty} P(Y_n > u)\, d\varphi(u) \\ &\qquad \leqslant \int_a^{\infty} \left[1 - \exp\left(-\sum_{n=1}^{\infty} P(Y_n > u)\right)\right] d\varphi(u) \leqslant E(\varphi(N)) \end{aligned}$$

by (3.4) and (2.5). Also by (2.5) we get

$$\sum_{n=1}^{\infty} \int_a^{\infty} P(Y_n > u)\, d\varphi(u) = \sum_{n=1}^{\infty} E(\varphi_a(Y_n)) - \varphi(a) \sum_{n=1}^{\infty} P(Y_n > a). \tag{3.7}$$

Combining (3.6) and (3.7) gives the right side in (3.1).

For the converse, observe that since φ is continuous we have

$$\varphi_a(N) \leqslant \sum_{n=1}^{\infty} \varphi_a(Y_n). \tag{3.8}$$

Since φ is nondecreasing and not identically zero, $\Sigma_{n=1}^{\infty} E(\varphi_a(Y_n)) < \infty$ implies the right side in (3.2), hence

$$\sum_{n=1}^{\infty} E(\varphi_a(Y_n)) < \infty \Rightarrow N < \infty \text{ a.s. and } E(\varphi_a(N)) < \infty.$$

The result now follows by observing that $E(\varphi(N)) \leqslant \varphi(a) + E(\varphi_a(N))$.

The assumptions on φ in the following theorem are satisfied by all continuous regularly varying functions, i.e. functions of the form $\varphi(u) = u^p \psi(u)$, $p \geqslant 0$, ψ a continuous slowly varying function near ∞. The exponential case is given in Theorem 3.8.

THEOREM 3.3. *Suppose* $M = \sup_n \|S_n\| < \infty$ *a.s., and let* $N = \sup_n \|X_n\|$. *If* $\varphi \in \Phi$ *and satisfies* $\varphi(4u) \leqslant c\varphi(u)$ *for some* $c > 0$ *then the following statements are equivalent*:

(a) $E(\varphi(N)) < \infty$;

(b) $E(\varphi(M)) < \infty$;

(c) $\exists a > 0$ *such that* $\Sigma_{n=1}^{\infty} E(\varphi_a(\|X_n\|)) < \infty$,

where $\varphi_a(x) = \varphi(x) I[x > a]$.

Furthermore, if $S = \Sigma_{n=1}^{\infty} X_n$ *converges a.s., then each of the above statements is equivalent to*

(d) $E(\varphi(\|S\|)) < \infty$ *and, for* $\varphi \in \Phi_0$, $\lim_{n\to\infty} E(\varphi(\|S - S_n\|)) = 0$.

We shall need the following lemma from Hoffmann-Jørgensen ([3, Theorem 3.1], [3a]). The proof is a modification of the argument in Kahane [5, p. 16]. We reproduce the proof for completeness.

LEMMA 3.4. *Let* X_n *be symmetric, independent, B-valued random variables on* (Ω, F, P). *Let* $F_k(u) = P(\|S_k\| > u)$, $G(u) = P(N > u)$. *Then for* $t > 0, s > 0$

$$F_k(2t + s) \leqslant G(s) + 4F_k^2(t). \tag{3.6$'$}$$

PROOF. Let $T = \inf\{n \geqslant 1: \|S_n\| > t\}$. Since $\|S_k\| > 2t + s$ implies $T \leqslant k$, we have

$$F_k(2t+s) = \sum_{j=1}^{k} P(T=j,\ \|S_k\| > 2t+s). \tag{3.7$'$}$$

If $T = j$ and $\|S_k\| > 2t + s$, then $\|S_{j-1}\| < t$, and $\|S_k - S_j\| = \|S_k - S_{j-1} - X_j\| \geqslant \|S_k\| - \|S_{j-1}\| - \|X_j\| \geqslant t + s - N$. Hence

$$\begin{aligned} P(T=j,\ \|S_k\| > 2t+s) &\leqslant P(T=j,\ \|S_k - S_j\| \geqslant t+s-N) \\ &\leqslant P(T=j,\ \|S_k - S_j\| \geqslant t) + P(T=j,\ N > s) \\ &= P(T=j)P(\|S_k - S_j\| > t) + P(T=j,\ N>s) \end{aligned} \tag{3.8$'$}$$

using independence at the last step. Summing on j and using (3.7$'$) we get

$$F_k(2t+s) \leqslant P(N>s) + \sum_{j=1}^{k} P(T=j)P(\|S_k - S_j\| > t). \tag{3.9}$$

By Lemma 2.1 we have

$$P(\|S_k - S_j\| > t) \leqslant 2P(\|S_k\| > t) \tag{3.10}$$

and

$$\sum_{j=1}^{k} P(T=j) = P\left(\max_{1\leqslant j\leqslant k} \|S_j\| > t\right) \leqslant 2P(\|S_k\| > t). \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) we get (3.6$'$).

PROOF OF THEOREM 3.3. We have already shown (a) $\Leftrightarrow$ (c) under less restrictive conditions on φ. To see that (b) $\Rightarrow$ (a), note that $\|X_n\| \leqslant \|S_n\| + \|S_{n-1}\|$. Hence taking the sup over $n \geqslant 1$, we have $N \leqslant 2M$, and so $\varphi(N) \leqslant \varphi(2M) \leqslant \varphi(4M) \leqslant c\varphi(M)$. We now show that (a) $\Rightarrow$ (b). Note that we may assume $\varphi(0) = 0$ without any loss of generality. We first assume that the X_j are symmetric random variables. Then by Lemma 3.4 we have

$$\int_0^\infty F_k(4t)\, d\varphi(t) \leqslant 4\int_0^\infty F_k^2(t)\, d\varphi(t) + \int_0^\infty G(t)\, d\varphi(t), \tag{3.12}$$

where $F_k(u) = P(\|S_k\| > u)$, $G(u) = P(N > u)$. Since $\varphi(0) = 0$, by integration by parts the left side in (3.12) equals $E(\varphi(\|S_k\|/4))$, which dominates $c^{-1}E(\varphi(\|S_k\|))$ since $\varphi(4t) \leqslant c\varphi(t)$. Hence (3.12) gives

$$\int_0^\infty (c^{-1}F_k(t) - 4F_k^2(t))\, d\varphi(t) \leqslant E(\varphi(N)). \tag{3.13}$$

Since $M < \infty$ a.s. by assumption, there exists a t_0 such that for $t \geqslant t_0$, $P(M > t) \leqslant (8c)^{-1}$. Hence for $t \geqslant t_0$, $F_k(t) \leqslant (8c)^{-1}$ for all $k \geqslant 1$. Hence

$$\int_{t_0}^\infty (c^{-1}F_k(t) - 4F_k^2(t))\, d\varphi(t) \geqslant \frac{1}{2c}\int_{t_0}^\infty F_k(t)\, d\varphi(t). \tag{3.14}$$

Combining this with (3.13), we get

$$E(\varphi(\|S_k\|)) \leqslant \varphi(t_0) + 2cE(\varphi(N)). \tag{3.15}$$

By Lemma 2.1

$$P\left(\max_{1 \leqslant n \leqslant k} \|S_n\| > t\right) \leqslant 2P(\|S_k\| > t). \tag{3.16}$$

This together with (3.15) gives

$$E\left(\varphi\left(\max_{1 \leqslant n \leqslant k} \|S_n\|\right)\right) \leqslant 2\varphi(t_0) + 4cE(\varphi(N)). \tag{3.17}$$

This shows (a) ⇒ (b) in the symmetric case. In the general case, we use symmetrization. Let $\{X_n, n \geqslant 1\}$ be the given sequence and $\{X'_n, n \geqslant 1\}$ an independent copy. Let $N' = \sup_n \|X'_n\|$ and $M' = \sup_n \|S'_n\|$, $S'_n = X'_1 + \cdots + X'_n$. Note that

$$\varphi\left(\operatorname{Sup}_n \|X_n - X'_n\|\right) \leqslant \varphi(N + N') \leqslant \varphi(2\max(N, N'))$$

$$\leqslant c\varphi(\max(N, N')) \leqslant c[\varphi(N) + \varphi(N')].$$

Hence

$$E(\varphi(N)) < \infty \Rightarrow E\left(\varphi\left(\operatorname{Sup}_n \|X_n - X'_n\|\right)\right) < \infty. \tag{3.18}$$

Applying our result for the symmetric case, we have

$$E\left(\varphi\left(\sup_n \|X_n - X'_n\|\right)\right) < \infty \Rightarrow E\left(\varphi\left(\sup_n \|S_n - S'_n\|\right)\right) < \infty. \tag{3.19}$$

It remains to check that

$$E\left(\varphi\left(\operatorname{Sup}_n \|S_n - S'_n\|\right)\right) < \infty \Rightarrow E(\varphi(M)) < \infty. \tag{3.20}$$

To see this, let $\varphi = 0$ on $(-\infty, 0)$, then

$$E\left(\varphi\left(\operatorname{Sup}_n \|S_n - S'_n\|\right)\right) \geqslant E(\varphi(M - M')), \tag{3.21}$$

and by the independence of M, M' and Fubini's theorem, we have

$$E(\varphi(M - M')) < \infty \Rightarrow \exists y \geqslant 0 \text{ such that } E(\varphi(M - y)) < \infty. \tag{3.22}$$

Since $M - y \geqslant M/2$ on the set $\{M > 2y\}$, we conclude that

$$E\left(\varphi\left(\operatorname{Sup}_n \|S_n - S'_n\|\right)\right) < \infty \Rightarrow \int_{\{M > 2y\}} \varphi(M/2)\, dP < \infty. \tag{3.23}$$

Since $E(\varphi(M/2)) \leqslant \varphi(y) + \int_{\{M>2y\}} \varphi(M/2)\, dP$, and by assumption $\varphi(4u) \leqslant c\varphi(u)$, it follows that

$$E\left(\varphi\left(\operatorname*{Sup}_n \|S_n - S_n'\|\right)\right) < \infty \Rightarrow E(\varphi(M)) < \infty. \tag{3.24}$$

Hence (3.18), (3.19) and (3.24) prove (a) $\Rightarrow$ (b) without the assumption of symmetry.

It remains to prove the last part of the theorem. Since $\|S\| \leqslant M$, we have that (b) $\Rightarrow E(\varphi(\|S\|)) < \infty$. In the symmetric case Lemma 2.1 gives $P(M > u) \leqslant 2P(\|S\| > u)$, and so $E(\varphi(\|S\|)) < \infty \Rightarrow$ (b). In the general case the validity of this conclusion follows via symmetrization. Hence (b) $\Leftrightarrow E(\varphi(\|S\|)) < \infty$. Since $2M \geqslant \|S_n - S\| \longrightarrow 0$ a.s., (b) implies (by the dominated convergence theorem) that when $\varphi \in \Phi_0$, $E(\varphi(\|S_n - S\|)) \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof of the theorem.

If $\varphi(u) = u^p$ for some $p \geqslant 0$, then, as we already observed, φ satisfies the conditions of Theorem 3.3, and we get the following important corollary. This should be compared with Hoffmann-Jørgensen's Theorem 3.1 [3], where he shows that $E(N^p) < \infty$, $p > 0 \Rightarrow E(M^q) < \infty$ for $0 \leqslant q < p$. (The improved version of Theorem 3.1 in [3a] also gives parts (a) and (b) of this corollary.)

COROLLARY 3.5. *Suppose $M < \infty$ a.s. If $p > 0$, then the following are equivalent*:

(a) $E(N^p) < \infty$;

(b) $E(M^p) < \infty$;

(c) $\sum_{n=1}^{\infty} E(\|X_n\|^p I[\|X_n\| > a]) < \infty$, *for some $a > 0$.*

Furthermore, if ΣX_n converges a.s. and S represents this sum, then each of the above statements is equivalent to

(d) $E(\|S\|^p) < \infty$ *and* $E(\|S_n - S\|^p) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

If the X_j's are independent, real-valued, random variables, then combining Theorem 3.3 with Kolmogorov's three-series theorem, we get

THEOREM 3.6. *Let $\{X_j, j \geqslant 1\}$ be a sequence of real-valued independent random variables on some probability space (Ω, F, P). If $\varphi \in \Phi$ and satisfies $\varphi(4u) \leqslant c\varphi(u)$ for some $c > 0$ and all $u \geqslant 0$, then*

ΣX_j *converges a.s.,*

$E[\varphi(|\Sigma X_j|)] < \infty$,

and if $\varphi \in \Phi_0$

$\lim_{n\to\infty} E[\varphi(|\Sigma_{j=n}^{\infty} X_j|)] = 0$

if and only if the following three series converge for some $c > 0$:

(a) $\Sigma_j E(X_j I[|X_j| \leqslant c])$,

(b) $\Sigma_j\{E(X_j^2 I[|X_j| \leqslant c]) - E^2(X_j I[|X_j| \leqslant c])\}$,
(c) $\Sigma_j E(\varphi(|X_j|)I[|X_j| > c])$.

REMARK 3.7. When $\varphi = 1$, this theorem reduces to the three-series theorem.

PROOF. First assume (a), (b), (c). Since φ is not identically zero, the other conditions on φ imply that $\varphi(u) > 0$ for $u > 0$. Since

$$\sum_j E(\varphi(|X_j|)I[|X_j| > c]) \geqslant \varphi(c)\sum_j P(|X_j| > c),$$

(c) implies that $\Sigma_j P(|X_j| > c) < \infty$. This together with (a) and (b) imply that ΣX_j converges by the three-series theorem. By Theorem 3.1 the condition (c) above then implies that $E(\varphi(|\Sigma X_j|)) < \infty$, and for $\varphi \in \Phi_0$, $E(\varphi(|\Sigma_{j=n}^{\infty} X_j|)) \longrightarrow 0$ as $n \longrightarrow \infty$. Conversely, if ΣX_j converges, then (a) and (b) follow from the three-series theorem and $E(\varphi(|\Sigma X_j|)) < \infty$ implies (c) by Theorem 3.3.

We next consider the case when $\varphi(x) = \exp(x)$. In this case it turns out that we do not quite get the expected analogue of Theorem 3.3 but Theorem 3.8 is satisfactory for a number of interesting cases.

THEOREM 3.8. *Let $\{X_n, n \geqslant 1\}$ be independent B-valued random variables on (Ω, F, P). Suppose $M = \sup_n \|S_n\| < \infty$ a.s., and for some $a > 0$, $\alpha > 0$ and $\delta > 0$*

$$\sum_{n=1}^{\infty} E\{\exp(\alpha\|X_n\|\log^{1+\delta}\|X_n\|)I[\|X_n\| > a]\} < \infty. \tag{3.25}$$

Then there exists an $\epsilon > 0$ such that $E[\exp(\epsilon M)] < \infty$.

For convenience in writing we will prove the theorem for $\delta = 1$; it should be obvious how to modify the argument for any $\delta > 0$. Actually, one can replace $\log^{1+\delta}\|X_n\|$ by a more general nondecreasing function of $\|X_n\|$. We will not go into that.

We first prove the following lemma.

LEMMA 3.9. *Let $\{X_n, n \geqslant 1\}$ be independent B-valued random variables on (Ω, F, P). Suppose $M = \sup_n \|S_n\| < \infty$ a.s. and that (3.25) holds. Let $F_k(u) = P(\|S_k\| > u)$ and $\gamma(u) = E[\exp(\alpha N\log^2 N)I(N > u)]$. Then for all $\epsilon > 0$, $t > 0$ and $k \geqslant 1$ we have*

$$\begin{aligned} E(\exp(\epsilon\|S_k\|)) \leqslant \exp(4\epsilon t) &+ \sum_{n=0}^{\infty} \exp(\epsilon 2^{n+3}t)(8F_k(t/4))^{2^{n+1}} \\ &+ \sum_{n=0}^{\infty} \exp(\epsilon t 2^{n+3}) \sum_{s=1}^{n+1} (8^2\gamma(ts^{-2}2^{s-2}))^{2^{n-s+1}} \\ &\cdot \exp(-\alpha ts^{-2}2^{n-1}\log^2(ts^{-2}2^{s-2})). \end{aligned} \tag{3.26}$$

PROOF. By Lemma 3.1 the assumption (3.25) is equivalent to

$$E(\exp(\alpha N \log^2 N)) < \infty. \tag{3.27}$$

From this we conclude that for $u > 0$

$$G(u) = P(N > u) \leqslant \gamma(u)\exp(-\alpha u \log^2 u). \tag{3.28}$$

Applying Lemma 3.4 with $t = 2^n t_1$ and $s = (n+1)^{-2}2^{n+1}t_1$ we get

$$F_k[2^{n+1}(1 + (n+1)^{-2})t_1] \leqslant 4F_k^2(2^n t_1) + G(2^{n+1}(n+1)^{-2}t_1). \tag{3.29}$$

Define t_2 by the equation $2^n t_1 = 2^n(1 + 1/n^2)t_2$, then

$$F_k(2^n t_1) \leqslant 4F_k^2(2^{n-1}t_2) + G(2^n n^{-2} t_2).$$

Substituting this in (3.29) and continuing recursively (making repeated use of the inequality $(4x + y)^n \leqslant (8x)^n + (8y)^n$, $x \geqslant 0, y \geqslant 0$) we get

$$\begin{aligned} &F_k[2^{n+1}(1 + (n+1)^{-2})t_1] \\ &\quad\leqslant (8F_k(t_{n+1}))^{2^{n+1}} + \sum_{j=0}^{n} [8^2 G(2^{n+1-j}t_{j+1}(n+1-j)^{-2})]^{2^j}, \end{aligned} \tag{3.30}$$

where t_j is given by

$$\begin{aligned} t_{j+1} &= \left(1 - \frac{1}{1 + (n+1-j)^2}\right) t_j \\ &= t_1\left(1 - \frac{1}{1+n^2}\right)\left(1 - \frac{1}{1 + (n-1)^2}\right) \cdots \left(1 - \frac{1}{1 + (n+1-j)^2}\right) \geqslant \frac{t_1}{4}. \end{aligned} \tag{3.31}$$

Hence we get from (3.30) and (3.31)

$$F_k(2^{n+2}t_1) \leqslant (8F_k(t_1/4))^{2^{n+1}} + \sum_{s=1}^{n+1} [8G(t_1 2^{s-2} s^{-2})]^{2^{n-s+1}}. \tag{3.32}$$

In subsequent uses of (3.32) we will substitute t for t_1. For $\epsilon > 0$, we have

$$\begin{aligned} E(\exp(\epsilon\|S_k\|)) &\leqslant -\sum_{n=2}^{\infty} \int_{(2^n t, 2^{n+1}t]} \exp(\epsilon x)\, dF_k(x) + \exp(4\epsilon t) \\ &\leqslant \exp(4\epsilon t) + \sum_{n=0}^{\infty} \exp(\epsilon 2^{n+3}t) F_k(2^{n+2}t). \end{aligned} \tag{3.33}$$

The estimate for $G(u)$ given in (3.28) is now used in (3.32), and the resulting estimate for $F_k(2^{n+2}t)$ is then used in (3.33) to get (3.26). This completes the proof of the lemma.

PROOF OF THEOREM 3.8. We will establish this only in the symmetric case. The general case is then established by the familiar symmetrization procedure. The proof consists of showing that one can pick t sufficiently large and ϵ sufficiently small, independent of k, such that the right side in (3.26) is finite. Note that $\gamma(u) \leqslant \gamma(0)$, where $\gamma(u)$ is defined in the statement of Lemma 3.9. Setting $\gamma_1 = \max(8^2\gamma(0), 2)$, we can dominate the right side in (3.26) by

$$\begin{aligned}\exp(4\epsilon t) &+ \sum_{n=0}^{\infty} \exp(\epsilon t 2^{n+3})\,(8F_k(t/4))^{2^{n+1}} \\ &+ \sum_{n=0}^{\infty} \exp(\epsilon t 2^{n+3}) \cdot \gamma_1^{2^{n+1}} \cdot \exp(-\alpha t 2^{n-2})\end{aligned} \tag{3.34}$$

provided t is chosen large enough so $s^{-2}\log^2(ts^{-2}2^{s-2}) \geqslant 2^{-1}$. Clearly such a choice of t does not depend on k. Now pick t, larger if necessary, so that $\gamma_1 \exp(-\alpha t/8) \leqslant ½$ and $8P(M > t/4) \leqslant ½$. (Here we use $M < \infty$ a.s.) Such a choice of t having been made, pick $\epsilon > 0$ sufficiently small so that $\exp(4\epsilon t) < 5/4$. For such $\epsilon > 0$, $t > 0$, it is clear that the expression in (3.34) is finite. Hence there exists $0 \leqslant A < \infty$, and $\epsilon > 0$, both independent of k, such that

$$E(\exp(\epsilon \|S_k\|)) \leqslant A. \tag{3.35}$$

By Lemma 2.1 and integration by parts this implies $E(\exp(\epsilon M_k)) \leqslant 2A$, independently of k. Using the monotone convergence theorem the proof is completed.

REMARK 3.10. Even when (3.25) is not satisfied it is possible to obtain results of the kind $E[\varphi(M)] < \infty$ for φ of exponential type. For instance suppose (3.25) is satisfied only when $\|X_n\|$ is replaced by $\|X_n\|^\alpha$ for some $0 < \alpha < 1$. Under these conditions we could show $E(\exp(\epsilon M^\alpha)) < \infty$. The proof begins with Lemma 3.4 but the recursion is carried out differently. The choices of t and s in (3.26) determine the size of the terms involving G which in turn determine the necessary conditions for the theorem.

The following result holds if one imposes even more restrictive conditions on the X_n. We are indebted to the referee for the proof of this theorem that is given here. It is considerably simpler than our original proof.

THEOREM 3.11. *Let $\{X_n, n \geqslant 1\}$ be independent B-valued random variables on $(\Omega, \mathcal{F}, P)$. If there exists a sequence $\{a_n\}$ of positive numbers such that*

$$\sup_\omega \|X_n(\omega)\| \leqslant a_n, \qquad \lim_n a_n = 0 \tag{3.36}$$

and $S = \sum_{j=1}^\infty X_j$ converges a.s., then for all $\epsilon > 0$, $E[\exp(\epsilon\|S\|)] < \infty$.

PROOF. As before, we may assume that the X_n's are symmetric. It is also clear that we may take $a_n \downarrow$. Let

$$M(k) = \sup_{n \geq 0} \left\| \sum_{j=k}^{k+n} X_j \right\|, \quad S(k) = \sum_{j=k}^{\infty} X_j, \tag{3.37}$$

then $M(k) \longrightarrow 0$ a.s. as $k \longrightarrow \infty$, and there exist positive numbers $t_k \longrightarrow 0$ such that

$$P[M(k) > t_k] \leq e^{-2} \leq P[M(k) \geq t_k]. \tag{3.38}$$

Moreover, by Lemmas 2.1 and 3.4 we have for $t > 0$

$$\begin{aligned} P[M(k) > 2t + a_k] &\leq 2P[\|S(k)\| > 2t + a_k] \\ &\leq 2P\left[\sup_{j \geq k} \|X_j\| > a_k\right] + 8P^2[\|S(k)\| > t] \leq 8P^2[M(k) > t]. \end{aligned} \tag{3.39}$$

From (3.38) and (3.39) it follows that there exist constants A_k such that, for $t \geq t_k$, $P[M(k) > t] \leq A_k e^{-\epsilon_k t}$ where $\epsilon_k = (t_k + a_k)^{-1} \longrightarrow \infty$ as $k \longrightarrow \infty$. If $\epsilon > 0$ is given, we may pick k large so that $\epsilon_k > \epsilon$, and with this choice of k, observing that $\|S\| \leq \Sigma_{j=1}^{k-1} a_j + M(k)$ we have, for $t \geq t_k$, $P[\|S\| > t] \leq B_k e^{-\epsilon_k t}$ for some $B_k < \infty$. This clearly suffices to conclude that $E[\exp(\epsilon\|S\|)] < \infty$.

4. A special case. In this section we will examine the results of the previous section in a special situation. We let

$$X_n = v_n Y_n, \quad n \geq 1, \tag{4.1}$$

where v_n are nonrandom elements of B and the Y_n are *independent real-valued* random variables. S_k, M_k, S, M, N have the same meaning as before with $X_n = v_n Y_n$. The questions that we are going to consider are of the following type: if the Y_n are independent, $\sup_n E(|Y_n|^p) < \infty$ for some $p > 0$, then does $M < \infty$ a.s. imply $E(M^p) < \infty$ or even $E(M^q) < \infty$ for $q < p$? The answer is that this is true only if additional conditions are imposed either on the distributions of $|Y_n|$ or on the sequence $\{\|v_n\|\}$. Otherwise, we will show by a counterexample that M need not have any finite moment even though Y_n does and $M < \infty$ a.s. The situation in the exponential case is somewhat different and is considered in Theorem 4.7.

The assumption that $M < \infty$ a.s. imposes a necessary condition on the distribution of Y_n and on $\|v_n\|$. Since $M < \infty$ a.s. implies $N < \infty$ a.s., it follows from Remark 3.2 that there exists $u > 0$ such that

$$\sum_n P(\|v_n\| \, |Y_n| > u) < \infty. \tag{4.2}$$

Using this observation we can show that if either the distribution of $|Y_n|$ or the sequence $\{\|v_n\|\}$ satisfies some regularity properties then moments of M exist. One such result is the following:

THEOREM 4.1. *Let Y_n in* (4.1) *be independent and identically distributed* *random variables. Let* $F(u) = P(|Y_1| \leqslant u)$ *and* $Q = 1 - F$. *Assume that* Q *is regularly varying and* $E|Y_1|^p < \infty$ *for some* $p > 0$. *Suppose* $M < \infty$ *a.s. Then* $E(M^{p-\epsilon}) < \infty$, $0 < \epsilon \leqslant p$.

PROOF. This is essentially Theorem 6.1 in [3]. It also follows easily from Corollary 3.5.

It is easy to see, by constructing a counterexample, that the theorem is false for $\epsilon = 0$.

In the next theorem we will show that if $P(|Y_1| > u)$ does not decrease to zero too fast as $u \longrightarrow \infty$, then the regularity of $\{\|v_n\|\}$ is enough to insure the existence of moments of M.

THEOREM 4.2. *Let* Y_n *in* (4.1) *be independent and identically distributed random variables. Assume that for some* $p > 0$, $E(|Y_1|^p) < \infty$ *but* $E(|Y_1|^q) = \infty$ *for* $q > p$. *Let* $\{\|v_n\|\}$ *be regularly varying and* $M < \infty$ *a.s., then* $E(M^{p-\epsilon}) < \infty$, $0 < \epsilon \leqslant p$.

PROOF. Without loss of generality we may assume $\|v_n\| \neq 0$ for all n. Furthermore, since rearrangement of terms will not affect the result we may assume $\{\|v_n\|\}$ is a nondecreasing sequence. Let $\alpha_0 = 0$; $\alpha_j = \|v_j\|^{-1}$ for $j \geqslant 1$. Since $M < \infty$ a.s. and Y_j identically distributed, (4.2) allows us to conclude that $\alpha_j \nearrow \infty$. We write $F(u) = P(|Y_1| \leqslant u)$; $Q = 1 - F$. The sequence $\{\alpha_j\}$ is regularly varying, hence

$$\alpha_j = j^\theta l(j) \tag{4.3}$$

for some $\theta \geqslant 0$ and l a slowly varying function. By [1, Corollary, p. 274] we can find a regularly varying sequence $\{\beta_j\}$ such that

$$\beta_0 = 0; \qquad \beta_j = \gamma j^\theta \exp\left[\int_1^j (\epsilon(y)/y)\, dy\right], \qquad j \geqslant 1 \tag{4.4}$$

and

$$\lim_{j \to \infty} \beta_j/\alpha_j = 1, \tag{4.5}$$

for some $\gamma > 0$, $\epsilon(y) \longrightarrow 0$. For $a > 0$, we have

$$\begin{aligned} E(|Y_1|^q) &= \sum_{j=0}^\infty \int_{(a\alpha_j, a\alpha_{j+1})} x^q\, dF(x) \leqslant \sum_{j=0}^\infty a^q \alpha_{j+1}^q [Q(a\alpha_j) - Q(a\alpha_{j+1})] \\ &\leqslant c_1 a^q \sum_{j=0}^\infty \beta_{j+1}^q [Q(a\alpha_j) - Q(a\alpha_{j+1})] \\ &\leqslant c_1 a^q \sum_{j=0}^\infty Q(a\alpha_j)[\beta_{j+1}^q - \beta_j^q] \leqslant c_2 a^q \sum_{j=0}^\infty Q(a\alpha_j) j^{\theta q - 1} l^q(j), \end{aligned} \tag{4.6}$$

for some positive constants c_1 and c_2, by using the representation in (4.4) for

β_j. By (4.2) we have for some $a > 0$

$$\sum_j Q(a\alpha_j) < \infty. \tag{4.7}$$

Since $E(|Y_1|^q) = \infty$ for $q > p$, (4.6) and (4.7) imply that $\theta \geqslant p^{-1}$. If $\theta > p^{-1}$, then $\Sigma_j \|v_j\|^p = \Sigma_j j^{-\theta p} l^{-1}(j) < \infty$ and consequently by Corollary 3.5 we have $E(M^p) < \infty$. Therefore assume $\theta = p^{-1}$. We now proceed to check condition (c) of Corollary 3.5 with $p - \epsilon$ in place of p. In order to check this note that we already have $\Sigma Q(a\alpha_j) < \infty$ for some $a > 0$ by (4.7). Also

$$\|v_n\|^{p-\epsilon} \int_{a\|v_n\|^{-1}}^{\infty} Q(u)\, du^{p-\epsilon} \leqslant n^{-1+\epsilon\theta} l^{-p+\epsilon}(n) \sum_{j=n}^{\infty} \int_{a_1\beta_j}^{a_1\beta_{j+1}} Q(u)\, du^{p-\epsilon}$$

$$\leqslant c_3 n^{-1+\epsilon\theta} l^{-p+\epsilon}(n) \sum_{j=n}^{\infty} Q(a_2\alpha_j) l^{p-\epsilon}(j) j^{-\epsilon\theta}$$

for some positive constants a_1, a_2 and c_3. Now

$$\sum_{n=1}^{\infty} n^{-1+\epsilon\theta} l^{-p+\epsilon}(n) \sum_{j=n}^{\infty} Q(a_2\alpha_j) l^{p-\epsilon}(j) j^{-\epsilon\theta}$$

$$= \sum_{j=1}^{\infty} \left(\sum_{n=1}^{j} n^{-1+\epsilon\theta} l^{-p+\epsilon}(n) \right) Q(a_2\alpha_j) l^{p-\epsilon}(j) j^{-\epsilon\theta} \leqslant c_4 \sum_{j=1}^{\infty} Q(a_2\alpha_j) < \infty$$

by using [1, Theorem 1, p. 273] and (4.7). Hence condition (c) of Corollary 3.5 is satisfied and the proof is complete.

We will now give a counterexample that shows that even though the Y_n's are independent identically distributed, $E(Y_1^2) < \infty$, and $M < \infty$ a.s. (indeed $\Sigma_n v_n Y_n$ converges a.s.), $E(M^\epsilon) = \infty$ for every $\epsilon > 0$.

EXAMPLE 4.3. We take $B = C[0, 1]$, the space of continuous functions on $[0, 1]$ with the sup norm. For $\varphi \in C[0, 1]$, we will denote its sup norm by $\|\varphi\|$. For $j \geqslant 0$, let $\varphi_j \geqslant 0$ be given by

$$\varphi_j^2(t) = \begin{cases} 0 & \text{for } t \notin (2^{-j-1}, 2^{-j}), \\ 2^{j+2} & \text{for } t = 3 \cdot 2^{-j-2}, \\ \text{linear in between.} \end{cases}$$

Thus the φ_j form an orthonormal system in $L^2[0, 1]$. Define

$$v_j = \lambda_j \varphi_j, \tag{4.8}$$

where the λ_j are positive real numbers and will be chosen suitably later. Let Y_j, $j \geqslant 1$, be independent identically distributed random variables on some (Ω, F, P), each with distribution F. We will pick F later so $E(Y_1^2) < \infty$. Let

$$Q(x) = P(|Y_1| > x), \tag{4.9}$$

and define

$$X(t) = \sum_{j=0}^{\infty} v_j(t) Y_j. \tag{4.10}$$

The following lemma gives necessary and sufficient conditions for the uniform convergence of the series in (4.10).

LEMMA 4.4. *Let v_j, Y_j and Q be as above. Then the series in* (4.10) *converges uniformly if and only if*

$$\sum_{j=0}^{\infty} Q(a/\|v_j\|) < \infty, \qquad \forall a > 0. \tag{4.11}$$

PROOF. Since the v_j have disjoint supports, the series in (4.10) converges uniformly a.s. if and only if for almost all ω, $\forall a > 0, \exists n_0(a, \omega)$ such that

$$\sup_{n \geq n_0} (\|v_j\| \, |Y_j|) \leq a. \tag{4.12}$$

By the Borel-Cantelli lemma, (4.11) holds if and only if (4.12) holds. Hence the proof.

We now proceed with our example. We write

$$\gamma_j = \|v_j\|^{-1}. \tag{4.13}$$

For $m \geq 1$ let $u_m = 2^{m(m-1)/2}$, and pick $\gamma_j > 0$ so that

$$\gamma_j = u_m \quad \text{for } u_0^2 + \cdots + u_m^2 \leq j < u_0^2 + \cdots + u_{m+1}^2. \tag{4.14}$$

Also define

$$Q(x) = \begin{cases} (m^2 u_{m+1}^2)^{-1} & \text{for } 2^{-(m-1)/2} u_m < x \leq 2^{-m/2} u_{m+1}, \\ 1 & \text{for } x \in [0, 1]. \end{cases} \tag{4.15}$$

We first check that $E(Y_1^2) < \infty$, which follows from

$$\int_0^\infty x Q(x)\, dx \leq 1 + \sum_{m=1}^{\infty} (m^2 u_{m+1}^2)^{-1} (2^{-m/2} u_{m+1})^2 < \infty.$$

We also check (4.11) which is equivalent to checking

$$\sum_j Q(a\gamma_j) < \infty \tag{4.16}$$

for all $a > 0$. From (4.14) we get for $0 < a \leq 1$

$$\sum_{j=1}^{\infty} Q(a\gamma_j) = \sum_{m=1}^{\infty} 2^{m(m+1)} Q(au_m). \tag{4.17}$$

Given $0 < a \leq 1$, there exists m_0 such that for $m \geq m_0$

$$2^{-(m-1)/2} u_m \leq a u_m \leq 2^{-m/2} u_{m+1},$$

hence

$$\sum_{m=m_0}^{\infty} 2^{m(m+1)}Q(au_m) \leqslant \sum_{m=m_0}^{\infty} 2^{m(m+1)}(m^2u_{m+1}^2)^{-1} < \infty$$

and (4.16) is checked and by Lemma 4.4 the series in (4.10) converges uniformly a.s.

We will now show that $E(\|S\|^\epsilon) = \infty$ for all $\epsilon > 0$, where $S = \Sigma\, v_j Y_j$. By Corollary 3.5 we have

$$E(\|S\|^\epsilon) < \infty \Leftrightarrow \sum_{n=1}^{\infty} E(\|v_n\|^\epsilon |Y_n|^\epsilon I[|Y_n| > a/\|v_n\|]) < \infty \tag{4.18}$$

for some $a > 0$. We will show that the sum on the right side diverges for all $a > 0$. Now

$$E(\|v_n\|^\epsilon |Y_n|^\epsilon I[|Y_n| > a/\|v_n\|]) = \|v_n\|^\epsilon \int_{(a/\|v_n\|,\, \infty)} u^\epsilon \, dP(|Y_1| \leqslant u)$$

$$\geqslant \epsilon\|v_n\|^\epsilon \int_{a/\|v_n\|}^{\infty} Q(u)u^{\epsilon-1}\, du = \epsilon\int_a^\infty Q(u/\|v_n\|)u^{\epsilon-1}\, du.$$

Hence recalling (4.13), (4.14) and (4.15) we see that the sum in (4.18) dominates

$$\epsilon \sum_{n=1}^{\infty} \int_a^\infty Q(x\gamma_n)x^{\epsilon-1}\, dx = \epsilon \sum_{m=1}^{\infty} 2^{m(m+1)} \int_a^\infty Q(xu_m)x^{\epsilon-1}\, dx$$

$$\geqslant \epsilon \sum_{m=n}^{\infty} 2^{m(m+1)} \int_a^{2^{(m-1)/2}} Q(xu_m)x^{\epsilon-1}\, dx$$

$$\geqslant \sum_{m=n}^{\infty} 2^{m(m+1)} Q(2^{(m-1)/2}u_m)(2^{\epsilon(m-1)/2} - a^\epsilon)$$

$$= \sum_{m=n}^{\infty} 2^{m(m+1)}(m^2u_{m+1}^2)^{-1}(2^{\epsilon(m-1)/2} - a^\epsilon),$$

where n is chosen large enough so $2^{(n-1)/2} \geqslant a$ and equality at the last step follows by observing that $2^{-(m-1)/2}u_m \leqslant 2^{(m-1)/2}u_m \leqslant 2^{-m/2}u_{m+1}$. The last sum clearly diverges and the demonstration of the counterexample is complete.

REMARK 4.5. By a similar argument, for any $p > 0$, one can choose the Y_n's such that $E(|Y_1|^p) < \infty$, $M < \infty$ a.s. and $E(M^\epsilon) = \infty$ for every $\epsilon > 0$.

We now consider the exponential case in the following theorem.

THEOREM 4.6. *Let X_n be as in* (4.1). *Suppose $M < \infty$ a.s. If for some $\alpha > 0$, $\beta > 0$, $c > 0$, and all $u \geqslant 1$*

$$\begin{gathered} P(|Y_n| > u) \leqslant \exp(-\alpha u \log^{1+c} u); \\ \sum_{n=1}^{\infty} \exp\left(-\frac{\beta}{\|v_n\|}\log^{1+c}\frac{1}{\|v_n\|}\right) < \infty, \end{gathered} \tag{4.19}$$

then there exists $\epsilon > 0$ such that $E(\exp(\epsilon M)) < \infty$.

PROOF. First observe that $M < \infty$ a.s. implies $N = \sup_n \|X_n\| < \infty$ a.s.

Hence by Remark 3.2 there exists $a > 0$ such that

$$\sum_n P\left(|Y_n| > \frac{a}{\|v_n\|}\right) < \infty. \tag{4.20}$$

Let $F_n(u) = P(X_n \leqslant u)$, $Q_n = 1 - F_n$. We apply Theorem 3.8. Hence we look at ($\delta > 0, b > 0$)

$$\sum_{n=1}^{\infty} \int_{[b,\infty)} \exp(\delta u \log^{1+c} u)\, dF_n(u)$$

$$= \sum_{n=1}^{\infty} \left\{ Q_n(b) \exp(\delta b \log^{1+c} b) + \int_b^\infty Q_n(u)\, d(e^{\delta u \log^{1+c} u}) \right\}$$

and for $b \geqslant \max(a, 1)$, using (4.19) and (4.20)

$$\leqslant A + B \sum_{n=1}^{\infty} \int_b^\infty \exp\left(-\frac{\alpha u}{\|v_n\|} \log^{1+c} \frac{u}{\|v_n\|}\right) \exp(2\delta u \log^{1+c} u)\, du,$$

where A, B are finite constants. By (4.19) there is a $\gamma > 0$ such that $\|v_n\| \leqslant \gamma$, hence we can dominate this last expression by

$$A + B \sum_{n=1}^{\infty} \int_b^\infty \exp\left(-\frac{\theta u}{\|v_n\|} \log^{1+c} \frac{u}{\|v_n\|}\right) du, \tag{4.21}$$

for some $\theta > 0$ by picking δ sufficiently small. Since b can be chosen as large as we wish, the convergence of

$$\sum_{n=1}^{\infty} \exp\left(-\frac{\beta}{\|v_n\|} \log^{1+c} \frac{1}{\|v_n\|}\right)$$

implies that of the expression in (4.21). Hence by Theorem 3.8 the result follows.

If the Y_n are uniformly bounded random variables, then we have the following immediate consequence of Theorem 3.8.

THEOREM 4.7. *Let X_n be as in* (4.1). *Suppose that $|Y_n| \leqslant A < \infty$ a.s. for all n, and that either*

(a) $\|v_n\| \leqslant B < \infty$ *for all n, or*

(b) Y_n *are identically distributed.*

Then $M < \infty$ a.s. implies that for some $\epsilon > 0$

$$E(\exp(\epsilon M)) < \infty. \tag{4.22}$$

PROOF. $M < \infty$ a.s. implies $\sup_n(\|v_n\| |Y_n|) < \infty$ a.s., hence under (b) we must have $\sup_n \|v_n\| < B$ for some $B < \infty$. Hence (b) implies (a) and it is enough to prove the result under (a). If (a) holds, then $\|X_n\| = \|v_n\| |Y_n| \leqslant AB$ a.s. for all n. Hence (4.22) follows from Theorem 3.8 by taking $a > AB$ in (3.25). This completes the proof.

Theorem 4.7 is proved in [5, Theorem 4, p. 27] when $\{Y_n\}$ is a Rademacher sequence. Under the added condition that ΣX_n converges to S, (4.22) is also obtained with $\|S\|$ replacing M. In the next theorem, which is a special case of Theorem 3.11, we have this result holding for all $\epsilon > 0$.

THEOREM 4.8. *Let* $\{X_n\}$ *be as in* (4.1). *Suppose the* Y_n *are independent identically distributed and* $\sup_n |Y_n| \leqslant A < \infty$ *a.s. If* $\Sigma_{n=1}^{\infty} v_n Y_n$ *converges to* S, *then*

(4.23)
$$\forall \epsilon > 0, \qquad E(\exp(\epsilon\|S\|)) < \infty.$$

REMARK 4.9. If $B = R^1$, $\{Y_i\}$ a Rademacher sequence, then it is known that $E(\exp(\epsilon\|S\|^2)) < \infty$ for all $\epsilon > 0$. It is likely that this is true when B is any separable Banach space. We will obtain this stronger result in §6 under additional conditions. In fact, this result will apply to any sequence of independent uniformly bounded, symmetric random variables $\{Y_i\}$.

5. Comparison theorems for sums of independent vector-valued random variables. Let $\{u_n\}$ be a sequence of elements in B and $\{\epsilon_n\}$ a Rademacher sequence of random variables on some probability space (Ω, F, P), i.e. the ϵ_n are independent, identically distributed random variables such that $P[\epsilon_1 = 1] = P[\epsilon_1 = -1] = ½$. Kahane [5, Theorem 5, p. 18] gives the following very useful result which he calls a *contraction principle*: If the random series $\Sigma_{n=1}^{\infty} \epsilon_n u_n$ converges in norm a.s. (or is bounded) and λ_n is a bounded scalar sequence, then the random series $\Sigma_{n=1}^{\infty} \epsilon_n \lambda_n u_n$ converges in norm a.s. (or is bounded). Our aim here is to generalize this contraction principle in several directions. The nature of the generalizations leads us to call our results comparison theorems.

Let $\{\eta_n, n \geqslant 1\}$ be a sequence of independent, real-valued, random variables on a probability space (Ω, F, P) and $\{u_n, n \geqslant 1\}$ be a sequence of elements in B. We will be concerned with the following questions: If the series $\Sigma_{k=1}^{\infty} \eta_k u_k$ is a.s. convergent (or a.s. bounded), and if $\{\xi_k\}$ is some other sequence of independent, real-valued random variables (possibly on a different probability space) such that ξ_k is "smaller" than η_k, in some sense, for all $k \geqslant 1$, then is the series $\Sigma_{k=1}^{\infty} \xi_k u_k$ a.s. convergent (or a.s. bounded)? Also, if φ is a nondecreasing, nonnegative convex function defined on $[0, \infty)$, and

(5.1)
$$E\left[\varphi\left(\sup_n \left\|\sum_{k=1}^{n} \eta_k u_k\right\|\right)\right] < \infty,$$

or if the series converges a.s., and

(5.2)
$$E\left[\varphi\left(\left\|\sum_{k=1}^{\infty} \eta_k u_k\right\|\right)\right] < \infty,$$

then do (5.1) and (5.2) remain valid if the η_k's are replaced by the "smaller" ξ_k's?

Our results include the above mentioned result of Kahane and its generalization by Hoffmann-Jørgensen [3]. Many of their ideas are used in our proofs. We also use a technique of Pisier [8]. Before stating the main results we introduce some notation.

If the random variables of the sequence $\{\eta_k, k \geqslant 1\}$ are symmetric, then they are said to be *uniformly nondegenerate* if there exist a, $b > 0$ such that

$$P[|\eta_k| \geqslant a] \geqslant b, \qquad \forall k \geqslant 1. \tag{5.3}$$

In general the η_k are called *uniformly nondegenerate* if there exist $c, d > 0$ such that

$$P[|\eta_k - \eta_k'| \geqslant c] \geqslant d, \qquad \forall k \geqslant 1, \tag{5.4}$$

where $\{\eta_k'\}$ is an independent copy of the sequence $\{\eta_k\}$. Clearly (5.3) can be included in (5.4) but it will be convenient to have both definitions.

REMARK 5.0. All our results that will be stated below involve two sequences of independent random variables $\{\eta_k\}$ and $\{\xi_k\}$ which could be given on different probability spaces. Since the results depend only on the finite-dimensional distributions of the η_k's, and the finite-dimensional distributions of the ξ_k's, and not on the structure of the underlying probability space(s), we can assume that both sequences are defined on the same probability space. (We can simply take the two sequences independent of each other.) We will often need to introduce a Rademacher sequence which is independent of $\{\eta_k\}$ and $\{\xi_k\}$. Hence even if the two sequences $\{\eta_k\}$ and $\{\xi_k\}$ were originally given on different probability spaces, without any loss of generality we will assume that there is a basic probability space $(\Omega, F, P) = (\Omega_1 \times \Omega_2, F_1 \times F_2, P_1 \times P_2)$ such that the sequences $\{\eta_k\}$ and $\{\xi_k\}$ are given on (Ω_1, F_1, P_1) and an independent Rademacher sequence $\{\epsilon_n\}$ on (Ω_2, F_2, P_2). They are then defined on the product space in the usual manner. E_1, E_2 and E will denote expectation operators with respect to P_1, P_2 and P. On occasion we will choose (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2) in some suitable manner.

Our first results are stated for symmetric η_k's. Remark 5.0 applies to each result.

THEOREM 5.1. *Let $\{u_k, k \geqslant 1\}$ be a sequence of elements of B and $\{\eta_k\}$ independent, symmetric, real-valued random variables on the probability space (Ω, F, P). Assume that the η_k's are uniformly nondegenerate. Let $\{\xi_k, k \geqslant 1\}$ be independent, symmetric, real-valued random variables on (Ω, F, P) such that, for some $x_0 \geqslant 0$, $0 < \alpha \leqslant 1$,*

$$P[|\eta_k| \geqslant x] \geqslant \alpha P[|\xi_k| \geqslant x], \tag{5.5}$$

for all $x \geqslant x_0$. Then

$$\sup_n \left\| \sum_{k=1}^n \eta_k u_k \right\| < \infty \ \textit{a.s.} \Rightarrow \sup_n \left\| \sum_{k=1}^n \xi_k u_k \right\| < \infty \ \textit{a.s.} \tag{5.6}$$

and

$$\sum_{k=1}^{\infty} \eta_k u_k \text{ converges a.s.} \Rightarrow \sum_{k=1}^{\infty} \xi_k u_k \text{ converges a.s.} \tag{5.7}$$

If in addition the ξ_k are uniformly bounded, then (5.5) is satisfied, and we get the immediate

COROLLARY 5.2. *Let $\{u_k\}$ and $\{\eta_k\}$ be as in Theorem 5.1. If $\{\xi_k\}$ is a sequence of independent, symmetric, real-valued, uniformly bounded random variables on $(\Omega, \mathcal{F}, P)$, then (5.6) and (5.7) hold.*

THEOREM 5.3. *Let φ be a nonnegative, nondecreasing, convex function on $[0, \infty)$. Let $\{u_k\}$, $\{\eta_k\}$, $\{\xi_k\}$ be as in Theorem 5.1. Suppose (5.5) holds with $x_0 = 0$. Then*

$$E\left[\varphi\left(\sup_n \left\| \sum_{k=1}^{n} \xi_k u_k \right\|\right)\right] \leqslant E\left[\varphi\left(\alpha^{-1} \sup_n \left\| \sum_{k=1}^{n} \eta_k u_k \right\|\right)\right]; \tag{5.8}$$

if $\Sigma_{k=1}^{\infty} \eta_k u_k$ converges a.s., so does $\Sigma_{k=1}^{\infty} \xi_k u_k$, and in this case (5.8) holds with $\|\Sigma_{k=1}^{\infty}\|$ replacing $\sup_n \|\Sigma_{k=1}^{n}\|$.

If $x_0 > 0$ in (5.5), let a, b be any values for which (5.3) holds. Then

$$\begin{aligned} &E\left[\varphi\left(\sup_n \left\| \sum_{k=1}^{n} \xi_k u_k \right\|\right)\right] \\ &\leqslant \frac{1}{2} E\left[\varphi\left(\frac{2x_0}{\alpha a b} \sup_n \left\| \sum_{k=1}^{n} \eta_k u_k \right\|\right)\right] + \frac{1}{2} E\left[\varphi\left(\frac{2}{\alpha} \sup_n \left\| \sum_{k=1}^{n} \eta_k u_k \right\|\right)\right]. \end{aligned} \tag{5.9}$$

If $\Sigma_{k=1}^{\infty} \eta_k u_k$ converges a.s. so does $\Sigma_{k=1}^{\infty} \xi_k u_k$ and (5.9) holds with $\|\Sigma_{k=1}^{\infty}\|$ replacing $\sup_n \|\Sigma_{k=1}^{n}\|$.

All the proofs will be given at the end of this section.

The symmetry of $\{\xi_k\}$ is essential to our method of proof; however, in most cases we can remove the symmetry assumption on $\{\eta_k\}$. That this cannot always be done will be shown by examples later in this section.

In the general case, when the η_k need not be symmetric, we will need another uniform condition, that there exist a $\gamma > 0$ and a $\delta > 0$ such that

$$P[|\eta_k| < \gamma] \geqslant \delta, \qquad \forall k \geqslant 1. \tag{5.10}$$

We then obtain

THEOREM 5.4. *Let $\{u_k\}$ be a sequence of elements of B, and $\{\eta_k\}$ a sequence of independent, real-valued random variables on $(\Omega, \mathcal{F}, P)$ that are uniformly nondegenerate. Also assume that $\{\eta_k\}$ satisfies (5.10) for some $\gamma > 0$ and $\delta > 0$. Let $\{\xi_k\}$ be independent, symmetric, real-valued, random variables on $(\Omega, \mathcal{F}, P)$ such that, for some $x_0 \geqslant 0$, $0 < \alpha \leqslant 1$,*

$$P[|\eta_k| \geq x] \geq \alpha P[|\xi_k| \geq x] \tag{5.11}$$

for all $x \geq x_0$. *Then*

$$\sup_n \left\| \sum_{k=1}^n \eta_k u_k \right\| < \infty \text{ a.s.} \Rightarrow \sup_n \left\| \sum_{k=1}^n \xi_k u_k \right\| < \infty \text{ a.s.} \tag{5.12}$$

and

$$\sum_{k=1}^\infty \eta_k u_k \text{ converges a.s.} \Rightarrow \sum_{k=1}^\infty \xi_k u_k \text{ converges a.s.} \tag{5.13}$$

REMARK 5.5′. If we take $\xi_k = \epsilon_k \eta_k$, where $\{\epsilon_k\}$ is a Rademacher sequence in (Ω, F, P) independent of $\{\eta_k\}$, then Theorem 5.4 shows that if the η_k satisfy (5.4) and (5.10), then $\Sigma_{k=1}^\infty \eta_k u_k$ converges a.s. (or is bounded a.s.) $\Rightarrow \Sigma_{k=1}^\infty \epsilon_k \eta_k u_k$ converges a.s. (or is bounded a.s.).

For use in an analogue of Theorem 5.3 we define

(5.14) $v_0 = \max(x_0, 2\gamma)$,

where x_0 appears following (5.11) and γ in (5.10).

THEOREM 5.5. *Let φ be a nonnegative, nondecreasing, convex function on* $[0, \infty)$. *Let $\{u_k\}$, $\{\eta_k\}$ and $\{\xi_k\}$ be defined as in Theorem* 5.4. *Then for c, d any values for which* (5.4) *holds, we have*

$$\begin{aligned} &2E\left[\varphi\left(\sup_n \left\| \sum_{k=1}^n \xi_k u_k \right\|\right)\right] \\ &\leq E\left[\varphi\left(c_1 \sup_n \left\| \sum_{k=1}^n \eta_k u_k \right\|\right)\right] + E\left[\varphi\left(c_2 \sup_n \left\| \sum_{k=1}^n \eta_k u_k \right\|\right)\right], \end{aligned} \tag{5.15}$$

where $c_1 = 4v_0/\alpha c d\delta$ *and* $c_2 = 8/\alpha\delta$, *with* v_0 *given in* (5.14) *and* γ, δ *as in* (5.10).

Furthermore, if $\Sigma_{k=1}^\infty \eta_k u_k$ *converges a.s., then so does* $\Sigma_{k=1}^\infty \xi_k u_k$ *and* (5.15) *holds with* $\|\Sigma_{k=1}^\infty\|$ *replacing* $\sup_n \|\Sigma_{k=1}^n\|$.

There is also an anlogue of Corollary 5.2, the proof of which follows by symmetrization from (5.4) and Theorem 5.1.

THEOREM 5.6. *Let $\{u_k\}$ be a sequence of elements in B, and $\{\eta_k\}$ be independent, real-valued random variables on (Ω, F, P) that are uniformly nondegenerate. Let $\{\xi_k\}$ be independent, symmetric, real-valued random variables on (Ω, F, P) that are uniformly bounded. Then* (5.12) *and* (5.13) *hold.*

We will now give some examples to show why (5.4) and (5.10) cannot be dropped in Theorem 5.4.

In the first example the η_k are nondegenerate, but not uniformly nondegenerate. ((5.4) does not hold.) Let $B = R^1$ and

$u_k = 0$ for $k = 1, 2, 3$,

$u_{2k} = (\log k)^{-1}, k \geq 2$,

$u_{2k+1} = (-\log k)^{-1}, k \geq 2$.

Let $\{\eta_k\}$ be independent random variables with

$$P[\eta_k = 1 + 2^{-k}] = P[\eta_k = 1 - 2^{-k}] = ½.$$

Then $\Sigma_{k=1}^{\infty} \eta_k u_k$ converges a.s., but $\Sigma_{k=1}^{\infty} \epsilon_k \eta_k u_k$ does *not* converge a.s. if $\{\epsilon_k\}$ is a Rademacher sequence independent of $\{\eta_k\}$. Hence by Remark 5.5′, Theorem 5.4 cannot hold.

In the next example $\{\eta'_k\}$ is uniformly nondegenerate but (5.10) does not hold. Let $\{\eta_k\}$ and $\{u_k\}$ be as above, and $\eta'_k = 2^k \eta_k$, $u'_k = u_k 2^{-k}$, then $\Sigma_{k=1}^{\infty} \eta'_k u'_k$ converges a.s., but $\Sigma_{k=1}^{\infty} \epsilon_k \eta'_k u'_k$ again does not converge a.s. and Theorem 5.4 cannot hold.

The essential role of (5.3) in Theorem 5.1 can be demonstrated in a similar manner.

We begin the proofs with a series of lemmas. The first lemma is an extension of [3, Lemma 4.1] (see remark below); our method of proof follows Kahane [5, p. 18].

LEMMA 5.7. *Let φ be a nonnegative, nondecreasing, convex function defined on $[0, \infty)$ and $\lambda_1, \lambda_2, \ldots$ real constants such that $\sup_k |\lambda_k| \leq c$. Let $\{u_k\}$ be a sequence of elements in B and $\{\eta_k\}$ independent, symmetric, real-valued random variables on (Ω, F, P). Then*

$$E\left[\varphi\left(\left\|\sum_{k=1}^{n} \lambda_k \eta_k u_k\right\|\right)\right] \leq E\left[\varphi\left(c\left\|\sum_{k=1}^{n} \eta_k u_k\right\|\right)\right] \tag{5.16}$$

for each n, and

$$E\left[\varphi\left(\sup_n \left\|\sum_{k=1}^{n} \lambda_k \eta_k u_k\right\|\right)\right] \leq E\left[\varphi\left(c \sup_n \left\|\sum_{k=1}^{n} \eta_k u_k\right\|\right)\right]. \tag{5.17}$$

PROOF. We will prove (5.17); the proof of (5.16) is similar. *Case* (i). $\lambda_k = 0$ *or* 1. In this case

$$\sum_{k=1}^{n} \lambda_k \eta_k u_k = \frac{1}{2}\left(\sum_{k=1}^{n} (2\lambda_k - 1)\eta_k u_k + \sum_{k=1}^{n} \eta_k u_k\right).$$

Therefore

$$\sup_n \left\|\sum_{k=1}^{n} \lambda_k \eta_k u_k\right\| \leq \frac{1}{2}\left(\sup_n \left\|\sum_{k=1}^{n} (2\lambda_k - 1)\eta_k u_k\right\| + \sup_n \left\|\sum_{k=1}^{n} \eta_k u_k\right\|\right).$$

Since φ is convex and nondecreasing, we get

$$\varphi\left(\sup_n \left\|\sum_{k=1}^{n} \lambda_k \eta_k u_k\right\|\right) \leq \frac{1}{2}\left[\varphi\left(\sup_n \left\|\sum_{k=1}^{n} (2\lambda_k - 1)\eta_k u_k\right\|\right) + \varphi\left(\sup_n \left\|\sum_{k=1}^{n} \eta_k u_k\right\|\right)\right].$$

Since the η_k are symmetric $\Sigma_{k=1}^{n}(2\lambda_k - 1)\eta_k u_k$ and $\Sigma_{k=1}^{n} \eta_k u_k$ are stochastically similar, consequently

$$E\left[\varphi\left(\sup_n\left\|\sum_{k=1}^n \lambda_k\eta_k u_k\right\|\right)\right] \leqslant E\left[\varphi\left(\sup_n\left\|\sum_{k=1}^n \eta_k u_k\right\|\right)\right]. \tag{5.18}$$

Case (ii). $0 \leqslant \lambda_k \leqslant 1$. (Since the η_k are symmetric we may take $\lambda_k \geqslant 0$, c can clearly be taken equal to 1 without any loss of generality.) Writing λ_k in binary expansion

$$\lambda_k = \sum_{i=1}^{\infty} 2^{-i}\lambda_{ki}, \qquad \lambda_{ki} = 1 \text{ or } 0,$$

we have

$$\sum_{k=1}^n \lambda_k\eta_k u_k = \sum_{k=1}^n\left(\sum_{i=1}^{\infty} 2^{-i}\lambda_{ki}\right)\eta_k u_k = \sum_{i=1}^{\infty} 2^{-i}\left(\sum_{k=1}^n \lambda_{ki}\eta_k u_k\right).$$

Again, since φ is convex and nondecreasing,

$$\varphi\left(\sup_n\left\|\sum_{k=1}^n \lambda_k\eta_k u_k\right\|\right) \leqslant \sum_{i=1}^{\infty} 2^{-i}\varphi\left(\sup_n\left\|\sum_{k=1}^n \lambda_{ki}\eta_k u_k\right\|\right).$$

Now taking expectations and using (5.18) we obtain (5.17) and the proof is complete.

REMARK 5.8. Lemma 5.7 is also valid with $\{\eta_k u_k\}$ replaced by $\{X_k\}$, where X_k are independent symmetric B-valued random variables.

LEMMA 5.9. *Let $\{u_k\}$ be a sequence of elements in B and $\{\eta_k\}$ independent, symmetric, real-valued random variables on (Ω, F, P) that are uniformly nondegenerate and uniformly bounded. Assume that*

$$M = \sup_n\left\|\sum_{k=1}^n \eta_k u_k\right\| < \infty \ \text{a.s.} \tag{5.19}$$

Then for any sequence $\{\xi_k\}$ of independent, symmetric, real-valued random variables on (Ω, F, P) that are uniformly bounded we have

$$M' = \sup_n\left\|\sum_{k=1}^n \xi_k u_k\right\| < \infty \ \text{a.s.}, \tag{5.20}$$

and $E(M) < \infty$, $E(M') < \infty$.

Furthermore, if $\Sigma_{k=1}^{\infty} \eta_k u_k$ converges a.s., then so does $\Sigma_{k=1}^{\infty} \xi_k u_k$ and both $\|\Sigma_{k=1}^{\infty} \eta_k u_k\|$ and $\|\Sigma_{k=1}^{\infty} \xi_k u_k\|$ have finite expectations.

PROOF. Since the η_k are uniformly nondegenerate, (5.19) implies that $\sup\|u_k\| \leqslant c < \infty$ for some constant c. Since the η_k are uniformly bounded we see from Theorem 3.3 that $E(M) < \infty$ and the condition on uniform nondegeneracy is equivalent to

$$E(|\eta_k|) = \beta_k > \beta > 0 \quad \text{for some } \beta. \tag{5.21}$$

Let $\{\epsilon_k\}$ be a Rademacher sequence as in Remark 5.0. Notice that $\{\eta_k u_k\}$ and $\{\epsilon_k |\eta_k| u_k\}$ are stochastically similar. Hence (recall Remark 5.0)

$$E\left(\sup_n \left\|\sum_{k=1}^n \eta_k u_k\right\|\right) = E_2\left(E_1\left(\sup_n \left\|\sum_{k=1}^n \epsilon_k |\eta_k| u_k\right\|\right)\right) \tag{5.22}$$
$$\geqslant E_2\left(\sup_n \left\|\sum_{k=1}^n \epsilon_k \beta_k u_k\right\|\right).$$

Since $E(M) < \infty$, (5.22) implies

$$E\left(\sup_n \left\|\sum_{k=1}^n \epsilon_k \beta_k u_k\right\|\right) < \infty. \tag{5.23}$$

By Lemma 5.7 ($\beta_k u_k$ replacing u_k, ϵ_k replacing η_k, and $\lambda_k = \beta_k^{-1} \leqslant \beta^{-1}$ by (5.21)) we get

$$E\left(\sup_n \left\|\sum_{k=1}^n \epsilon_k u_k\right\|\right) < \infty. \tag{5.24}$$

To go from $\{\epsilon_k\}$ to $\{\xi_k\}$ in (5.24) we make essentially the same argument as above. By Lemma 5.7

$$E_2\left(\sup_n \left\|\sum_{k=1}^n \epsilon_k |\xi_k| u_k\right\|\right) \leqslant c_1 E_2\left(\sup_n \left\|\sum_{k=1}^n \epsilon_k u_k\right\|\right), \tag{5.25}$$

where $\sup_k |\xi_k| \leqslant c_1$ a.s. (P_1). Using (5.24) and taking expectation with respect to P_1 in (5.25) shows that $E(M') < \infty$, which of course implies (5.26). This finishes the proof of the first part of the lemma.

If $\Sigma_{k=1}^\infty \eta_k u_k$ converges a.s., then we can complete the proof of the lemma if we show that $\Sigma_{k=1}^\infty \xi_k u_k$ converges a.s. The reason being that we can then use Lemma 2.1 to show that M and M' are finite a.s. We have just proved that this implies M and M' have finite expectations, which in turn implies, by the dominated convergence theorem, that $\|\Sigma_{k=1}^\infty \eta_k u_k\|$ and $\|\Sigma_{k=1}^\infty \xi_k u_k\|$ have finite expectations.

Assume that $\Sigma_{k=1}^\infty \eta_k u_k$ converges a.s., as above, the fact that the η_k are uniformly nondegenerate implies $\|u_k\| \leqslant c < \infty$ uniformly in k. It follows from Theorem 3.3 that

$$\lim_{n\to\infty} E\left(\left\|\sum_{k=n}^\infty \eta_k u_k\right\|\right) = 0. \tag{5.26}$$

By Lemma 2.1 this implies (via integration by parts) that

$$\lim_{n\to\infty} E\left(\sup_{m \geqslant n} \left\|\sum_{k=n}^m \eta_k u_k\right\|\right) = 0. \tag{5.27}$$

Introducing $\{\epsilon_k\}$ as in Remark 5.0 and writing $\beta_k = E|\eta_k|$, the argument that leads to (5.23) shows that (5.27) implies

$$\lim_{n\to\infty} E\left(\sup_{m\geq n}\left\|\sum_{k=n}^{m} \epsilon_k\beta_k u_k\right\|\right) = 0. \tag{5.28}$$

Since (5.21) still applies, we use Lemma 5.7 with $\lambda_k = |\xi_k(\omega)| \leq c_1\beta^{-1}$, where $|\xi_k(\omega)| \leq c_1$ a.s. for all k, and get

$$E\left(\sup_{m\geq n}\left\|\sum_{k=n}^{m} \epsilon_k\beta_k u_k\right\|\right) \geq \beta c_1^{-1} E\left(\sup_{m\geq n}\left\|\sum_{k=n}^{m} \epsilon_k|\xi_k|u_k\right\|\right),$$

therefore by (5.28) we have

$$\lim_{n\to\infty} E\left(\sup_{m\geq n}\left\|\sum_{k=n}^{m} \xi_k u_k\right\|\right) = 0.$$

Hence there exists a sequence of integers $\eta_k \nearrow \infty$ such that

$$P\left(\sup_{m\geq n_k}\left\|\sum_{k=n_k}^{m} \xi_k u_k\right\| > 2^{-k}\right) \leq 2^{-k}.$$

This implies that $\Sigma_{k=1}^{\infty} \xi_k u_k$ converges a.s. by the Borel-Cantelli lemma. The proof of the lemma is complete.

LEMMA 5.10. *Let $\{u_k\}$ be a sequence of elements in B and $\{\eta_k\}$ be a sequence of independent, symmetric, real-valued random variables on $(\Omega, \mathcal{F}, P)$. Let $\{\xi_k\}$ be a sequence of independent, symmetric, real-valued random variables on $(\Omega, \mathcal{F}, P)$ satisfying*

$$P[|\eta_k| \geq x] \geq P[|\xi_k| \geq x] \tag{5.29}$$

for all $x \geq 0$. Then

$$\sup_n \left\|\sum_{k=1}^{n} \eta_k u_k\right\| < \infty \text{ a.s.} \Rightarrow \sup_n \left\|\sum_{k=1}^{n} \xi_k u_k\right\| < \infty \text{ a.s.} \tag{5.30}$$

and

$$\sum_{k=1}^{\infty} \eta_k u_k \text{ converges a.s.} \Rightarrow \sum_{k=1}^{\infty} \xi_k u_k \text{ converges a.s.} \tag{5.31}$$

PROOF. The main idea in the proof is to define $\{\eta_k\}$ and $\{\xi_k\}$ on the same probability space in such a way that $|\xi_k| \leq |\eta_k|$ a.s. and the finite-dimensional distributions of each sequence are preserved. Let $F_k(x) = P[|\eta_k| \leq x]$, $G_k(x) = P[|\xi_k| \leq x]$, and for $y \in [0, 1]$, let $f_k(y) = \inf\{x\colon F_k(x) \geq y\}$, $g_k(y) = \inf\{x\colon G_k(x) \geq y\}$. In Remark 5.0 let $(\Omega_1, \mathcal{F}_1, P_1)$ be such that on it are defined a Rademacher sequence $\{\epsilon_k'\}$ and a sequence of independent random variables $\{\psi_k\}$, which is independent of $\{\epsilon_k'\}$, such that, for $0 \leq u \leq 1$, $k \geq 1$, $P_1(\psi_k \leq u) = u$. We then define, for $k \geq 1$, $\tilde{\eta}_k = f_k(\psi_k)$, $\tilde{\xi}_k = g_k(\psi_k)$. Clearly $\{\epsilon_k'\tilde{\eta}_k\}$ is stochastically equivalent to $\{\eta_k\}$ and $\{\epsilon'_k\tilde{\xi}_k\}$ stochastically equivalent to $\{\xi_k\}$. Furthermore, given $k \geqslant 1$ we have by (5.29), $f_k(x) \geqslant g_k(x)$ and $|\epsilon'_k\tilde{\xi}_k| = |\tilde{\xi}_k| \leqslant |\tilde{\eta}_k| = |\epsilon'_k\tilde{\eta}_k|$ a.s. Hence, without loss of generality, we may assume that $\{\eta_k\}$ and $\{\xi_k\}$ are defined on (Ω_1, F_1, P_1) and for $k \geqslant 1$

$$|\xi_k| \leqslant |\eta_k| \text{ a.s.} \tag{5.32}$$

Let $\{\epsilon_k\}$ be a Rademacher sequence on (Ω_2, F_2, P_2) and let $(\Omega, F, P) = (\Omega_1 \times \Omega_2, F_1 \times F_2, P_1 \times P_2)$ as in Remark 5.0. We then have

$$\sup_n \left\| \sum_{k=1}^n \eta_k u_k \right\| < \infty \text{ a.s.} \Rightarrow \sup_n \left\| \sum_{k=1}^n \epsilon_k \eta_k u_k \right\| < \infty \text{ a.s.}$$

Hence for almost all $\omega_1 \in \Omega_1$ (with respect to P_1),

$$\sup_n \left\| \sum_{k=1}^n \epsilon_k \eta_k(\omega_1) u_k \right\| < \infty \text{ a.s.}$$

We now apply Lemma 5.9 with $\eta_k = \epsilon_k$, u_k replaced by $\eta_k(\omega_1)u_k$ and ξ_k replaced by $\epsilon_k\xi_k(\omega_1)\eta_k(\omega_1)^{-1}$ ($= 0$ if $\eta_k(\omega_1) = \xi_k(\omega_1) = 0$). Since ξ_k is symmetric and uniformly bounded, by (5.32) we conclude that

$$\sup_n \left\| \sum_{k=1}^n \epsilon_k \xi_k(\omega_1) u_k \right\| < \infty \text{ a.s. } (P_2)$$

for almost all $\omega_1 \in \Omega_1(P_1)$. This shows (5.30). The proof for (5.31) is similar.

PROOF OF THEOREM 5.1. To begin let us assume that $x_0 = 0$ in (5.5). Let (Ω_2, F_2, P_2) be a probability space on which are defined a Rademacher sequence $\{\epsilon_n\}$ and a sequence $\{\alpha_k\}$, independent of $\{\epsilon_n\}$, of independent, identically distributed random variables such that $P_2(\alpha_1 = 1) = \alpha$, $P_2(\alpha_1 = 0) = 1 - \alpha$. Note that (Remark 5.0) $\{\eta_k\}$ and $\{\xi_k\}$ are on (Ω_1, F_1, P_1). Form the product space $(\Omega_1 \times \Omega_2, F_1 \times F_2, P_1 \times P_2)$, so that $\{\eta_k\}$ and $\{\xi_k\}$ are independent of $\{\epsilon_k\}$ and $\{\alpha_k\}$. Let $\xi'_k = \alpha_k\xi_k$, $k \geqslant 1$. We have, for $x > 0$, $P[|\xi'_k| \geqslant x] = \alpha P[|\xi_k| \geqslant x]$. Therefore by (5.5) we get (with $x_0 = 0$) $P[|\eta_k| \geqslant x] \geqslant P[|\xi'_k| \geqslant x]$ for all $x \geqslant 0$. Assume $\sup_n \|\Sigma_{k=1}^n \eta_k u_k\| < \infty$ a.s. By Lemma 5.10 this implies

$$\sup_n \left\| \sum_{k=1}^n \alpha_k \xi_k u_k \right\| < \infty \text{ a.s.} \tag{5.33}$$

Since $\{\alpha_k \xi_k\}$ and $\{\epsilon_k\alpha_k|\xi_k|\}$ are stochastically equivalent, we may replace $\alpha_k\xi_k$ by $\epsilon_k\alpha_k|\xi_k|$ in (5.33) and by Fubini's theorem for almost all $\omega_1 \in \Omega_1$ (with respect to P_1)

$$(5.34)\qquad \sup_n \left\| \sum_{k=1}^{n} \epsilon_k \alpha_k |\xi_k(\omega_1)| u_k \right\| < \infty \text{ a.s. } (P_2).$$

Since $\{\alpha_k \epsilon_k\}$ is uniformly nondegenerate, we apply Lemma 5.9 with $\eta_k = \alpha_k \epsilon_k$, $|\xi_k(\omega_1)| u_k$ for u_k and ϵ_k for ξ_k to conclude that

$$(5.35)\qquad \sup_n \left\| \sum_{k=1}^{n} \epsilon_k |\xi_k(\omega_1)| u_k \right\| < \infty \text{ a.s. } (P_2)$$

for almost all $\omega_1 \in \Omega_1$. Hence

$$(5.36)\qquad \sup_n \left\| \sum_{k=1}^{n} \xi_k u_k \right\| < \infty \text{ a.s.}$$

This proves (5.6) when $x_0 = 0$ in (5.5).

If $x_0 > 0$ in (5.5), let

$$\zeta_k = \begin{cases} \eta_k & \text{if } |\eta_k| > x_0, \\ 0 & \text{otherwise;} \end{cases} \qquad \zeta_k' = \begin{cases} x_0 & \text{if } 0 < \eta_k \leqslant x_0, \\ -x_0 & \text{if } -x_0 \leqslant \eta_k < 0, \\ 0 & \text{otherwise.} \end{cases}$$

By (5.5) we have $P[|\zeta_k + \zeta_k'| \geqslant x] \geqslant \alpha P[|\xi_k| \geqslant x]$ for all $x \geqslant 0$. So if we can show that $\sup_n \|\Sigma_{k=1}^n \eta_k u_k\| < \infty$ a.s. implies

$$(5.37)\qquad \sup_n \left\| \sum_{k=1}^{n} (\zeta_k + \zeta_k') u_k \right\| < \infty$$

then we will have by what we showed above that

$$(5.38)\qquad \sup_n \left\| \sum_{k=1}^{n} \xi_k u_k \right\| < \infty \text{ a.s.}$$

Assume $\sup_n \|\Sigma_{k=1}^n \eta_k u_k\| < \infty$ a.s. Since $P[|\eta_k| \geqslant x] \geqslant \alpha P[|\zeta_k| \geqslant x]$ for all $x \geqslant 0$, we have

$$(5.39)\qquad \sup_n \left\| \sum_{k=1}^{n} \zeta_k u_k \right\| < \infty \text{ a.s.}$$

Hence it is enough to show that

$$(5.40)\qquad \sup_n \left\| \sum_{k=1}^{n} \zeta_k' u_k \right\| < \infty \text{ a.s.}$$

For this we use the uniform nondegeneracy of the η_k's. Let a, b be numbers for which (5.3) holds. Let

$$\zeta_k'' = \begin{cases} a & \text{if } \eta_k \geqslant a, \\ -a & \text{if } \eta_k \leqslant -a, \\ 0 & \text{otherwise.} \end{cases}$$

Now the ζ_k'' are independent, symmetric random variables and (by (5.3)) $P[|\eta_k| \geqslant x] \geqslant bP[|\zeta_k''| \geqslant x]$ for all $x \geqslant 0$; hence again by the special case considered above we have

$$\sup_n \left\| \sum_{k=1}^{n} \zeta_k'' u_k \right\| < \infty \quad \text{a.s.} \tag{5.41}$$

Since the ζ_k'' are also uniformly nondegenerate and uniformly bounded, we now conclude by Lemma 5.9 that (5.41) implies (5.40). This proves (5.6). The assertion (5.7) follows by similar arguments which are omitted.

We need the following extension of Lemma 5.7 (5.16) in the proof of Theorem 5.3.

LEMMA 5.11. *Given the hypotheses of Lemma 5.7 assume that* $\Sigma_{k=1}^{\infty} \eta_k u_k$ *converges a.s. Then* $\Sigma_{k=1}^{\infty} \lambda_k \eta_k u_k$ *converges a.s. and*

$$E\left[\varphi\left(\left\| \sum_{k=1}^{\infty} \lambda_k \eta_k u_k \right\|\right)\right] \leqslant E\left[\varphi\left(c\left\| \sum_{k=1}^{\infty} \eta_k u_k \right\|\right)\right].$$

PROOF. Since $\Sigma_{k=1}^{\infty} c\eta_k u_k$ converges a.s., by Theorem 5.1 $\Sigma_{k=1}^{\infty} \lambda_k \eta_k u_k$ converges a.s. because $P[|c\eta_k| \geqslant x] \geqslant P[|\lambda_k \eta_k| \geqslant x]$ for all $x \geqslant 0$. Once we know that both series converge a.s. we can use the proof of Lemma 5.7 with $\|\Sigma_{k=1}^{\infty}\|$ replacing $\sup_n \|\Sigma_{k=1}^{n}\|$.

PROOF OF THEOREM 5.3. *Case* (i). $x_0 = 0, \alpha = 1$. As in the proof of Lemma 5.10 we can assume that both $\{\eta_k\}$ and $\{\xi_k\}$ are defined on the same probability space (Ω_1, F_1, P_1) and that, for all $k \geqslant 1$, $|\eta_k| \geqslant |\xi_k|$ a.s. We define a Rademacher sequence $\{\epsilon_n\}$ on a probability space (Ω_2, F_2, P_2) and form the product space $(\Omega_1 \times \Omega_2, F_1 \times F_2, P_1 \times P_2)$ as before. We have

$$E\left[\varphi\left(\sup_n \left\| \sum_{k=1}^{n} \eta_k u_k \right\|\right)\right] = E_1\left[E_2\left\{\varphi\left(\sup_n \left\| \sum_{k=1}^{n} \epsilon_k |\eta_k(\omega_1)| u_k \right\|\right)\right\}\right], \tag{5.42}$$

where $\omega_1 \in \Omega_1$ is held fixed when E_2 operates. We now apply Lemma 5.7 (5.16) with $\lambda_k = |\xi_k(\omega_1)||\eta_k(\omega_1)|^{-1}$ ($= 0$ if $\xi_k(\omega_1) = \eta_k(\omega_1) = 0$). Since $|\lambda_k| \leqslant 1$ we see that the right-hand side in (5.42) dominates

$$E_1\left[E_2\left\{\varphi\left(\sup_n \left\| \sum_{k=1}^{n} \epsilon_k |\xi_k(\omega_1)| u_k \right\|\right)\right\}\right] = E\left[\varphi\left(\sup_n \left\| \sum_{k=1}^{n} \xi_k u_k \right\|\right)\right].$$

This proves Case (i).

Case (ii). $x_0 = 0, 0 < \alpha \leqslant 1$. Let independent random variables $\alpha_k, k \geqslant 1$, be defined on (Ω_2, F_2, P_2) such that $P_2(\alpha_k = 1) = \alpha, P_2(\alpha_k = 0) = 1 - \alpha$, $k \geq 1$, and let $\xi'_k = \alpha_k \xi_k$. Since, for all $x \geq 0, P[|\eta_k| \geq x] \geq P[|\xi'_k| \geq x]$ we apply Case (i) to get $(E_2(\alpha_k) = \alpha)$

$$\begin{aligned} E\left[\varphi\left(\sup_n \alpha^{-1}\left\|\sum_{k=1}^n \eta_k u_k\right\|\right)\right] &\geq E\left[\varphi\left(\sup_n \alpha^{-1}\left\|\sum_{k=1}^n \xi'_k u_k\right\|\right)\right] \\ &= E_1\left[E_2\left(\varphi\left(\sup_n \alpha^{-1}\left\|\sum_{k=1}^n \alpha_k\xi_k u_k\right\|\right)\right)\right] \\ &\geq E_1\left[\varphi\left(\sup_n \alpha^{-1}\left\|\sum_{k=1}^n E_2(\alpha_k)\xi_k u_k\right\|\right)\right] \\ &= E_1\left[\varphi\left(\sup_n \left\|\sum_{k=1}^n \xi_k u_k\right\|\right)\right], \end{aligned}$$

where Jensen's inequality applies since φ is convex. This completes the proof of (5.8). The assertion after (5.8) follows similarly by using Lemma 5.11 in place of Lemma 5.7.

We now consider the case $x_0 > 0$. Define ζ_k and ζ'_k as in the proof of Theorem 5.1. Then for all $x \geq 0$ we have

$$P[|\zeta_k + \zeta'_k| \geq x] \geq \alpha P[|\xi_k| \geq x]. \tag{5.43}$$

Applying the first part of this theorem $(x_0 = 0)$, we get

$$E\left[\varphi\left(\sup_n \left\|\sum_{k=1}^n \xi_k u_k\right\|\right)\right] \leq E\left[\varphi\left(\alpha^{-1} \sup_n \left\|\sum_{k=1}^n (\zeta_k + \zeta'_k) u_k\right\|\right)\right], \tag{5.44}$$

and using the facts that φ is nondecreasing and convex, we dominate the right side in (5.44) by

$$\frac{1}{2}E\left[\varphi\left(\frac{2}{\alpha}\sup_n \left\|\sum_{k=1}^n \zeta_k u_k\right\|\right)\right] + \frac{1}{2}E\left[\varphi\left(\frac{2}{\alpha}\sup_n \left\|\sum_{k=1}^n \zeta'_k u_k\right\|\right)\right]. \tag{5.45}$$

Since $|\zeta_k| \leq |\eta_k|$, and $|\zeta'_k| \leq x_0$, applying (5.8) to each part in the sum (5.45) we dominate (5.45) by

$$\frac{1}{2}E\left[\varphi\left(\frac{2}{\alpha}\sup_n \left\|\sum_{k=1}^n \eta_k u_k\right\|\right)\right] + \frac{1}{2}E\left[\varphi\left(\frac{2x_0}{\alpha}\sup_n \left\|\sum_{k=1}^n \epsilon_k u_k\right\|\right)\right], \tag{5.46}$$

where $\{\epsilon_k\}$ is a Rademacher sequence. (We know that $\sup_n \|\Sigma_{k=1}^n \epsilon_k u_k\| < \infty$ a.s. by Corollary 5.2.) Let $\delta = 2x_0\alpha^{-1}$. To get (5.9) it remains to check that

$$E\left[\varphi\left(\delta \sup_n \left\|\sum_{k=1}^n \epsilon_k u_k\right\|\right)\right] \leq E\left[\varphi\left(\frac{\delta}{ab}\sup_n \left\|\sum_{k=1}^n \eta_k u_k\right\|\right)\right]. \tag{5.47}$$

But this follows by observing that (5.3) implies $P[|\eta_k a^{-1}| \geqslant x] \geqslant bP[|\epsilon_k| \geqslant x]$ for *all* $x \geqslant 0, k \geqslant 1$.

For the analogue of (5.9) when $\Sigma_{k=1}^{\infty} \eta_k u_k$ converges a.s. we use the same proof, this time working with the assertion following (5.8). This completes the proof of Theorem 5.3.

The proofs for general η_k follow easily from the following lemma.

LEMMA 5.12. *Let η be a real-valued random variable satisfying* (5.10) *with γ and δ, and let η' be an independent copy of η, then* $P[|\eta - \eta'| \geqslant x/2] \geqslant \delta P[|\eta| \geqslant x]$ *for $x \geqslant 2\gamma$.*

PROOF. By (5.10) and independence of η, η'

$$\delta P[|\eta| \geqslant x] \leqslant P[|\eta| \geqslant x, |\eta'| < \gamma] \leqslant P[|\eta - \eta'| \geqslant x - \gamma] \leqslant P[|\eta - \eta'| \geqslant x/2]$$

for $x \geqslant 2\gamma$.

The proofs of Theorems 5.4, 5.5 and 5.6 follow by symmetrizing $\{\eta_k\}$ and applying the corresponding result in the symmetric case with the help of Lemma 5.12. We will give the details only for the proof of Theorem 5.4.

PROOF OF THEOREM 5.4. Suppose $\sup_n \|\Sigma_{k=1}^n \eta_k u_k\| < \infty$ a.s. Let $\{\eta'_k\}$ be an independent copy of $\{\eta_k\}$. Clearly

$$\sup_n \left\| \sum_{k=1}^n 2(\eta_k - \eta'_k) u_k \right\| < \infty \text{ a.s.}$$

The sequence $\{2(\eta_k - \eta'_k)\}$ is symmetric and satisfies (5.3) for some $a > 0$ and $b > 0$. Also by Lemma 5.12 and (5.11), $P[|2(\eta_k - \eta'_k)| \geqslant x] \geqslant \alpha\delta P[|\xi_k| \geqslant x]$ for $x \geqslant x_0$. We can apply Theorem 5.1 to $2(\eta_k - \eta'_k)$ and $\{\xi_k\}$ to obtain $\sup_n \|\Sigma_{k=1}^n \xi_k u_k\| < \infty$ a.s. This proves (5.12). A similar proof applies to (5.13).

6. **Applications.** We will now apply the results of the previous sections to a variety of problems in Gaussian processes and series of random functions.

6(a). *Gaussian processes.* We will derive a theorem of Landau and Shepp [6].

Let $\{\varphi_n\}$ be a sequence of elements of B and let $\{Y_n\}$ be a sequence of independent, identically distributed, random variables on $(\Omega, \mathcal{F}, P)$. For $\omega \in \Omega$ we write as before

$$S_n(\omega) = \sum_{j=1}^n \varphi_j Y_j(\omega). \tag{6.1}$$

We then have

THEOREM 6.1. *Suppose $M = \sup_n \|S_n\| < \infty$ a.s. If Y_1 is Gaussian with mean* 0, *variance* 1, *then there exists $\epsilon > 0$ such that $E(\exp(\epsilon M^2)) < \infty$.*

PROOF. By applying Theorem 3.8 we will first show that $E(\exp(\epsilon M)) < \infty$ for some $\epsilon > 0$. Therefore by (3.25), taking $\delta = 1$, we need to show that for some $\alpha > 0, a > 0$

$$\sum_{n=1}^{\infty} \frac{1}{\|\varphi_n\|} \int_a^{\infty} \exp\left(\alpha u \log^2 u - \frac{u^2}{2\|\varphi_n\|^2}\right) du < \infty. \tag{6.2}$$

(Note that $\|\varphi_n\| \neq 0$ may be assumed for all n.) Since $M < \infty$ a.s. it follows from Remark 3.2 that

$$\sum_{n=1}^{\infty} P(\|\varphi_n\| \, |Y_n| > \lambda) < \infty \tag{6.3}$$

for some $\lambda < 0$. Using the well-known Gaussian tail estimate (6.3) implies

$$\sum_{n=1}^{\infty} \frac{1}{\|\varphi_n\|} \exp(-\lambda^2/2\|\varphi_n\|^2) < \infty. \tag{6.4}$$

For $a \geqslant 1$ the sum in (6.2) is dominated by

$$\sum_{n=1}^{\infty} \frac{1}{\|\varphi_n\|} \int_a^{\infty} \exp\left\{-\frac{u^2}{2\|\varphi_n\|^2}(1 - \alpha\|\varphi_n\|^2)\right\} du \tag{6.5}$$

and since $\|\varphi_n\|$ must be bounded, the sum in (6.5) converges for suitable $\alpha > 0$ by (6.4). Hence we have for some $\epsilon > 0$

$$E(\exp(\epsilon M)) < \infty. \tag{6.6}$$

The proof of the theorem is now completed by the following argument which was shown to us by S. R. S. Varadhan.

Let $\{Y_{n,j}, n \geqslant 1\}, j \geqslant 1$, be independent copies of the sequence $\{Y_n, n \geqslant 1\}$, and let $S_{n,j} = \Sigma_{k=1}^n \varphi_k Y_{k,j}$. Then for $a > 0$, m a positive integer

$$P(\|S_n\| > a\sqrt{m}) = P\left(\left\|\frac{S_{n,1} + \cdots + S_{n,m}}{\sqrt{m}}\right\| > a\sqrt{m}\right), \tag{6.7}$$

by noting that the $Y_{n,j}$'s are independent $N(0, 1)$ and hence S_n has the same distribution as $(S_{n,1} + \ldots + S_{n,m})m^{-1/2}$. Using Chebychev's inequality we get

$$P(\|S_{n,1} + \cdots + S_{n,m}\| > am) \leqslant [E(\exp(\epsilon\|S_n\|))]^m \exp(-\epsilon am). \tag{6.8}$$

Since $\|S_n\| \leqslant M$, combining (6.6), (6.7) and (6.8) we get

$$P(\|S_n\| > a\sqrt{m}) \leqslant [E(\exp(\epsilon M))]^m \exp(-\epsilon am) \tag{6.9}$$

where ϵ does not depend on n. Hence by taking a sufficiently large, we get $0 < \rho < 1$, independent of k, such that, for $m \geqslant 1$,

$$P(\|S_n\| > a\sqrt{m}) \leqslant \rho^m. \tag{6.10}$$

This implies that $\exists \epsilon_1 > 0, A < \infty$, independent of n, such that

$$E(\exp(\epsilon_1 \|S_n\|^2)) \leqslant A. \tag{6.11}$$

Since S_n are symmetric, by Lemma 2.1, (6.11) gives

$$E(\exp(\epsilon_1 \|M_n\|^2)) \leqslant 2A. \tag{6.12}$$

The result now follows by the monotone convergence theorem.

The following corollary of the above theorem was proved by Landau and Shepp [6]. For a more general setting and a different proof see Fernique [2] (see also [9]).

COROLLARY 6.2. *Let $\{X_n, n \geqslant 1\}$ be an arbitrary sequence of real-valued Gaussian random variables on (Ω, F, P). Let $Z = \sup_n |X_n|$. Then*

$$Z < \infty \text{ a.s.} \Rightarrow E(\exp(\epsilon Z^2)) < \infty$$

for some $\epsilon > 0$.

PROOF. Let $T = \{1, 2, \ldots\}$ with the discrete metric, i.e. $d(i, j) = 1$ (0) for $i \neq j$ $(i = j)$, d denoting the metric. Take $B = C(T) =$ all real bounded sequences with the norm on B the sup norm. We can use the Gram-Schmidt process, and for each $n \in T$, write

$$X_n = \sum_{j=1}^{\infty} \varphi_j(n) Y_j, \tag{6.13}$$

where the Y_j are mutually orthogonal $N(0, 1)$ random variables, and $\varphi_j(n)$ fixed real numbers determined by the joint distributions of the X_n, and

$$E(X_n^2) = \sum_{j=1}^{\infty} \varphi_j(n)^2. \tag{6.14}$$

Since the Y_j's are uncorrelated and Gaussian, they are mutually independent. Hence the series in (6.13) converges a.s. for each n. By Lemma 2.1, for all $a > 0$ and N positive integer,

$$P\left(\sup_{k \geqslant 1} \sup_{1 \leqslant n \leqslant N} \left|\sum_{j=1}^{k} \varphi_j(n) Y_j\right| > a\right) \leqslant 2P\left(\sup_{1 \leqslant n \leqslant N} \left|\sum_{j=1}^{\infty} \varphi_j(n) Y_j\right| > a\right).$$

Letting $N \to \infty$, we have for all $a > 0$

$$P\left(\sup_{N \geqslant 1} \sup_{k \geqslant 1} \sup_{1 \leqslant n \leqslant N} \left|\sum_{j=1}^{\infty} \varphi_j(n) Y_j\right| > a\right) \leqslant 2P\left(\sup_{N \geqslant 1} |X_N| > a\right),$$

and since $\sup_{N \geqslant 1} |X_N| = Z < \infty$ a.s. by assumption, we have

$$\sup_{N \geqslant 1} \sup_{k \geqslant 1} \sup_{1 \leqslant n \leqslant N} \left|\sum_{j=1}^{k} \varphi_j(n) Y_j\right| < \infty \text{ a.s.} \tag{6.15}$$

The expression in (6.15) equals $\sup_{k \geqslant 1} \sup_{n \geqslant 1} |\Sigma_{j=1}^{k} \varphi_j(n) Y_j| = M$, in the notation of Theorem 6.1. Hence $Z < \infty$ a.s. implies $M < \infty$ a.s. and Theorem 6.1 applies to give an $\epsilon > 0$ such that $E(\exp(\epsilon M^2)) < \infty$; since $Z \leqslant M$, the corollary follows.

6(b). *Sums of independent Banach space valued random variables.* The following theorem is an immediate corollary of Corollary 3.5.

THEOREM 6.3. *Let $\{X_k\}$ be independent random variables with values in a separable Banach space B. Let $\{a_k\} \in l^p$, $0 < p < \infty$. Assume that $\Sigma a_k X_k$ converges a.s. and that $E(\|X_k\|^p) \leqslant c < \infty$ for all k. Then $E(\|\Sigma a_k X_k\|^p) < \infty$.*

6(c). *Random Fourier series.* For $0 \leqslant t \leqslant 2\pi$ consider

$$\xi(t) = \sum_{n=0}^{\infty} a_n \eta_n \cos(nt + \Phi_n) \tag{6.16}$$

where $\{\eta_n e^{i\Phi_n}\}$ is a sequence of independent complex valued random variables (η_n and Φ_n are real random variables); $E(\eta_n^2) \equiv 1$ and $\{a_n\} \in l^2$. Since $\{a_n\} \in l^2$, for a fixed t the sum in (6.16) converges a.s. In the study of these series it is natural to require that $\{a_n\} \in l^2$.

THEOREM 6.4. *Assume that the series in* (6.16) *converges uniformly a.s. and let $\|\xi\| = \sup_t |\xi(t)|$. Then*

(i) *$E(|\eta_n|^p) \leqslant c < \infty$, $2 \leqslant p < \infty$, implies $E(\|\xi\|^p) < \infty$.*

(ii) *$P(|\eta_n| > u) \leqslant \exp(-\alpha u \log^{1+\delta} u)$ for some $\alpha > 0$, $\delta > 0$, and all $u \geqslant 1$ implies $E(\exp(\epsilon\|\xi\|)) < \infty$ for some $\epsilon > 0$.*

(iii) *$|\eta_n| \leqslant A < \infty$ a.s. implies $E(\exp(\lambda\|\xi\|)) < \infty$ for all $\lambda < \infty$.*

PROOF. (i) follows from Theorem 6.3, (ii) from Theorem 3.8 and (iii) from Theorem 3.11.

REMARK. An extension of Theorem 6.4 (iii), under additional hypotheses, is given in §6(d).

We now consider the question of whether the series in (6.16) converge uniformly a.s. To simplify the discussion we will assume that $\eta_n e^{i\Phi_n}$, $n = 0, 1, \ldots,$ are also symmetric. Let

$$\sigma^2(h) = \sum_{n=0}^{\infty} a_n^2 \sin^2 \frac{nh}{2},$$

$$\bar{\sigma}^2(h) = \text{nondecreasing rearrangement of } \sigma^2(h), \qquad h \in [0, 2\pi],$$

(see [4] for exact definition) and

$$I(\bar{\sigma}) = \int_0^{1/2} \frac{\bar{\sigma}(h)}{h(\log 1/h)^{1/2}}\, dh.$$

If $\{\eta_n\}$ are independent Gaussian random variables with $E\eta_n = 0$, $E\eta_n^2 = 1$ then $I(\bar{\sigma}) < \infty$ if and only if the series in (6.16) converges uniformly a.s. (see [7, §3] for further discussion and references). In [7] it is also shown that $I(\bar{\sigma}) < \infty$ implies that the series in (6.16) converges uniformly a.s. if the η_k are sub-Gaussian;

however, as far as we know, no inference has been made about the behavior of (6.16) for non-Gaussian η_k when $I(\bar{\sigma}) = \infty$. We can do this readily by applying Theorem 5.1.

THEOREM 6.5. *Consider the series in* (6.16) *with the additional assumption that* $\eta_n e^{i\Phi_n}$, $n = 0, 1, \ldots$, *are symmetric. Suppose there exist an* $\alpha > 0$ *and* x_0 *such that*

$$P[|\eta_k| \geqslant x] \geqslant \alpha \int_x^\infty e^{-u^2/2}\, du$$

for all $x \geqslant x_0$. *Then* $I(\bar{\sigma}) = \infty$ *implies that the series in* (6.16) *is unbounded a.s.*

PROOF. For $\Phi_n \equiv 0$ the proof is an immediate consequence of Theorem 5.1 and the above mentioned result for Gaussian Fourier series. This result extends to the general symmetric $\eta_n e^{i\Phi_n}$ by Theorem 3.1 in [7]. The dichotomy between uniform convergence and unboundedness can be found in [5, p. 49].

6(d). *Uniformly bounded random variables.* By strengthening the hypothesis of Theorem 4.8 we prove the results referred to in Remark 4.9.

THEOREM 6.6. *Let* $\{\eta_k\}$ *be independent standard Gaussian random variables and* $\{u_k\}$ *a sequence of elements in B. Let* $\{\xi_k\}$ *be independent symmetric real-valued random variables satisyfing* $\sup_k |\xi_k| \leqslant c < \infty$ *a.s. Then if* $\Sigma_{k=1}^\infty \eta_k u_k$ *converges a.s., so does* $\Sigma_{k=1}^\infty \xi_k u_k$ *and*

$$E(e^{\lambda\|\Sigma_{k=1}^\infty \xi_k u_k\|^2}) < \infty \tag{6.17}$$

for all real λ.

PROOF. The fact that $\Sigma_{k=1}^\infty \xi_k u_k$ converges a.s. follows immediately from Corollary 5.2. It follows from Fernique [2] that since $\Sigma_{k=1}^\infty \eta_k u_k$ converges a.s., there exist $\lambda_n > 0$ such that

$$E(e^{\lambda_n\|\Sigma_{k=n}^\infty \eta_k u_k\|^2}) < \infty, \tag{6.18}$$

where $\lambda_n \to \infty$ as $n \to \infty$. If the ξ_k are symmetric and satisfy (5.3) with a, b, let $v_0 = \max(2, 2c/ab)$. Clearly $P[|\eta_k| \geqslant x] > P[|\xi_k| \geqslant x]$ for $x > c$. Given any real λ we can pick n sufficiently large by (6.18) so that

$$E[e^{2v_0^2\lambda\|\Sigma_{k=n}^\infty \eta_k u_k\|^2}] < \infty.$$

It follows from (5.9) that

$$E[e^{2\lambda\|\Sigma_{k=n}^\infty \xi_k u_k\|^2}] < \infty. \tag{6.19}$$

Since $\Sigma_{k=1}^\infty \xi_k u_k$ converges a.s. implies that $\sup_k \|u_k\| \leqslant A$ (for some constant A), we have

$$E[e^{\lambda\|\Sigma_{k=1}^{n-1}\xi_k u_k\|^2}] \leqslant e^{\lambda n^2 A^2 c^2} < \infty, \tag{6.20}$$

and (6.19) and (6.20) give (6.17).

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