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WELL-ORDERING OF CERTAIN NUMERICAL POLYNOMIALS

BY

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ABSTRACT. An ordering is introduced in the set of numerical polynomials (in one variable) which is then shown to induce a well-ordering on a certain subset of numerical polynomials, namely those which occur as the differential dimension polynomials of differential algebraic varieties, or equivalently, those which come from initial subsets of $\mathbf{N}^m$.

Let $\mathbf{N}$ be the set of natural numbers (including 0), $\mathbf{Z}$ be the ring of rational integers and $\mathbf{Q}$ be the field of rational numbers. A *numerical polynomial* in the indeterminate X is a polynomial $\omega \in \mathbf{Q}[X]$ such that $\omega(s) \in \mathbf{N}$ for every sufficiently large $s \in \mathbf{N}$. The numerical polynomials considered in this note are those coming from initial subsets of $\mathbf{N}^m$ ($m \in \mathbf{N}$). Specifically we consider $\mathbf{N}^m$ as an ordered set relative to the product order, that is, $(a_1, \ldots, a_m) \leqslant (b_1, \ldots, b_m)$ in $\mathbf{N}^m$ if and only if $a_i \leqslant b_i$ in $\mathbf{N}$ for every i. A subset V of $\mathbf{N}^m$ is *initial* if for every $\mathbf{v}, \mathbf{v}' \in \mathbf{N}^m$ such that $\mathbf{v}' \leqslant \mathbf{v}$, $\mathbf{v} \in V$ implies $\mathbf{v}' \in V$.

REMARK 1. Let V be an initial subset of $\mathbf{N}^m$ and define $E = E(V)$ to be the set of minimal points of the complement of V in $\mathbf{N}^m$. Then E is finite since every infinite sequence of points in $\mathbf{N}^m$ has an infinite increasing subsequence. Moreover V is the set of all points of $\mathbf{N}^m$ that are not greater than or equal to any point of E.

Kolchin [1, Chapter 0, Lemma 16, p. 51] showed (in view of the above remark) that for any initial subset V of $\mathbf{N}^m$, there exists a numerical polynomial ω_V such that for every sufficiently large $s \in \mathbf{N}$, $\omega_V(s)$ is the number of points $(v_1, \ldots, v_m) \in V$ with $v_1 + \cdots + v_m \leqslant s$. If $\mathcal{V}$ is a finite sequence $V_1, \ldots, V_n$ of initial subsets $V_i \subset \mathbf{N}^{m_i}$, we define the *numerical polynomial* of $\mathcal{V}$ to be $\omega_{\mathcal{V}} = \omega_{V_1} + \cdots + \omega_{V_n}$. In this paper, we shall show that the set Ω of numerical polynomials of the form $\omega_{\mathcal{V}}$ for some such $\mathcal{V}$ is well-ordered relative to the total order introduced on the numerical polynomials as follows: we say $\omega_1 \leqslant \omega_2$ if $\omega_1(s) \leqslant \omega_2(s)$ for all sufficiently large $s \in \mathbf{N}$. The origin of this problem comes from differential algebra where numerical polynomials of the form $\omega_{\mathcal{V}}$ occur as the differential dimension polynomials of differential algebraic varieties (see

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Kolchin [1, Chapter III, §5]). Using this well-ordering property one can presumably prove certain results in differential algebra by induction on the differential dimension polynomial just as one can in algebraic geometry by induction on the dimension of an algebraic variety.

Let V be an initial subset of $\mathbf{N}^m$. We shall first develop some properties of V and, as a by-product, give a different proof of Kolchin's result mentioned earlier. From now on, v_i $(1 \leqslant i \leqslant m)$ will always denote the ith coordinate of an element $\mathbf{v}$ of $\mathbf{N}^m$. If $T \subset \mathbf{N}^m$, then $T(s)$ denotes the set of all points $\mathbf{t} \in T$ with $t_1 + \cdots + t_m \leqslant s$. Let $\mathbf{N}_m = \{1, \ldots, m\}$. Given a subset $J \subset \mathbf{N}_m$ and an element $\mathbf{a} \in \mathbf{N}^m$, we define the symbol $\mathbf{a}_J$ to be the set of all points $\mathbf{b} \in \mathbf{N}^m$ such that $b_j = a_j$ for all $j \notin J$. Observe that if $\mathbf{b} \in \mathbf{a}_J$, then $\mathbf{b}_J = \mathbf{a}_J$. A subset $K \subset \mathbf{N}^m$ is said to be *k-dimensional* if $K = \mathbf{a}_J$ for some $\mathbf{a} \in \mathbf{N}^m$ and $J = J(K) \subset \mathbf{N}_m$ with Card $J = k$. Call $J(K)$ the *direction* of K. A subset K of $\mathbf{N}^m$ is *properly k-dimensional* in V if K is a k-dimensional subset of V and is not contained in any $(k + 1)$-dimensional subset of V.

REMARK 2. For any two subsets $\mathbf{a}_J$, $\mathbf{a}'_{J'}$ of $\mathbf{N}^m$, we have $\mathbf{a}_J \subset \mathbf{a}'_{J'}$ if and only if $J \subset J'$ and $a'_{j'} = a_{j'}$ for all $j' \notin J'$.

REMARK 3. Let $\mathbf{a}_J$ be a k-dimensional subset of V and $\mathbf{b} \in \mathbf{N}^m$ such that $\mathbf{b} \leqslant \mathbf{a}$. Then $\mathbf{b}_J$ is a k-dimensional subset of V containing $\mathbf{b}$. In particular, if $\mathbf{b} \leqslant \mathbf{a}$ in $\mathbf{N}^m$ and $\mathbf{a}$ belongs to some k-dimensional subset of V, then $\mathbf{b}$ belongs to some k-dimensional subset of V.

Let V be an initial subset of $\mathbf{N}^m$ and let $d_k(V)$ be the number of subsets of $\mathbf{N}^m$ properly k-dimensional in V. Note that $d_m(V) \leqslant 1$ and equality holds if and only if $V = \mathbf{N}^m$, in which case $d_k(V) = 0$ for all $k \neq m$.

PROPOSITION 1. *Let V be an initial subset of $\mathbf{N}^m$. Then $d_k(V)$ is finite for all $k \in \mathbf{N}$.*

PROOF. Clearly $d_k(V) = 0$ for all $k > m$ and the case $k = m$ has been noted above. Assume $k < m$ and let $E = E(V)$ (see Remark 1). For each $i \in \mathbf{N}_m$, let $\bar{e}_i = \max e_i$ $(\mathbf{e} \in E)$. If $\mathbf{a}_J$ is properly k-dimensional in V, we claim that $a_j < \bar{e}_j$ for all $j \notin J$. For otherwise, there would exist a $j \notin J$ such that $a_j \geqslant \bar{e}_j$ and since $\mathbf{a}_J \subset V$, we would have, by Remark 1, $\mathbf{a}_J \subset \mathbf{a}_{J \cup \{j\}} \subset V$, a contradiction. It follows from our claim that for each subset J of $\mathbf{N}_m$ consisting of k elements, the number of subsets of the form $\mathbf{a}_J$ that are properly k-dimensional in V is $\leqslant \prod_{j \notin J} \bar{e}_j$. This proves the proposition.

The sequence $\{d_k(V)\}_{k \in \mathbf{N}}$ will be referred to as the *dimension sequence* of V.

PROPOSITION 2. *Let $\mathcal{J}$ be a set of subsets of $\mathbf{N}_m$. Let $K = \mathbf{a}_{J(K)}$ be a k-dimensional subset of an initial subset V of $\mathbf{N}^m$ such that for every $\mathbf{v} \in K$, $\mathbf{v}_J \subset V$*

for some $J \in \mathfrak{J}$. Then there exists $J_0 \in \mathfrak{J}$ such that $\mathbf{a}_{J(K) \cup J_0}$ is a subset of V containing K.

PROOF. The case $k = 0$ is trivial. Assume $k > 0$. We first show that there exists $J_0 \in \mathfrak{J}$ with $\mathbf{v}_{J_0} \subset V$ for all $\mathbf{v} \in K$. Otherwise, for each $J \in \mathfrak{J}$, there would be an element $\mathbf{v}(J)$ of K with $\mathbf{v}(J)_J \not\subset V$. Since $\mathfrak{J}$ is finite, there would exist $\mathbf{w} \in K$ with $\mathbf{w} \geqslant \mathbf{v}(J)$ for every $J \in \mathfrak{J}$. But then by Remark 3, $\mathbf{w}_J \not\subset V$ for every $J \in \mathfrak{J}$, which would contradict our assumption on K. Now for every $\mathbf{b} \in \mathbf{a}_{J(K) \cup J_0}$, the element $\mathbf{v}$ defined by $v_j = b_j$ if $j \in J(K)$ and $v_j = a_j$ otherwise belongs to K and $\mathbf{b} \in \mathbf{v}_{J_0}$. Hence $\mathbf{a}_{J(K) \cup J_0} = \bigcup_{\mathbf{v} \in K} \mathbf{v}_{J_0} \subset V$.

COROLLARY. *Let V be an initial subset of $\mathbf{N}^m$. A k-dimensional subset K of V is properly k-dimensional in V if and only if K contains some element that does not belong to any $(k + 1)$-dimensional subset of V.*

PROOF. The "if" part is clear. Suppose $K = \mathbf{a}_{J(K)}$ is a k-dimensional subset of V but no such element exists. By the proposition, there exists a subset J_0 of $\mathbf{N}_m$ with Card $J_0 = k + 1$ such that $\mathbf{a}_{J(K) \cup J_0}$ is a subset of V containing K. This shows that K is not properly k-dimensional in V and proves the corollary.

PROPOSITION 3. *Let V be an initial subset of $\mathbf{N}^m$ with dimension sequence $\{d_k(V)\}_{k \in \mathbf{N}}$. Then for any h $(0 \leqslant h \leqslant m)$ such that $d_h(V) \neq 0$, there exists an initial subset $V_1 \subset V$ such that $d_k(V_1) = d_k(V)$ if $k > h$, $d_h(V_1) = d_h(V) - 1$ and $d_k(V_1) = 0$ if $k < h$.*

PROOF. If $h = m$, we may take $V_1 = \varnothing$. Suppose $h < m$. Let $K = \mathbf{a}_{J(K)}$ be properly h-dimensional in V such that $\Sigma_{j \notin J(K)} a_j$ is maximum. Let $\mathfrak{J}_h$ be the set of subsets of $\mathbf{N}_m$ with h elements. Let V_1 be the set of all elements $\mathbf{v} \in V$ such that $\mathbf{v}_J \subset V$ for some $J \in \mathfrak{J}_h$ and $\mathbf{v}_J \neq K$. We first show that V_1 is initial. Suppose $\mathbf{v} \in V_1$, $\mathbf{v}' \in \mathbf{N}^m$ and $\mathbf{v}' \leqslant \mathbf{v}$. Then $\mathbf{v}_J \subset V$ for some $J \in \mathfrak{J}_h$ and $\mathbf{v}_J \neq K$. By Remark 3, $\mathbf{v}'_J \subset V$. If $\mathbf{v}'_J = K$, we would have $J = J(K)$ and $\Sigma_{j \notin J} v_j \geqslant \Sigma_{j \notin J} v'_j = \Sigma_{j \notin J(K)} a_j$. Moreover $\mathbf{v}'_J$, and by Remark 3 also $\mathbf{v}_J$, would be properly h-dimensional in V. By the choice of K, this would imply that $v_j = v'_j$ for all $j \notin J$ and hence $\mathbf{v}_J = \mathbf{v}'_J = K$, a contradiction. Hence $\mathbf{v}'_J \neq K$ and $\mathbf{v}' \in V_1$, proving that V_1 is initial.

Now a k-dimensional $(k > h)$ subset of V is always contained in V_1. Consequently every subset of $\mathbf{N}^m$ that is properly k-dimensional in V_1 $(k \geqslant h)$ remains properly k-dimensional in V. This shows that $d_k(V_1) = d_k(V)$ for all $k > h$. It is clear that if L is an h-dimensional subset of V and if $L \neq K$, then $L \subset V_1$. Hence $d_h(V) - 1 \leqslant d_h(V_1) \leqslant d_h(V)$. We observe that if $\mathbf{v} \in V_1$, then $\mathbf{v}_J \subset V_1$ for some $J \in \mathfrak{J}_h$ and $\mathbf{v}_J \neq K$, and that if further $\mathbf{v} \in K$, then $J \neq J(K)$.

Suppose $K = \mathbf{a}_{J(K)} \subset V_1$. Applying Proposition 2, there exists J_0 with

Card $J_0 = h$ and $J_0 \neq J(K)$ such that $\mathbf{a}_{J(K) \cup J_0}$ is a subset of V_1 containing K. This contradicts the fact that K is properly h-dimensional in V. Hence $K \not\subset V_1$, proving that $d_h(V_1) = d_h(V) - 1$.

To complete the proof, let $K' = \mathbf{a}'_{J(K')}$ be a k-dimensional subset of V_1 with $k < h$. Applying Proposition 2, there exists $J_0 \subset \mathbf{N}_m$ with Card $J_0 = h$ such that $\mathbf{a}'_{J(K') \cup J_0}$ is a subset of V_1 containing K'. Hence K' is not properly k-dimensional in V_1 and $d_k(V_1) = 0$ for all $k < h$.

COROLLARY 1. *Let V be an initial subset of $\mathbf{N}^m$ with dimension sequence $\{d_k(V)\}_{k \in \mathbf{N}}$. Then there exist initial subsets V_{hi} $(0 \leqslant h \leqslant m, 0 \leqslant i \leqslant d_h(V))$ of V with the following properties*:

(a) $V_{00} = V$, $V_{m d_m(V)} = \varnothing$ *and for every h* $(0 \leqslant h \leqslant m)$

$$V_{h0} \supset V_{h1} \supset \cdots \supset V_{h d_h(V)} = V_{h+1\ 0}.$$

(b) *For every h* $(0 \leqslant h \leqslant m)$

$$\begin{aligned} d_k(V_{hi}) &= d_k(V), && \text{for } k > h,\ 1 \leqslant i \leqslant d_h(V), \\ d_h(V_{hi}) &= d_h(V) - i, && \text{for } 0 \leqslant i \leqslant d_h(V), \text{ and} \\ d_k(V_{hi}) &= 0 && \text{for } k < h,\ 0 \leqslant i \leqslant d_h(V). \end{aligned}$$

(c) *Every subset of $\mathbf{N}^m$ that is properly k-dimensional in V_{hi} $(k \geqslant h)$ remains properly k-dimensional in any $V_{h'i'}$ such that $V_{hi} \subset V_{h'i'}$, and in particular, in V.*

PROOF. Repeated application of Proposition 3.

COROLLARY 2. *Let V be an initial subset of $\mathbf{N}^m$ and suppose $V = V_0 \supset V_1 \supset \cdots \supset V_m$ is a chain of initial subsets of V such that for every h $(0 \leqslant h \leqslant m)$, $d_k(V_h) = d_k(V)$ if $k \geqslant h$ and $d_k(V_h) = 0$ if $k < h$. Then for each such h,*

(a) $V_h = \{\mathbf{v} \in V \mid \mathbf{v}$ *belongs to some h-dimensional subset of V*$\}$;

(b) *there exists a numerical polynomial $\beta_h(X)$ of degree $< h$ such that for all sufficiently large $s \in \mathbf{N}$,*

$$\operatorname{Card}(V_h(s) - V_{h+1}(s)) = d_h(V)\binom{s+h}{h} - \beta_h(s)$$

($V_{m+1} = \varnothing$ *by convention) and $\beta_h(X) = 0$ if $d_h(V) = 0$.*

An immediate consequence of Corollaries 1 and 2 is the result of Kolchin mentioned earlier.

COROLLARY 3. *Let V be an initial subset of $\mathbf{N}^m$. There exists a numerical polynomial $\omega_V(X)$ of degree $\leqslant m$ such that* Card $V(s) = \omega_V(s)$ *for all sufficiently* *large* $s \in \mathbf{N}$. *Furthermore, if* m' *is the largest integer* k *such that* $d_k(V) \neq 0$, *then* $\omega_V(X) = d_{m'}(V)\binom{X+m'}{m'}$ + *terms of lower degree.*

PROOF OF COROLLARIES 2 AND 3. Part (a) of Corollary 2 is obvious and Corollary 3 follows from Corollaries 1 and 2 since

$$\operatorname{Card} V(s) = \sum_{h=0}^{m} \operatorname{Card}(V_h(s) - V_{h+1}(s)).$$

To prove part (b) of Corollary 2, set $d_k = d_k(V)$ and let V_{hi} $(0 \leqslant h \leqslant m, 0 \leqslant i \leqslant d_h)$ be initial subsets of V satisfying the conditions of Corollary 1. By part (a) of Corollary 2, $V_h = V_{h0}$ $(0 \leqslant h \leqslant m)$. If $d_h = 0$, then $V_h = V_{h+1}$ and we may take $\beta_h(X) = 0$. If $h < m$, let K_{hi} $(1 \leqslant i \leqslant d_h)$ be the unique subset of $V_{h\,i-1}$ that is properly h-dimensional in $V_{h\,i-1}$ but not properly h-dimensional in V_{hi}. Clearly $K_{hi} \not\subset V_{hi}$. Observe that $K_{hi} \cap V_{hi}$ is an initial subset of K_{hi} (K_{hi} being identified with $\mathbf{N}^h$). Define $\alpha_{hi} = \Sigma_{j \notin J} a_j$, where $\mathbf{a}_J = K_{hi}$. By induction, we may assume that Corollary 2 (and hence also Corollary 3) holds for smaller values of m so that there exists a numerical polynomial $\omega_{hi}(X)$ such that for all sufficiently large $s \in \mathbf{N}$, $\operatorname{card}(K_{hi} \cap V_{hi}(s)) = \omega_{hi}(s - \alpha_{hi})$. Since $K_{hi} \cap V_{hi} \neq K_{hi}$, degree $\omega_{hi}(X) < h$. It follows from $K_{hi} - V_{hi} = V_{h\,i-1} - V_{hi}$ that for sufficiently large $s \in \mathbf{N}$,

$$\operatorname{Card}(V_{h\,i-1}(s) - V_{hi}(s)) = \binom{s - \alpha_{hi} + h}{h} - \omega_{hi}(s - \alpha_{hi})$$

and hence

$$\operatorname{Card}(V_h(s) - V_{h+1}(s)) = \sum_{i=1}^{d_h(V)} \left[\binom{s - \alpha_{hi} + h}{h} - \omega_{hi}(s - \alpha_{hi})\right].$$

This completes the proof.

PROPOSITION 4. *Let* $\{d_k\}_{k \in \mathbf{N}}$ *be a sequence of natural numbers such that* $d_k = 0$ *for all sufficiently large* $k \in \mathbf{N}$.

(a) *For any fixed* $m \in \mathbf{N}$, *there exist only finitely many initial subsets of* $\mathbf{N}^m$ *with dimension sequence* $\{d_k\}_{k \in \mathbf{N}}$.

(b) *There exist only finitely many numerical polynomials of the form* ω_V, *where* V *is an initial subset of* $\mathbf{N}^m$ *for some* $m \in \mathbf{N}$ *having dimension sequence* $\{d_k\}_{k \in \mathbf{N}}$.

PROOF. (a) Let m' be the largest integer k such that $d_k \neq 0$. Part (a) is trivial if $m < m'$. For $m \geqslant m'$, it suffices to show by induction on $m - i$ that for $i = 0, \ldots, m$ the set U_i of initial subsets V of $\mathbf{N}^m$ with $d_k(V) = d_k$ for $i \leqslant k \leqslant m$ and $d_k(V) = 0$ otherwise is finite. We have

$$V_m = \begin{cases} \emptyset & \text{if } m = m' \text{ and } d_{m'} > 1, \\ \{\emptyset\} & \text{if } d_m = 0 \text{ (for example, } m > m'), \\ \{\mathbf{N}^m\} & \text{if } m = m' \text{ and } d_{m'} = 1. \end{cases}$$

Suppose $i < m$ and assume that V_j is finite for all j with $i < j \leqslant m$. Let $V \in V_i$ and let V' be the set of all points in V that belongs to some $(i+1)$-dimensional subset of V. By Corollaries 1 and 2 of Proposition 3, $V' \in V_{i+1}$. Let $E' = E(V')$ (see Remark 1) and for $1 \leqslant j \leqslant m$, let $\bar{e}_j = \max e_j$ $(\mathrm{e} \in E')$. Let $J_1, \ldots, J_p$ be all the subsets of $\mathbf{N}_m$ consisting of i elements. Let t_h be the number of subsets of $\mathbf{N}^m$ properly i-dimensional in V and having direction J_h. Of course $\Sigma_{h=1}^p t_h = d_i$. We claim that if $K = \mathrm{a}_{J_h}$ is a subset of $\mathbf{N}^m$ properly i-dimensional in V, then $a_j \leqslant t_h + \bar{e}_j$ for all $j \notin J_h$. Suppose this is false so that $a_j > t_h + \bar{e}_j$ for some $j \notin J_h$. For any $s \in \mathbf{N}$ with $s \leqslant t_h$, let $L_s = (a_1, \ldots, a_{j-1}, \bar{e}_j + s, a_{j+1}, \ldots, a_m)_{J_h}$. By Remark 3, L_s is an i-dimensional subset of V. Since K is properly i-dimensional in V, it follows from the Corollary of Proposition 2 that K contains some point v such that $\mathrm{v} \notin V'$. By Remark 1, $\mathrm{v} \geqslant \mathrm{e}'$ for some $\mathrm{e}' \in E'$. Let

$$\mathrm{v}' = (v_1, \ldots, v_{j-1}, \bar{e}_j + s, v_{j+1}, \ldots, v_m).$$

Then $\mathrm{v}' \in L_s$ and $\mathrm{v}' \geqslant \mathrm{e}'$. Applying Remark 1 and the Corollary of Proposition 2 again, we see that L_s is properly i-dimensional in V. Since s runs through 0 to t_h, this contradicts the definition of t_h and establishes our claim.

It is now easy to see that V_i is finite so that the induction step is completed and part (a) is proved.

(b) For each $m \in \mathbf{N}$, let Ω_m be the set of numerical polynomials of the form ω_V where V is an initial subset of $\mathbf{N}^m$ with dimension sequence $\{d_k\}_{k \in \mathbf{N}}$. By part (a), Ω_m is finite for all m. Let $m_0 = \Sigma_{k=0}^{m'}(k+1)d_k$. Then $m_0 \geqslant m'$ and, to complete the proof, it suffices to show that $\Omega_m = \Omega_{m_0}$ for all $m \geqslant m_0$. The canonical mapping from $\mathbf{N}^{m_0}$ into $\mathbf{N}^m$ given by $(x_1, \ldots, x_{m_0}) \mapsto (x_1, \ldots, x_{m_0}, 0, \ldots, 0)$ clearly maps each initial subset V of $\mathbf{N}^{m_0}$ onto an initial subset of $\mathbf{N}^m$ having the same numerical polynomial and dimension sequence as V. Thus $\Omega_{m_0} \subset \Omega_m$. Conversely, let V be an initial subset of $\mathbf{N}^m$ with dimension sequence $\{d_k\}_{k \in \mathbf{N}}$. Let I be the set of indices j $(1 \leqslant j \leqslant m)$ such that V contains an element with nonzero jth coordinate. Let J_0 be the union of all directions $J(K)$ of K as K runs through the set of subsets of $\mathbf{N}^m$ that are properly k-dimensional in V for some k $(0 \leqslant k \leqslant m)$. For each $j \in I - J_0$, the point $\mathrm{v}^{(j)}$ whose ith coordinate is δ_{ij} (Kronecker's δ) belongs to V and hence belongs to some K_j which is properly k_j-dimensional in V for some k_j. Hence $K_j = \mathrm{v}^{(j)}_{J(K_j)}$. Since $K_j = K_{j'}$ implies $j = j'$, it follows that

$$\text{Card } I \leqslant \text{Card } J_0 + \sum_{k=0}^{m'} d_k \leqslant m_0.$$

Let $I = \{j_1, \ldots, j_q\}$, $q \leqslant m_0$. The mapping from V into $\mathbf{N}^{m_0}$ defined by $(v_1, \ldots, v_m) \mapsto (v_{j_1}, \ldots, v_{j_q}, 0, \ldots, 0)$ maps V onto an initial subset of $\mathbf{N}^{m_0}$ having the same numerical polynomial and dimension sequence as V. Hence $\Omega_m \subset \Omega_{m_0}$ and this completes the proof of the proposition.

Recall that if $\mathcal{V}$ is a finite sequence of initial subsets V_i, $V_i \subset N^{m_i}$ ($1 \leqslant i \leqslant n$), then $\omega_{\mathcal{V}} = \Sigma_{i=1}^{n} \omega_{V_i}$. In what follows, we shall identify $\mathcal{V}$ with the infinite sequence $\{V_i\}_{i \in \mathbf{N}}$ where $V_i = \varnothing$ for $i > n$. We observe for use below that any numerical polynomial $\omega(X)$ of degree $\leqslant m$ can be written uniquely as $\omega(X) = \Sigma_{h=0}^{m} a_h \binom{X+h}{h}$ where $a_h \in \mathbf{Z}$ and that if

$$\omega_1(X) = \sum_{h=0}^{m} a_h \binom{X+h}{h} \quad \text{and} \quad \omega_2(X) = \sum_{h=0}^{m} b_h \binom{X+h}{h}$$

then $\omega_1(X) \leqslant \omega_2(X)$ if and only if $(a_m, \ldots, a_0) \leqslant (b_m, \ldots, b_0)$ relative to the lexicographic order on $\mathbf{Z}^{m+1}$.

PROPOSITION 5. *The set Ω of numerical polynomials of the form $\omega_{\mathcal{V}}$ for some finite sequence $\mathcal{V}$ of initial subsets is well-ordered relative to the above ordering.*

PROOF. Let $\Pi \subset \Omega$, $\Pi \neq \varnothing$. To show that Π has a smallest element, it suffices to show that every descending sequence $\pi = \{\pi_i\}_{i \in \mathbf{N}}$ in Π is stable. Let π be such a descending sequence. Since it is enough to show that some subsequence of π is stable, we may assume that each π_i has the same degree m and write

$$\pi_i = \sum_{k=0}^{m+1} a_{ik} \binom{X+k}{k} \qquad (a_{ik} \in \mathbf{Z} \text{ and } a_{i\,m+1} = 0).$$

Consider for each h ($0 \leqslant h \leqslant m+1$), the statement

$(*)_h$ That there exist a subsequence (which we shall still denote by $\pi = \{\pi_i\}_{i \in \mathbf{N}}$) of π and a corresponding sequence $\{\mathcal{V}_i\}_{i \in \mathbf{N}}$, where each $\mathcal{V}_i$ is a sequence $\{V_{ij}\}_{j \in \mathbf{N}}$ of initial subsets V_{ij} of $\mathbf{N}^{m_{ij}}$, $V_{ij} = \varnothing$ for all sufficiently large j, satisfying the following conditions:

$(\mathrm{I})_h$ $\pi_i = \omega_{\mathcal{V}_i}$ for all $i \in \mathbf{N}$.

$(\mathrm{II})_h$ For each k, $h \leqslant k \leqslant m+1$, a_{ik} is independent of i (that is, $a_{1k} = a_{2k} = \cdots$).

$(\mathrm{III})_h$ If d_{ijk} denotes the number of subsets of $\mathbf{N}^{m_{ij}}$ properly k-dimensional in V_{ij}, then for each $j \geqslant 1$ and each k, $h \leqslant k \leqslant m+1$, d_{ijk} is independent of i.

$(\mathrm{IV})_h$ If $V_{ijk} = \{\mathbf{v} \in V_{ij} \mid \mathbf{v}$ belongs to some k-dimensional subset of $V_{ij}\}$ and if $\beta_{ijk}(X)$ (see Corollary 2 of Proposition 3) is the numerical polynomial of degree $< k$ such that the equation

$$\operatorname{Card}(V_{ijk}(s) - V_{ij\,k+1}(s)) = d_{ijk}\binom{s+k}{k} - \beta_{ijk}(s)$$

holds for all natural numbers s larger than some suitable $n_{ij} \in \mathbf{N}$, then for each $j \geqslant 1$ and each k, $h \leqslant k \leqslant m+1$, $\beta_{ijk}(X)$ is independent of i.

It suffices to show by induction on $m + 1 - h$ that $(*)_h$ holds for each $h \leqslant m+1$, for the statement $(*)_0$ will then provide a subsequnece of π that is stable. The case $h = m+1$ is trivial. Assume that $h < m+1$ and that we have chosen a subsequence $\{\pi_i\}_{i \in \mathbf{N}}$ and $\{V_i\}_{i \in \mathbf{N}}$ satisfying conditions $(\mathrm{I})_{h+1}$, $(\mathrm{II})_{h+1}$, $(\mathrm{III})_{h+1}$ and $(\mathrm{IV})_{h+1}$. Now

$$\begin{aligned}\omega_{V_i}(s) = \sum_{j=1}^{\infty} \omega_{V_{ij}}(s) &= \sum_{j=1}^{\infty}\sum_{k=0}^{m} \operatorname{Card}(V_{ijk}(s) - V_{ij\,k+1}(s)) \\ &= \sum_{k=0}^{h}\sum_{j=1}^{\infty}\left[d_{ijk}\binom{s+k}{k} - \beta_{ijk}(s)\right] \\ &\quad + \sum_{k=h+1}^{m}\sum_{j=1}^{\infty}\left[d_{ijk}\binom{s+k}{k} - \beta_{ijk}(s)\right]\end{aligned}$$

for all $s \geqslant \max_j n_{ij}$. Since $\{\omega_{V_i}(X)\}_{i \in \mathbf{N}}$ form a descending sequence (by $(\mathrm{I})_{h+1}$) and the polynomials obtained by replacing s by X in the second double sum of the right-hand side are independent of i (by $(\mathrm{III})_{h+1}$ and $(\mathrm{IV})_{h+1}$), it follows that the polynomials obtained by replacing s by X in the first double sum of the right-hand side form a descending sequence as i varies. Since $\deg \beta_{ijk} < k$, we may assume, after taking a subsequence, that $\Sigma_{j=1}^{\infty} d_{ijh}$ is independent of i, say with common value d_h. Now for every i, the number of indices j such that $d_{ijk} \neq 0$ for a fixed $k \geqslant h$ is finite; for example, this number is at most d_h when $k = h$. By $(\mathrm{III})_{h+1}$, there exists a natural number j_0 such that $d_{ijk} = 0$ for all $i \geqslant 1$, $j \geqslant j_0$ and $k > h$. Thus for every i, after permuting a finite number of the initial subsets V_{ij} with $j \geqslant j_0$ and reindexing, we may assume that $d_{ijh} = 0$ for all $j > j_0 + d_h$. Let $p = j_0 + d_h$ and let S be the set of all $(x_1, \ldots, x_p) \in \mathbf{N}^p$ such that $x_1 + \cdots + x_p = d_h$. Since for each i, $(d_{i1h}, \ldots, d_{iph}) \in S$ and S is finite, we may, after taking a subsequence, assume that d_{ijh} is independent of i for all j. It follows from the corollaries of Proposition 3 that each V_{ijh} (as i varies) has the same dimension sequence and therefore by Proposition 4(b), we may assume, again after a finite number of steps involving taking subsequences, that $\omega_{V_{ijh}}(X)$ is independent of i for all j. But we have

$$\omega_{V_{ijh}}(X) = \sum_{k=h}^{m}\left[d_{ijk}\binom{X+k}{k} - \beta_{ijk}(X)\right]$$

so that $\beta_{ijh}(X)$ is independent of i. Finally

$$\pi_i(X) = \omega_{v_i}(X) = \sum_{k=h}^{m} \sum_{j=1}^{\infty} \left[d_{ijk}\binom{X+k}{k} - \beta_{ijk}(X) \right] + (\text{terms of degree} < h)$$

from which it follows that a_{ik} is independent of i for every k $(h \leqslant k \leqslant m+1)$. This completes the induction.

REFERENCE

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