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THE TRIGONOMETRIC HERMITE-BIRKHOFF INTERPOLATION PROBLEM

BY

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ABSTRACT. The classical Hermite-Birkhoff interpolation problem, which has recently been generalized to a special class of Haar subspaces, is here considered for trigonometric polynomials. It is shown that a slight weakening of the result (conservativity and Pólya conditions) established for those special Haar subspaces also holds for trigonometric polynomials after one rephrases the statement of the problem, the underlying assumptions, and the result itself appropriately to reflect the inherent differences between algebraic polynomials (which the special class of Haar subspaces essentially are) and the periodic trigonometric polynomials. Furthermore, simple necessary and sufficient conditions for poisedness of one-rowed incidence matrices analogous to the Pólya conditions for two-rowed incidence matrices in the algebraic version are proved, and an elementary necessary condition for the poisedness of an arbitrary (trigonometric) incidence matrix stated.

Recently there has been considerable interest in the classical Hermite-Birkhoff interpolation problem; we cite [1], [2], [4], [5], [10], [11], [13]–[16], [19]–[26], as examples. Variants of the Hermite-Birkhoff interpolation problem have also been proposed and studied, both for algebraic polynomials (e.g., [3], [8], [9]) and for more general subspaces of functions [6], [7], [17]. In particular, Ikebe [7] shows that for the classical Hermite-Birkhoff interpolation problem with a Haar subspace whose successive derivatives are Haar and decrease in magnitude with each derivative (like algebraic polynomials) replacing the algebraic polynomials, that an incidence matrix is necessarily poised whenever it is conservative and satisfies the Pólya conditions. In this paper we replace the algebraic polynomials in the statement of the Hermite-Birkhoff interpolation problem by trigonometric polynomials and obtain an analogous result. Certain differences are present, however, due to the inherent difference between trigonometric and algebraic polynomials. One of these differences is that the degree of a trigonometric polynomial

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does *not* drop as it is differentiated (save the constant polynomial). The fact that the derivative of an algebraic polynomial does drop is what leads to the Pólya conditions in the usual algebraic polynomial Hermite-Birkhoff interpolation problem; its relevancy for trigonometric polynomials is therefore immediately in question, and save for the requirement that an incidence matrix must have a one in its first column in order to be poised, is in fact totally irrelevant (although no papers have appeared treating the trigonometric version of the Hermite-Birkhoff interpolation problem, save the special case of lacunary interpolation, examples showing the Pólya conditions in the trigonometric version not to be a necessary condition have been constructed by many people and are fairly widely known). Furthermore, the restriction that an n-incidence matrix should have only n columns is superfluous. Secondly, trigonometric polynomials are not invariant under scalar expansion, although they are under translation. Consequently in the trigonometric version only one of the k points at which point and derivative point evaluations are specified in a $k \times n$ matrix may be taken to be any given point (say 0), while in the algebraic polynomial version two of those points may be arbitrarily specified (say to 0 and 1). This leads one to the realization that the simplest nontrivial incidence matrix for the trigonometric case will have one row, while the analogous case for algebraic polynomials has two rows (so-called Pólya systems, after Pólya [19] who solved the algebraic version for two-rowed incidence matrices).

Thirdly, for the real algebraic polynomial Hermite-Birkhoff interpolation problem the points at which the point and derivative point evaluations are specified lie on the real line, which has a natural *antisymmetric* ordering, while for the trigonometric version the points at which the evaluations are specified should actually be viewed as lying on the unit circle, which has a *symmetric* natural ordering. Consequently the notion of conservativity of incidence matrices when trigonometric polynomials are involved must be slightly changed from the definition in the case of algebraic polynomials (or the almost algebraic polynomial subspaces of Ikebe).

Fourthly, algebraic polynomials may be viewed as a graded algebra with each direct summand having dimension one, while one views trigonometric polynomials similarly with each direct summand after the first (which has dimension one) having dimension two. Thus while it is natural to consider any positive integer in the algebraic case, in the trigonometric case it is natural to consider n only to be an odd positive integer.

1. **Preliminaries.** By the *trigonometric Hermite-Birkhoff interpolation problem* we mean the following.

TRIGONOMETRIC HERMITE-BIRKHOFF INTERPOLATION PROBLEM. *Given*

(i) *positive integers* k *and* n, *with* $k \leqslant n$ *and* n *odd*,

(ii) *a set* I *of* n *ordered pairs* (i, j) *with* $1 \leqslant i \leqslant k$, $0 \leqslant j < \infty$,

(iii) k *real points* $x_1, x_2, \ldots, x_k$, $0 \leqslant x_1 < x_2 < \ldots < x_k < 2\pi$, *determine the* (*subspace of*) *real trigonometric polynomials* $p(x)$ *of degree at most* $(n-1)/2$ *which satisfy the interpolation conditions*

$$p^{(j)}(x_i) = 0 \quad \textit{for } (i, j) \in I.$$

The interpolation conditions are usually posed via a semi-infinite ($k \times \infty$) matrix

$$E = \|e_{ij}\|_{i=1;\, j=0}^{k}{}^{\infty}, \quad \text{where } e_{ij} = \begin{cases} 1 & \text{if } (i, j) \in I, \\ 0 & \text{otherwise,} \end{cases}$$

which is called the *n-incidence matrix* [21] associated with the given Hermite-Birkhoff interpolation problem.

By the *essential columns* of E we mean those columns, and the columns preceding those columns, which have a nonzero entry in them. In other words, if the qth column of E is the last column of E having a one in it, then the essential columns of E are just the first $q + 1$ columns of E. Since all the information in an incidence matrix is contained in its essential columns, it is conventional to display only the essential columns of an incidence matrix (similarly it is conventional to assume without mention that none of the k rows of an incidence matrix is identically zero).

A $(2m + 1)$-incidence matrix is said to be *poised* in case the dimension of the subspace satisfying it is zero (i.e., the zero polynomial is the only trigonometric polynomial of degree at most m satisfying the associated interpolation conditions).

Suppose that $E = \|e_{ij}\|$ is a k-rowed n-incidence matrix having q essential columns. We define the standard numbers

$$m_\nu = \sum_{\mu=1}^{k} e_{\mu\nu}, \quad M_\nu = \sum_{\mu=1}^{\nu} m_\mu \qquad (\nu = 0, \ldots, q)$$

and say that E satisfies the *weak Pólya condition* in case $m_0 = M_0 > 0$, that E satisfies the *Pólya conditions* in case $M_\nu \geqslant \nu + 1$ $(\nu = 0, \ldots, q - 1)$, and that E satisfies the *strong Pólya conditions* in case $M_\nu \geqslant \nu + 2$ $(\nu = 0, \ldots, q - 2)$.

By a *sequence* in an incidence matrix $E = \|e_{ij}\|$ we mean a maximal (with respect to length) sequence of consecutive ones in a row of E. Equivalently, if $e_{i,j-1} = 0 = e_{i,p+1}$ while $e_{i\nu} = 1$ $(\nu = j, \ldots, p)$, then E has a sequence of length $p - j + 1$ in its ith row, namely the sequence $e_{i,j}, \ldots, e_{i,p}$. If $p - j + 1$

is odd, we call this sequence an *odd sequence*; otherwise $e_{i,j}, \ldots, e_{i,p}$ is an *even sequence.* Furthermore we refer to the sequence $e_{i,j}, \ldots, e_{i,p}$ for short as the (i, j)-sequence of E. We say that an (i, j)-sequence is (*trigonometrically*) *supported* in case there is an element $e_{\mu\nu}$ of E which is one, where $\mu \neq i$ and $\nu < j$ (to say that the (i, j)-sequence is (*algebraically*) *supported* means that there must exist two elements $e_{\mu\nu} = e_{\rho\tau} = 1$ where $\mu < i < \rho$ and $\nu, \tau < j$).

We say that the incidence matrix E is *conservative* in case E has no supported odd sequences. We say that E is *strongly conservative* (*or Ferguson*) in case any (i, j)-sequence of E is even whenever $i \geqslant 1$ (equivalently, E is strongly conservative if and only if E consists only of Hermite data (sequences beginning in the zeroth column) and even sequences).

We use the notation $|E|$ to mean the number of ones in E; i.e., if E is an n-incidence matrix, then $|E| = n$.

If A and B are matrices, we shall say that A is an *extension matrix* of B whenever B is a submatrix of A.

Notice that incidence matrices have columns indexed from zero, while ordinary matrices (including row vectors) have columns indexed from one. In the proof of the theorem below, the E_μ and F_μ are viewed as incidence matrices; the G_μ are viewed as matrices.

Finally, an incidence matrix E is said to be *Hermite* (or have *Hermite-type data*) whenever all nonzero entries of E occur in sequences originating in the zeroth column of E.

2. A sufficient condition for poisedness. In the proof of our theorem we will require the following well-known periodic version of Rolle's theorem.

LEMMA 1. *If $f \in C^1(R)$ is 2π periodic, and has ρ distinct real zeroes x_i, with $0 \leqslant x_1 < x_2 < \ldots < x_\rho$, then $f'(x)$ has* (*at least*) *ρ distinct real zeroes y_i such that $x_1 < y_1 < x_2 < y_2 < \ldots < x_\rho < y_\rho < x_1 + 2\pi$. In particular, $f'(x)$ also has at least ρ distinct real zeroes in the half-open interval* $[0, 2\pi)$.

We also will require the result that the trigonometric polynomials of degree at most n form a Haar subspace of dimension $2n + 1$. Said otherwise, a trigonometric polymonial of degree at most n which vanishes $2n + 1$ times (counting multiple zeroes) on $[0, 2\pi)$ is in fact the zero polynomial; equivalently, Hermite incidence matrices are poised.

THEOREM 1. *Any* $(2n + 1)$-*incidence matrix E which is strongly conservative and satisfies the weak Pólya condition is poised with respect to trigonometric polynomials of degree at most n.*

PROOF. If not, let $p(x)$ be a nonzero trigonometric polynomial of degree at most n which satisfies the interpolation conditions specified by E with respect to the points $0 \leqslant x_1 < \ldots < x_k < 2\pi$. Suppose next that E has q essential columns. We construct a sequence of incidence matrices $E_0, \ldots, E_q$ such that, for each $\nu = 0, \ldots, q$,

(i) E_ν has $2q - \nu$ columns,

(ii) $|E_\nu| \geqslant |E|$,

(iii) E_ν is strongly conservative,

(iv) E_ν satisfies the weak Pólya condition, and

(v) $h^{(\nu)}$ satisfies the interpolation conditions specified by E_ν at the points $x_1, \ldots, x_{k(\mu)}$.

We begin by letting E_0 be the first $2q$ columns of E. Suppose now that $E_0, \ldots, E_\mu$ have been specified ($\mu = 0, \ldots, q - 1$).

Let $z_1, \ldots, z_\rho$ be the points of E_μ at which zeroes of $p^{(\mu)}(x)$ are specified. By Lemma 1, $p^{(\mu+1)}(x)$ will have ρ zeroes $y_1, \ldots, y_\rho$ such that $z_1 < y_1 < \ldots \lesssim z_\rho < y_\rho < z_1 + 2\pi$. Let $\{y_1, \ldots, y_\rho\} = S \cup N$, where $S \subseteq \{x_1, \ldots, x_{k(\mu)}\} + \pi Z$, $N \subseteq [z_1, \ldots, z_1 + 2\pi)\backslash S$. Suppose $y_i \in S$, $y_i = x_{j(i)}$. Since $y_i \notin \{z_1, \ldots, z_\rho\}$, if $e_{j1} = 1$, then the strong conservativity of E_μ requires that the $(j, 1)$-sequence of E_μ be even. But y_i is a Rolle zero of $p^{(\mu+1)}(x)$ such that $z_i < y_i < z_{i+1}$ (identifying $z_{\rho+1}$ with $z_i + 2\pi$), whence there is a point ξ between z_i and z_{i+1} at which $p^{(\mu)}(x)$ takes on a relative extrema, whence at which $p^{(\mu+1)}(x)$ has an odd-order zero. If $\xi = y_i = x_j$, then $p'(x)$ actually satisfies an additional zero than specified by the $(j, 1)$-sequence of E_μ. If $\xi \neq y_i$, replace y_i by ξ, whence again either $y_i \notin S$, $y_i \in S$ and $e_{j1} = 0$, or $y_i \in S$, $e_{j1} = 1$, and the $(j, 1)$-sequence of E_μ is even but $p^{(\mu+1)}(x)$ actually satisfies an additional zero at y_i than specified in the $(j, 1)$-sequence.

Define a new incidence matrix F_μ from E_μ as follows:

(i) if $e_{il} = 1$, then $f_{il} = 1$,

(ii) if $y_m \in S$ and $e_{j(m),1} = 1$, set $f_{j(m),1} = 1$ $(m = 1, \ldots, \rho)$,

(iii) if $y_m \in S$ and $e_{j(m),1} = 1$, set $f_{j(m),\tau+1} = 1$,

where τ is the length of the $(j(m), 1)$-sequence of E_μ $(m = 1, \ldots, \rho)$.

Define an extension matrix G_μ of F_μ as follows:

If $y_m \in N$ and $x_{i(m)} < y_k < x_{i(m)+1}$, insert an extra row $(0\ 1\ 0\ 0 \ldots 0)$ into F_μ between the $i(m)$th and $(i(m) + 1)$st rows of F_μ $(m = 1, \ldots, \rho)$.

Finally, let $E_{\mu+1}$ be G_μ less its first column. Since the net result of the change from E_μ to $E_{\mu+1}$ is to add as many ones to the later columns of E_μ as are in the zeroth column of E_μ, $|E_{\mu+1}| = |E_\mu|$. Since the added ones always form part of Hermite-type data in the new incidence matrix $E_{\mu+1}$, $E_{\mu+1}$ still is strongly conservative. Since we have deleted one column from E_μ, the number of columns of $E_{\mu+1}$ is $(2q-\mu)-1$. Since we always add at least one to the first column of E_μ whenever the first column of E_μ has no ones, the zeroth column of $E_{\mu+1}$ has a nonzero entry, and thus $E_{\mu+1}$ satisfies the weak Pólya condition. Finally $p^{(\mu+1)}(x)$ satisfies (by construction) the interpolation conditions specified by $E_{\mu+1}$ at the points $\{x_i(\mu+1)\}_{i=1}^{k(\mu+1)}$, where $\{x_i(\mu+1)\}_{i=1}^{k(\mu+1)} = \{x_i(\mu)\}_{i=1}^{k(\mu)} \cup N$, ordered such that

$$x_1(\mu+1) = x_1(\mu) < x_2(\mu+1)$$
$$< \ldots < x_{k(\mu+1)}(\mu+1) < x_1(\mu+1) + 2\pi.$$

To conclude the proof, notice that E_q has Hermite-type data only, and as such is poised with respect to trigonometric polynomials of degree at most $(|E_q|-1)/2 = n$. But $p^{(q)}(x)$ is a trigonometric polynomial of degree at most n which satisfies E_q, whence $p^{(q)}(x) \equiv 0$; i.e., $p(x)$ must be a constant polynomial. By the weak Pólya condition, $p(x)$ must be the constant zero polynomial, a contradiction. $\square$

3. **Some examples.** At the beginning of this paper we altered substantially the usual statement and assumptions present in the Hermite-Birkhoff interpolation problem when algebraic polynomials are being dealt with. We justified these changes on an a priori theoretical analysis of the difference between algebraic and trigonometric polynomials. Whether it is valid to make these changes can only be justified on their usefulness in developing the solution of the Hermite-Birkhoff interpolation problem in the trigonometric case. We feel the theorem in §2 supports this view. In any case, the inappropriateness of the usual (algebraic) statement and assumptions may be easily illustrated by some examples.

EXAMPLE 1. $E = \left\|\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right\|$ is poised with respect to algebraic polynomials of degree two, but not with respect to trigonometric polynomials of degree one (e.g., $x_1 = 0, x_2 = \pi, p(x) = 1 - \cos x$).

EXAMPLE 2. $E = \|1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\|$ is not poised with respect to the algebraic polynomials of degree four (the Pólya conditions are not satisfied) but is with respect to trigonometric polynomials of degree two.

EXAMPLE 3. The 2-incidence matrix $E = \|1\ 1\|$ is poised with respect to algebraic polynomials of degree one (since E is Hermite) but not with respect to either of the subspaces $\langle 1, \sin x\rangle, \langle 1, \cos x\rangle$.

EXAMPLE 4. The 5-incidence matrix

$$E = \left\|\begin{matrix}1 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{matrix}\right\|$$

has four essential columns, does not satisfy the Pólya conditions, but is poised with respect to trigonometric polynomials of degree two.

Notice the fact that the incidence matrices in Examples 2 and 4 are poised with respect to trigonometric polynomials of degree two follows from Theorem 1.

We should inquire as to the sharpness of Theorem 1. For instance, may the hypothesis of strong conservativity be weakened to mere conservativity? Examples 5 and 6 provide a negative answer:

EXAMPLE 5.

$$E = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{Vmatrix}$$

is trigonometrically conservative, not strongly conservative, and is not poised (e.g., $x_1 = 0, x_2 = \pi, p(x) = 32 \sin x + \sin 2x$).

EXAMPLE 6.

$$E = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{Vmatrix}$$

is not poised (e.g., $x_1 = 0, x_2 = \pi, p(x) = 243 \sin x - \sin 3x$).

It is clear many examples behaving like Examples 5 and 6 above may be generated. These examples show strongly that the idea of conservativity alone for trigonometric incidence matrices does not play the same role as conservativity of the analogous algebraic version of the problem–notice that the $p(x)$ specified in the two examples have a zero at π which actually supports the odd sequences in the first row of the two incidence matrices, and in a sense one might as well be viewing the incidence matrices

$$\begin{Vmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{Vmatrix},$$

respectively, in place of those specified in the examples. In other words, the role of conservativity in the trigonometric version of the Hermite-Birkhoff interpolation problem is completely analogous to the role played by the Pólya and strong Pólya conditions in the trigonometric version of the problem.

Lest one feel that the nonpoisedness in Examples 5 and 6 may be due to the nonspecification of an even number of zeroes of any trigonometric polynomial which would satisfy the given incidence matrices (remember that continuous periodic functions necessarily have an even number of zeroes), notice that the incidence matrix

$$E = \begin{Vmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

is not poised either.

Rather than provide a polynomial satisfying E, we note

THEOREM 2. *A necessary condition for a two-rowed* $(2n + 1)$*-incidence matrix* E *to be poised is that*

(i) E *satisfy the weak Pólya condition,*

(ii) *the number of* $e_{ij} = 1$ *where* j *is odd is at most* n,

(iii) *the number of* $e_{ij} = 1$ *where* $j > 0$ *is even is at most* n.

Finally, to show that strong conservativity is not necessary for poisedness, consider

EXAMPLE 7. The one-rowed incidence matrix $E = \|1\ 0\ 1\ 0\ 0\ 1\|$ is trigonometrically poised.

Example 7 follows from Theorem 3 of the next section.

4. **One-rowed incidence matrices.** We say that a $(2n + 1)$-incidence matrix E *preserves parity* whenever E has precisely $n - (m_0 - 1)/2$ nonzero entries specified in its odd columns.

THEOREM 3. *If* E *is a one-rowed* $(2n + 1)$*-incidence matrix, a necessary and sufficient condition for* E *to be trigonometrically poised is that* E *satisfy the weak Pólya condition and preserve parity.*

PROOF OF NECESSITY. If E does not satisfy the weak Pólya condition, then any nonzero constant polynomial trivially satisfies E. Thus we may assume that $m_0 = 1$.

Suppose $E = (e_{ij})$, with $e_{ij} = 1$ if and only if $j \in J_o \cup J_e$, J_o having only odd integers, J_e having only even integers. If

$$t(x) = a_0 + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx),$$

then $t(x)$ satisfies E if and only if $t^{(j)}(0) = 0$ $(j \in J_o \cup J_e)$, if and only if

$$0 = \delta_{0j} a_0 + \sum_{k=1}^{n} (-1)^{j/2} k^j a_k \qquad (j \in J_e), \tag{1}$$

$$0 = \sum_{k=1}^{n} (-1)^{(j+1)/2} k^j b_k \qquad (j \in J_o), \tag{2}$$

where δ_{0j} is the Kronecker delta.

But both (1) and (2) are a system of linear homogeneous equations, (1) having $n + 1$ unknowns a_k and (2) having n unknowns b_k. In particular, therefore, the a_k and b_k are necessarily all zero only if (1) consists of $n + 1$ equations and (2) consists of n equations; i.e., that J_o have precisely n entries, which is equivalent to E preserving parity.

PROOF OF SUFFICIENCY. It suffices to show that the matrices of coefficients in both (1) and (2) are nonsingular. In actuality, it suffices to show that the matrices ($n \geqslant 1$)

$$J_n = (j_{ik})_{n \times n}, \quad \text{where } j_{ik} = k^{2\nu_i} \tag{3}$$

are nonsingular whenever the ν_i are any nonnegative integers, $0 \leqslant \nu_0 < \nu_1 < \ldots < \nu_n$. But the matrices (3) are special cases of the matrices $(e^{x_i \nu_j})$ which are known [12] to be nonsingular, since the exponential functions $e^{\nu_i x}, \ldots, e^{\nu_n x}, \ldots$ form a Markov system on any interval. □

The statement that E preserves parity may be rephrased as requiring that some rearrangement of the indices of the columns which have the nonzero entries in the one-rowed incidence matrix E have the alternating parity property (APP) as defined by E. Passow [18].

The proof of Theorem 2 is now obvious; in fact,

THEOREM 4. *In order that a* $(2n + 1)$*-incidence matrix E be poised it is necessary that*

(i) *E satisfies the weak Pólya condition,*

(ii) *no two rows of E have between them more than n evaluations specified in (positive) even columns,*

(iii) *no two rows of E have between them more than n evaluations specified in odd columns,*

(iv) *no two rows of E have between them more than* $n + 1$ *evaluations specified in even columns.*

5. **A closing comment.** Several alternative computational proofs to the slick sufficiency proof of Theorem 3 can be produced with a little patience.

Examples 5 and 6 are not due to the author.

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