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ON THE FIXED POINT INDICES AND NIELSEN NUMBERS OF FIBER MAPS ON JIANG SPACES

BY

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ABSTRACT. Let $T = \{E, P, B\}$ be a locally trivial fiber space, where E, B and $P^{-1}(b)$ for each $b \in B$ are compact, connected ANR's (absolute neighborhood retracts). If $f: E \longrightarrow E$ is a fiber (preserving) map then f induces f': $B \longrightarrow B$ and f_b: $P^{-1}(b) \longrightarrow P^{-1}(b)$ for each $b \in B$ such that $Pf = f'P$.

If E, B and $P^{-1}(b)$ for each $b \in B$ satisfy the Jiang condition then $N(f) \cdot P(T, f) = N(f') \cdot N(f_b)$, and $i(f) = i(f') \cdot i(f_b) \cdot P(T, f)$ for each $b \in B$.

If, in addition, the inclusion map i: $P^{-1}(b) \longrightarrow E$ induces a monomorphism $i_\#$: $\pi_1(P^{-1}(b)) \longrightarrow \pi_1(E)$ and f' induces a fixed point free homomorphism $f'_\#$: $\pi_1(B) \longrightarrow \pi_1(B)$, then $N(f) = N(f') \cdot N(f_b)$ and $i(f) = i(f') \cdot i(f_b)$ for each $b \in B$.

As an application, we prove: Let $T = \{E, P, CP(n)\}$ be a principal torus bundle over an n-dimensional complex projective space $CP(n)$. If $f: E \longrightarrow E$ is a fiber map such that for some $b \in CP(n)$, f_b: $P^{-1}(b) \longrightarrow P^{-1}(b)$ is homotopic to a fixed point free map, then there exists a map g: $E \longrightarrow E$ homotopic to f and fixed point free.

1. **Introduction.** In the study of fixed point theorems of a continuous map $f: X \longrightarrow X$, where X is a compact, connected ANR (absolute neighborhood retract), there are a number of interesting numbers that are associated with f. In particular we are interested in the Lefschetz number $L(f)$, the Nielsen number $N(f)$, and the fixed point index $i(F)$ for the fixed point class F of f. In many cases the Nielsen number is more useful than the Lefschetz number. For example, if X is a manifold M^n, $n \geqslant 3$, and $f: X \longrightarrow X$ is such that $N(f) = 0$, then there is a map g: $X \longrightarrow X$ homotopic to f and fixed point free, while $L(f) = 0$ does not give any information. The trouble with the Nielsen number is that it is rather hard to compute. In 1964, Jiang [11] made Nielsen numbers substantially easier to compute in certain cases. Then R. Brown [3], [4], [5] extensively studied the product theorem of Nielsen numbers for fiber (preserving) maps $f: E \longrightarrow E$, where E is the total space of a fiber space $T = \{E, P, B\}$ in order to find a way to compute Nielsen numbers.

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The purpose of this paper is to introduce some new product theorems for Nielsen numbers and the fixed point indices for fiber maps.

In §2 we give definitions and known results for the convenience of readers.

In §3 we give algebraic preliminaries which deal with the Reidemeister number of a homomorphism h from a group G into itself.

§4 contains our major results, Theorems 4 and 7:

THEOREM 4. *Let* $\mathcal{T} = \{E, P, B\}$ *be a fiber space and* $f: E \longrightarrow E$ *be a fiber map. If* E, B *and* $P^{-1}(b)$ *for each* $b \in B$ *satisfy the Jiang condition, then* $N(f) \cdot \mathcal{P}(\mathcal{T}, f) = N(f') \cdot N(f_b)$ *and* $i(f) = i(f') \cdot i(f_b) \cdot \mathcal{P}(\mathcal{T}, f)$ *for each* $b \in B$. *(See* §4 *for the definition of* $\mathcal{P}(\mathcal{T}, f)$.)

Thus $\mathcal{P}(\mathcal{T}, f)$ plays the role of an obstruction to the product theorem. That is, it is only if $\mathcal{P}(\mathcal{T}, f) = 1$ that we have $N(f) = N(f') \cdot N(f_b)$ and $i(f) = i(f') \cdot i(f_b)$ for each $b \in B$.

THEOREM 7. *If* E, B *and* $P^{-1}(b)$ *satisfy the Jiang condition,* $i: P^{-1}(b) \longrightarrow E$ *induces the monomorphism* $i_{\#}: \pi_1(P^{-1}(b)) \longrightarrow \pi_1(E)$, *and if* $f'_{\#}: \pi_1(B) \longrightarrow \pi_1(B)$ *is a fixed point free, then* $\mathcal{P}(\mathcal{T}, f) = 1$ *and we have* $N(f) = N(f')N(f_b)$ *and* $i(f) = i(f') \cdot i(f_b)$ *for each* $b \in B$.

This theorem and its corollary generalize Brown and Fadell's theorem [3], [5] if the spaces involved satisfy the Jiang condition.

In the last section we study more concrete situations. We study fixed point properties of fiber maps for $\mathcal{T} = \{E, P, CP(n)\}$, where $P^{-1}(b) \simeq T^k$ for each $b \in CP(n)$. Some applications are given toward the end. In particular, the following corollary is proved.

Let $\mathcal{T} = \{E, P, CP(n)\}$ be a principal torus bundle over the n-dimensional complex projective space $CP(n)$. If $f: E \longrightarrow E$ is a fiber map such that for some $b \in CP(n)$, $f_b: P^{-1}(b) \longrightarrow P^{-1}(b)$ is homotopic to a fixed point free map, then there exists a map $g: E \longrightarrow E$ which is homotopic to f and fixed point free.

This paper is self-contained, but for more information on notations and terminology, readers are referred to [2], [3] and [7].

Throughout this paper we assume all spaces are compact, connected ANR's and each fiber space $\mathcal{T} = \{E, P, B\}$ is orientable and locally trivial. Also we assume $f: E \longrightarrow E$ such that $L(f) \neq 0$, unless otherwise stated, even though numbers of theorems and corollaries are true without this condition.

Finally, I am grateful to Professor R. F. Brown for his comments and encouragement. Also, we would like to thank the referee for his nice suggestions.

2. **Definitions and known results.** Let $f: X \longrightarrow X$ be a continuous map. Denote the set of all $x \in X$ such that $f(x) = x$ by $\Phi(f)$. Two points $x, y \in \Phi(f)$

are said to be equivalent if there is a path $C: I \longrightarrow X$ such that $C(0) = x$, $C(1) = y$, and C is homotopic to $f(C)$ such that x and y are fixed throughout the homotopy. This relation is an equivalence relation which divides $\Phi(f)$ into equivalence classes $F_1, \ldots, F_n$. For each $j = 1, \ldots, n$, there is an open set U_j in X such that $F_j \subset U_j$ and $\mathrm{cl}(U_j) \cap \Phi(f) = F_j$, where "cl" denotes closure. Define the fixed point index $i(F_j)$ of the fixed point class F_j by $i(F_j) = i(X, f, U_j)$.

A fixed point class F is said to be essential if $i(F) \neq 0$ and inessential if $i(F) = 0$. The Nielsen number $N(f)$ of the map f is defined to be the number of fixed point classes of f that are essential.

Let $\mathrm{Map}(X, X)$ be the space of all continuous maps from X into X with the compact open topology. For a fixed base point $x_0 \in X$, the evaluation map E defined by $Ef = f(x_0)$ for every $f: X \longrightarrow X$ maps $(\mathrm{Map}(X, X), \mathrm{id}) \longrightarrow (X, x_0)$ and induces the homomorphism $E_\#: \pi_1(\mathrm{Map}(X, X), \mathrm{id}) \longrightarrow \pi_1(X, x_0)$. Denote the image of $E_\#$ by $T(X) \subset \pi_1(X, x_0)$. If $E_\#$ is onto then we say that the space X satisfies the Jiang condition and call X a Jiang space.

If X satisfies the Jiang condition then the fundamental group $\pi_1(X)$ is abelian, and for any continuous map $f: X \longrightarrow X$ the fixed point indices $i(F_j)$ are the same for every fixed point class F_j of f. If we denote this number by $i(f)$ then it is known that $L(f) = i(f)N(f)$ [11]. It is well known that any H-space, lens space, or quotient space of a topological group with respect to a connected compact Lie group satisfies the Jiang condition.

In an effort to devise a method of computing the Nielsen number, R. Brown [3] studied the fiber maps. Let us recall some definitions from [3] and [14]. Let $\mathcal{T} = \{E, P, B\}$ be a locally trivial fiber space, and $\Omega_p = \{(e, \omega) \in E \times B^I \mid P(e) = \omega(0)\}$. Define $P': E^I \longrightarrow \Omega_p$ by $P'(\alpha) = (\alpha(0), P(\alpha))$. A map $\lambda: \Omega_p \longrightarrow E^I$ is called a lifting if the composition $P'\lambda: \Omega_p \longrightarrow \Omega_p$ is the identity map.

A lifting function λ is said to be regular if $\omega \in B^I$ a constant path implies that $\lambda(e, \omega)$ is a constant path. It is known that a fiber space $\mathcal{T} = \{E, P, B\}$ possesses a regular lifting function if B is metric.

Let $f: E \longrightarrow E$ be a fiber (-preserving) map and λ be the lifting for $\mathcal{T}$. For $b \in B$ a pathwise connected space, define maps $f_b: P^{-1}(b) \longrightarrow P^{-1}(b)$ as follows. Let $\omega \in B^I$ such that $\omega(0) = f'(b)$ and $\omega(1) = b$, then for $e \in P^{-1}(b)$, $f_b(e) = \lambda(f(e), \omega)(1)$. Thus any fiber map $f: E \longrightarrow E$ induces $f': B \longrightarrow B$ and $f_b: P^{-1}(b) \longrightarrow P^{-1}(b)$ for each $b \in B$ such that $Pf = f'P$, that is the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ \downarrow P & & \downarrow P \\ B & \xrightarrow{f'} & B \end{array}$$

Let $\mathcal{T} = \{E, P, B\}$ be a fiber space with lifting function λ and choose $b \in B$. A loop $\omega \in B^I$, such that $\omega(0) = \omega(1) = b$, induces a map $\bar{\omega}: P^{-1}(b) \longrightarrow P^{-1}(b)$ by $\bar{\omega}(e) = \lambda(e, \omega)(1)$. The fiber space $\mathcal{T}$ is said to be orientable if the induced homomorphism $\bar{\omega}_*: H_*(P^{-1}(b), Z) \longrightarrow H_*(P^{-1}(b), Z)$ is the identity homomorphism for every loop ω based at b. If $\mathcal{T}$ is orientable then it is known that $L(f) = L(f') \cdot L(f_b)$ for each $b \in B$ [4], [8], [12]. However, the similar result for the Nielsen number does not hold in general. Nevertheless, R. Brown and E. Fadell [3], [5] were able to show the following theorem:

Let $\mathcal{T} = \{E, P, B\}$ be a locally trivial fiber space with fiber F, where E, B and F are connected finite polyhedra, and let $f: E \longrightarrow E$ be a fiber map. If one of the following conditions is satisfied:

(a) $\pi_1(B) = \pi_2(B) = 0$,

(b) $\pi_1(F) = 0$,

(c) $\mathcal{T}$ is trivial and either $\pi_1(B) = 0$ or $f = f' \times f_b$,

then $N(f) = N(f') \cdot N(f_b)$ for all $b \in B$.

3. **Algebraic preliminaries.** Let G be a group and $h: G \longrightarrow G$ a homomorphism. Two elements α and β of G are h-equivalent if there exists $\gamma \in G$ such that $\alpha = \gamma\beta h(\gamma^{-1})$. The Reidemeister number $R(h)$ of h is defined to be the cardinality of the set of equivalence classes of G under h-equivalence.

It is known that if G is an abelian group and $h: G \longrightarrow G$ is a homomorphism, then the h-equivalence classes form a group equal to coker$(1 - h)$. Therefore, $R(h) = \text{Ord}(\text{coker}(1 - h))$. For more details on Reidemeister numbers, readers are referred to [2].

DEFINITION. Given a commutative square

$$S = \begin{array}{ccc} A & \xrightarrow{h} & B \\ \varphi\downarrow & & \downarrow\psi \\ A & \xrightarrow{h} & B \end{array}$$

of abelian groups and homomorphisms, define $\mathcal{P}(S)$ to be the order of the quotient group $h^{-1}(\psi(B))/\varphi(A)$.

REMARK. At this point we would like to thank Professor R. F. Brown for suggesting this definition and the notation $\mathcal{P}(S)$.

LEMMA 1. *$\mathcal{P}(S) \equiv \text{Ord}(h^{-1}(\psi(B))/\varphi(A))$ is equal to* $\text{Ord}(\ker h^{\#})$, *where $h^{\#}: A/\varphi(A) \longrightarrow B/\psi(B)$ is the map on the cokernels induced by $h: A \longrightarrow B$.*

LEMMA 2. *Let us assume that we have the following commutative diagram of abelian groups,*

$$\begin{array}{ccccccc} A & \xrightarrow{h} & B & \xrightarrow{k} & C & \longrightarrow & 0 \\ \varphi\downarrow & & \psi\downarrow & & \omega\downarrow & & \\ A & \xrightarrow{h} & B & \xrightarrow{k} & C & \longrightarrow & 0 \end{array}$$

where the rows are exact. Let S be the square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ 1-\varphi\downarrow & & \downarrow 1-\psi \\ A & \xrightarrow{h} & B \end{array}$$

induced by the above commutative diagram. Then

$$R(\psi)P(S) = R(\varphi)\cdot R(\omega).$$

PROOF. It is not so hard to see that the diagram in the hypothesis induces the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{h} & B & \xrightarrow{k} & C & \longrightarrow & 0 \\ 1-\varphi\downarrow & & 1-\psi\downarrow & & 1-\omega\downarrow & & \\ A & \xrightarrow{h} & B & \xrightarrow{k} & C & \longrightarrow & 0 \end{array}$$

This diagram induces the following short exact sequence

$$0 \longrightarrow \operatorname{coker}(1-\varphi)/\ker(h^{\#}) \xrightarrow{\bar{h}^{\#}} \operatorname{coker}(1-\psi) \xrightarrow{k^{\#}} \operatorname{coker}(1-\omega) \longrightarrow 0,$$

where $\bar{h}^{\#}$ and $k^{\#}$ are obvious homomorphisms induced by h and k on cokernels.

Thus the above short exact sequence gives $R(\psi) = R(\varphi)\cdot R(\omega)/\operatorname{Ord}(\ker h^{\#})$ or $R(\psi)\cdot P(S) = R(\varphi)\cdot R(\omega)$ by Lemma 1.

REMARK. Note that $\ker h^{\#}$ for

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \varphi\downarrow & & \downarrow\psi \\ A & \xrightarrow{h} & B \end{array}$$

and $\ker h^{\#}$ for

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ 1-\varphi\downarrow & & \downarrow 1-\psi \\ A & \xrightarrow{h} & B \end{array}$$

are not isomorphic in general.

LEMMA 3. *In Lemma 2 if h is a monomorphism and ω is a fixed point free homomorphism (i.e., $0 \in C$ is the only element ω fixes), then $P(S) = 1$ and $R(\psi) = R(\varphi)\cdot R(\omega)$.*

PROOF. Since h is a monomorphism we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{h} & B & \xrightarrow{k} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1-\varphi} & & \downarrow{\scriptstyle 1-\psi} & & \downarrow{\scriptstyle 1-\omega} & & \\ 0 & \longrightarrow & A & \xrightarrow{h} & B & \xrightarrow{k} & C & \longrightarrow & 0 \end{array}$$

This commutative diagram induces the following exact sequence of the cokernels of vertical maps:

(I)
$$0 \longrightarrow \ker(h^{\#}) \longrightarrow \operatorname{coker}(1-\varphi) \xrightarrow{h^{\#}} \operatorname{coker}(1-\psi) \xrightarrow{k^{\#}} \operatorname{coker}(1-\omega) \longrightarrow 0.$$

On the other hand we define a homomorphism Δ: $\ker(1-\omega) \longrightarrow \operatorname{coker}(1-\varphi)$ in the following way.

Let $\beta \in \ker(1-\omega)$. Then since k is onto we have some $e \in B$ such that $k(e) = \beta$. Then since $k(1-\psi)(e) = (1-\omega)k(e) = (1-\omega)(\beta) = 0$, there exists $\gamma \in A$ such that $h(\gamma) = (1-\psi)(e)$. Define $\Delta(\beta) = [\gamma] \in \operatorname{coker}(1-\varphi)$. It is easy to see that this is a well-defined homomorphism, and we have the following exact sequence [10]:

(II)
$$\ker(1-\omega) \xrightarrow{\Delta} \operatorname{coker}(1-\varphi) \xrightarrow{h^{\#}} \operatorname{coker}(1-\psi) \xrightarrow{k^{\#}} \operatorname{coker}(1-\omega) \longrightarrow 0.$$

From I and II we can see that $\ker(h^{\#})$ will vanish if $\ker(1-\omega) = 0$. $\ker(1-\omega) = 0$ if and only if $(1-\omega)$ is a monomorphism, that is, $(1-\omega)(\beta) = 0$ if and only if $\beta = 0$. This in turn says that $\omega(\beta) = \beta$ if and only if $\beta = 0$. In other words, $0 \in C$ is the only element ω fixes.

Thus we proved that if h is a monomorphism and ω: $C \longrightarrow C$ is fixed point free, then $\ker(h^{\#}) = 0$ and $P(S) = 1$. Note that I and II become the same short exact sequence. Now since $R(\psi) = \operatorname{Ord}(\operatorname{coker}(1-\psi))$, we have $R(\psi) = R(\varphi) \cdot R(\omega)$.

4. **The main results and some examples.** We assume throughout this section that we have $L(f) \neq 0$. There is, therefore, at least one essential fixed point class of f. We choose, once and for all, a fixed point x_0 of f which is in an essential fixed point class. Denote $\pi_1(X, x_0)$ by $\pi_1(X)$.

Define the Reidemeister number $R(f)$ of f to be the Reidemeister number of the induced homomorphism $f_{\#}$: $\pi_1(X) \longrightarrow \pi_1(X)$. Jiang's main result [11] asserts that if X is a Jiang space then $N(f) = R(f)$.

Let $T = \{E, P, B\}$ be a locally trivial fiber space and assume E, B, and

$P^{-1}(b)$ for each $b \in B$, are path-connected and have abelian fundamental groups. Thus we identify fundamental groups with their first integral homology groups, which are base point free. Since $\mathcal{T}$ is locally trivial, we have a regular path lifting λ. Let $f: E \longrightarrow E$ be a fiber map; then the map $f_b: P^{-1}(b) \longrightarrow P^{-1}(b)$ is defined by $f_b(e) = \lambda(f(e), \omega)(1)$ for $e \in P^{-1}(b)$ where ω is a path from $f'(b)$ to b.

Let $h_t(e) = \lambda(f(e), \omega)(t)$ for $t \in [0, 1]$ and $e \in P^{-1}(b)$. Then $\{h_t\}$ defines a homotopy from the composition $P^{-1}(b) \overset{i}{\subset} E \xrightarrow{f} E$ to the composition $P^{-1}(b) \xrightarrow{f_b} P^{-1}(b) \overset{i}{\subset} E$, where i is the inclusion map. Then

$$\begin{array}{ccc} H_1(P^{-1}(b)) & \xrightarrow{i_*} & H_1(E) \\ \downarrow{\scriptstyle f_{b*}} & & \downarrow{\scriptstyle f_*} \\ H_1(P^{-1}(b)) & \xrightarrow{i_*} & H_1(E) \end{array}$$

commutes. Also we have the commutative diagram

$$\begin{array}{ccc} \pi_1(P^{-1}(b), e_0) & \xrightarrow{i_\#} & \pi_1(E, e_0) \\ \Big| & & \downarrow{\scriptstyle \alpha - f_\#} \\ \Big|\; 1 - f_{b\#} & & \pi_1(E, e_1) \\ \downarrow & & \downarrow{\scriptstyle \alpha^{-1}} \\ \pi_1(P^{-1}(b), e_0) & \xrightarrow{i_\#} & \pi_1(E, e_0) \end{array}$$

where we take e_0 to be a fixed point of f_b and f, without loss of generality, and $\alpha: \pi_1(E, e_0) \longrightarrow \pi_1(E, e_1)$ to be a canonical isomorphism. Thus $\alpha^{-1}(\alpha - f_\#) = (1 - \alpha^{-1}f_\#)$.

If $\mathcal{T}$ is an orientable fiber space then we can take α to be an identity isomorphism. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) \\ S = 1 - f_{b\#} \downarrow & & 1 - f_\# \downarrow \\ \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) \end{array}$$

Readers are urged to compare this argument with that of [13, Lemma 1].

Define $\mathcal{P}(\mathcal{T}, f)$ to be $\mathcal{P}(S)$. It is not so hard to see that $\mathcal{P}(\mathcal{T}, f)$ is independent of the choice of $b \in B$.

In the theorem of Brown and Fadell, if E, B and F satisfy the Jiang condition then we have $L(f) = i(f)N(f)$, $L(f')N(f')$ and $L(f_b) = i(f_b) \cdot N(f_b)$ for each $b \in B$. By replacing these numbers in $L(f) = L(f') \cdot L(f_b)$ we get $i(f)\,N(f) = i(f')N(f')i(f_b)N(f_b)$. Dividing this by $N(f) = N(f') \cdot N(f_b)$ we get $i(f) = i(f') \cdot i(f_b)$. Unfortunately, this equality does not hold in general. But this observation strongly suggests

THEOREM 4. *Let $\mathcal{T} = \{E, P, B\}$ be a fiber space and $f: E \longrightarrow E$ be a fiber map. If E, B and $P^{-1}(b)$ for each $b \in B$ satisfy the Jiang condition, then $N(f) \cdot \mathcal{P}(\mathcal{T}, f) = N(f') \cdot N(f_b)$ and $i(f) = i(f') \cdot i(f_b)\,\mathcal{P}(\mathcal{T}, f)$ for each $b \in B$.*

PROOF. Any fiber map $f: E \longrightarrow E$ induces the following commutative diagram whose rows are exact fiber homotopy sequences:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \pi_2(B) & \longrightarrow & \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(B) & \longrightarrow 0 \\ & \downarrow & & \downarrow{\scriptstyle 1 - f_{b\#}} & & \downarrow{\scriptstyle 1 - f_\#} & & \downarrow{\scriptstyle 1 - f'_\#} & \\ \cdots \longrightarrow & \pi_2(B) & \longrightarrow & \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(B) & \longrightarrow 0 \end{array}$$

Thus from Lemma 2 we have $R(f) \cdot \mathcal{P}(\mathcal{T}, f) = R(f') \cdot R(f_b)$ for each $b \in B$. Since E, B and $P^{-1}(b)$ are Jiang spaces and $L(f) \neq 0$, we have $N(f) = R(f)$, $N(f') = R(f')$ and $N(f_b) = R(f_b)$. Thus we have $N(f) \cdot \mathcal{P}(\mathcal{T}, f) = N(f') \cdot N(f_b)$ for each $b \in B$.

For the fixed point indices we have

$$i(f)N(f) = L(f) = L(f') \cdot L(f_b) = i(f') \cdot i(f_b) \cdot N(f') \cdot N(f_b)$$
$$= i(f') \cdot i(f_b) \cdot \mathcal{P}(\mathcal{T}, f) \cdot N(f).$$

Thus $i(f) = i(f') \cdot i(f_b)\,\mathcal{P}(\mathcal{T}, f)$ for each $b \in B$.

EXAMPLE 1. Let us examine the example given in [5] in our context.

Let $S^1 \xrightarrow{i} S^3 \xrightarrow{P} S^2$ be the Hopf fibration. If $f: S^3 \longrightarrow S^3$ is a fiber map then f induces the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \pi_2(S^2) & \longrightarrow & \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & 0 \\ & \downarrow{\scriptstyle (1 - f'_\#)} & & \downarrow{\scriptstyle (1 - f_{b\#})} & & \\ 0 \longrightarrow & \pi_2(S^2) & \longrightarrow & \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & 0. \end{array}$$

Let $\deg f' = d$ and suppose $d \neq -1$ so that f' has a fixed point $b \in S^2$. Then f_b also has degree d, and $N(f) = N(f') = 1$ as long as $d \neq \pm 1$ since S^2 and S^3 are simply connected.

Thus

$$\ker i^\# = \frac{\tau_1(P^{-1}(b))}{(1 - f_{b\#})\pi_1(P^{-1}(b))} = \frac{Z}{(1 - f_{b\#})Z}.$$

Thus $\mathcal{P}(\mathcal{T}, f) = |1 - d|$ This shows that for $|d| \geqslant 3$, $\ker i^\# \neq 0$, thus $N(f) \neq N(f') \cdot N(f_b)$.

COROLLARY 5. *Let $\mathcal{T} = \{E, P, B\}$ be a fiber space, and let $f: E \longrightarrow E$ be a fiber map. If E, B and $P^{-1}(b)$ for some $b \in B$ satisfy the Jiang condition and the fundamental group $\pi_1(P^{-1}(b))$ is trivial, then $i(f) = i(f') \cdot i(f_b)$ and $N(f) = N(f')$.*

As an application of Theorem 4 we consider the case where $E = G$ is a compact connected Lie group, $H \subset G$ a closed connected subgroup, and $B = G/H$ the homogeneous space. Call a map $f: G \longrightarrow G$ fiber preserving if $f(xH) \subseteq f(x)H$ for all $x \in G$. Such a map induces a map $f': B \longrightarrow B$. If f has a fixed point, then $f(x_0H) \subset x_0H$ for some x_0; let f_0 be the restriction of f to x_0H. Let $\mathcal{T} = \{G, P, G/H\}$.

COROLLARY 6. *Let G be a compact, connected Lie group, H a closed connected subgroup and $f: G \longrightarrow G$ a fiber map such that $L(f) \neq 0$. Then $N(f)\,\mathcal{P}(\mathcal{T}, f) = N(f') \cdot N(f_0)$ and $i(f) = i(f') \cdot i(f_0) \cdot \mathcal{P}(\mathcal{T}, f)$.*

DEFINITION. Let $\mathcal{T} = \{E, P, B\}$ be a locally trivial fiber space. If the inclusion map $i: P^{-1}(b) \longrightarrow E$ induces a monomorphism $i_\#: \pi_1(P^{-1}(b)) \longrightarrow \pi_1(E)$ for each $b \in B$, then $\mathcal{T}$ is said to be an injective fibering.

If the spaces involved satisfy the Jiang condition we have the following theorem which has a nonempty intersection with the theorem of Brown and Fadell mentioned in the introduction.

THEOREM 7. *Let $\mathcal{T} = \{E, P, B\}$ be an injective fibering and $f: E \longrightarrow E$ a fiber map. If the spaces E, B and $P^{-1}(b)$ for some $b \in B$ satisfy the Jiang condition, and if $f'_\#: \pi_1(B) \longrightarrow \pi_1(B)$ is fixed point free, then $N(f) = N(f') \cdot N(f_b)$ and $i(f) = i(f') \cdot i(f_b)$ for each $b \in B$.*

PROOF. Since $\mathcal{T}$ is an injective fibering, $f: E \longrightarrow E$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(B) & \longrightarrow & 0 \\
 & & \big\downarrow (1 - f_{b\#}) & & \big\downarrow (1 - f_\#) & & \big\downarrow (1 - f'_\#) & & \\
0 & \longrightarrow & \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(B) & \longrightarrow & 0
\end{array}$$

From Lemma 3 we have $\mathcal{P}(\mathcal{T}, f) = 1$ so Theorem 1 gives the desired result.

COROLLARY 8. *Let $\mathcal{T} = \{E, P, B\}$ be an injective fibering and $f: E \longrightarrow E$ a fiber map. If the spaces E, B and $P^{-1}(b)$ for some $b \in B$ satisfy the Jiang condition and $\pi_1(B) = 0$, then $N(f) = N(f_b)$ and $i(f) = i(f') \cdot i(f_b)$ for each $b \in B$,*

REMARK. In Example 1, $\pi_1(S^2) = 0$ but $i: S^1 \longrightarrow S^3$ does not induce a monomorphism $i_\#: \pi_1(S^1) \longrightarrow \pi_1(S^3)$.

EXAMPLE 2. Let $\{E, P, L_{2n+1}(m)\}$ be a principal circle bundle over a $(2n+1)$-dimensional lens space. It is known that E, $L_{2n+1}(m)$ and $P^{-1}(b) = S^1$ satisfy the Jiang condition. Let $f: E \longrightarrow E$ be a fiber map. Then the homotopy exact sequence for the fiber space gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(L_{2n+1}(m)) & \longrightarrow & 0 \\ & & \downarrow (1 - f_{b\#}) & & \downarrow (1 - f_\#) & & \downarrow (1 - f'_\#) & & \\ 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(L_{2n+1}(m)) & \longrightarrow & 0 \end{array}$$

Thus from Theorem 7 if $0 \in \pi_1(L_{2n+1}(m)) \simeq Z_m$ is the only element $f'_\#$ fixes, then $N(f) = N(f') \cdot N(f_b)$ and $i(f) = i(f') \cdot i(f_b)$.

This will happen if $f'_\#$ maps a generator x to x^r where $(r, m) = 1$ and $(r-1, m) = 1$. We would like to point out that if m is odd then it always has a fixed point free automorphism.

5. **Applications.** In this section we investigate fixed point properties of fiber maps on some particular manifolds. At this point we would like to remind readers that if the base space is simply connected then the fiber space is orientable.

THEOREM 9. *Let $T = \{E, P, CP(n)\}$ be a principal circle bundle over n-dimensional complex projective space $CP(n)$, and let $f: E \longrightarrow E$ be a fiber map. Then $i(f')$ divides $i(f)$ and $N(f)$ divides $N(f_b)$.*

PROOF. T is classified by $[CP(n), CP(\infty)] \simeq H^2(CP(n); Z) \simeq Z$. We are interested in topological classification of T. The two extreme cases are $E = S^{2n+1}$ and $E = CP(n) \times S^1$. It is known that $N(f_b) = |L(f_b)|$ for each $b \in CP(n)$ [2]. Thus we have $i(f_b) \pm 1$. If $E = S^{2n+1}$ then, since E and $CP(n)$ are simply connected, we have $N(f) = 0 = N(f')$ or $N(f) = 1 = N(f')$. Thus $i(f)N(f) = i(f')N(f') \cdot i(f_b)N(f_b)$ becomes $i(f) = \pm i(f')N(f_b)$ for each $b \in CP(n)$, and $i(f')$ divides $i(f)$. If $E = CP(n) \times S^1$ then E satisfies the Jiang condition and the result follows from the product theorem [3], [5].

For other cases, we have, for each $i \in Z$, $E \simeq L_{2n+1}(|i|; 1, \ldots, 1)$, $(2n+1)$-dimensional lens space. Denote it by $L(|i|)$ for short. This fact can be proved easily from the same argument given in [14] for the case $n = 1$, that is $CP(1) = S^2$. It is known that any lens space satisfies the Jiang condition [2], [11]. Thus we have $i(f)N(f) = \pm i(f')N(f_b)$ for each $b \in CP(n)$ since $CP(n)$ is simply connected. What we want to show now is that $N(f)$ divides $N(f_b)$ for each $b \in CP(n)$.

$f: E \longrightarrow E$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_2(CP(n)) & \xrightarrow{|i|_\#} & \pi_1(S^1) & \longrightarrow & \pi_1(L(|i|)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (1-f'_\#)} & & \downarrow{\scriptstyle (1-f_{b\#})} & & \downarrow{\scriptstyle (1-f_\#)} & & \\ 0 & \longrightarrow & \pi_2(CP(n)) & \xrightarrow{|i|_\#} & \pi_1(S^1) & \longrightarrow & \pi_1(L(|i|)) & \longrightarrow & 0, \end{array}$$

where $|i|_\#$ is multiplication by $|i|$. Here we have $\pi_2(CP(n)) \simeq Z$, $\pi_1(S^1) \simeq Z$, and $\pi_1(L(|i|)) \simeq Z_{|i|}$. Since all spaces involved are Jiang spaces we have that $N(f) = \operatorname{Ord}(\operatorname{Coker}(1 - f_\#))$ and $N(f_b) = \operatorname{Ord}(\operatorname{Coker}(1 - f_{b\#}))$ for each $b \in CP(n)$ [2], [11].

If $\deg f' = d$, then $\deg f_b = d$ and $N(f_b) = |1 - d|$. Now we have $\deg f = d \bmod |i|$. Let it be equal to k. Thus we have $N(f) = (1 - k, |i|)$ [2]. Hence $i(f) \cdot (1 - k, |i|) = \pm i(f')|1 - d|$. Since $1 - k \equiv 1 - d \bmod |i|$, we conclude that $(1 - k, |i|)$ divides $1 - d$, that is, $N(f)$ divides $N(f_b)$. This also proves that $i(f')$ must divide $i(f)$.

Corollary 10. *Let (S^1, M^n) be a free action of a circle group on a simple connected closed n-dimensional manifold M^n. Let $M^* = M^n/S^1$ be the orbit space and $f\colon M^n \longrightarrow M^n$ be an orbit preserving map. Then $i(f) = \pm i(f')N(f_b)$ for each $b \in M^*$.*

Theorem 11. *Let $\mathcal{T} = \{E, P, CP(n)\}$ be a principal torus T^k-bundle over the n-dimensional complex projective space $CP(n)$. If $f\colon E \longrightarrow E$ is a fiber map, then $i(f)N(f| = \pm i(f')N(f_b)$ for each $b \in B$.*

A proof of this theorem can be deduced from that of the next corollary with the fact $N(f_b) = |L(f_b)|$ given in [1]. Note that $N(f) = 0$ if $N(f') = 0$; also we would like to remind readers that if n is even then $N(f') = 1$ always in Theorems 9 and 11.

Corollary 12. Let $\mathcal{T} = \{E, P, CP(n)\}$ be a principal torus T^k-bundle over $CP(n)$. If $f\colon E \longrightarrow E$ is a fiber map such that for some $b \in CP(n)$, $f_b\colon P^{-1}(b) \longrightarrow P^{-1}(b)$ is homotopic to a fixed point free map, then there exists a map $g\colon E \longrightarrow E$ which is homotopic to f and fixed point free.

Proof. Let f'_b be a fixed point free map on $P^{-1}(b)$ which is homotopic to some f_b. Clearly $L(f'_b) = 0$ and since the Lefschetz number is invariant under homotopy, $L(f_b) = 0$. Now since this bundle is an orientable bundle we have $L(f) = L(f') \cdot L(f_b)$ and $L(f) = 0$.

If $\mathcal{T}$ is a trivial bundle then $E = CP(n) \times T^k$ and $T(E) = \pi_1(E)$ follows trivially, $L(f) = 0$ implies $N(f) = 0$, and there exists a fixed point free map $g\colon E \longrightarrow E$ homotopic to f (see [15]).

For other extreme cases we have $E = S^{2n+1} \times T^{k-1}$ [12]. Thus $T(E) = \pi_1(E)$ and the result follows as in the trivial case.

Now we consider other general cases. We can consider E to be a total space of a T^{k-1} bundle over the total space of the circle bundle over $CP(n)$. The total spaces of the circle bundles over $CP(n)$ are classified as in the proof of Theorem 9. Let $L(|i|)$ be one of them. It is known [12] that any total space of a T^{k-1}-principal bundle over $L(|i|)$ is given by $L(d) \times T^{k-1}$ for some divisor d of $|i|$. Now

$$T(E) = T(L(d) \times T^{k-1}) \simeq T(L(d)) \oplus T(T^{k-1})$$
$$\simeq \pi_1(L(d)) \oplus \pi_1(T^{k-1}) \simeq \pi_1(L(d) \times T^{k-1}).$$

The second and third isomorphisms follow from [9] and [11] respectively. Thus E satisfies the Jiang condition and we have $N(f) = 0$ since $L(f) = 0$. Now since E satisfies Wecken's condition [15], there is a fixed point free map: $g: E \longrightarrow E$ homotopic to f.

Note. If we proved Theorem 11 first then Corollary 12 follows immediately from $i(f)N(f) = \pm i(f')N(f_b)$.

THEOREM 13. *Let $T = \{E, P, B\}$ be a fiber space over an aspherical manifold B with fiber T^k. Furthermore, assume that $\pi_1(E)$ and $\pi_1(B)$ are abelian groups and $f: E \longrightarrow E$ is a fiber map. Then $i(f) = \pm i(f')$.*

PROOF. Since T is a fiber space, where $\pi_1(E)$ is abelian, over an aspherical manifold B we can associate an effective fixed point free T^k action on E such that $E/T^k \simeq B$. It is not so hard to see that E is also aspherical [6]. Now define $h: T^k \longrightarrow E$ by $h(t) = tx$, where $x \in E$. This map induces $h^x_\#: \pi_1(T^k) \longrightarrow \pi_1(E)$ and which is known to be a monomorphism. That is, $\pi_1(E)$ is at least as large as $\pi_1(T^k)$. Now if $\pi_1(E)$ is abelian then the Chern class $C_1 \in H^2(B; Z \oplus \cdots \oplus Z)$ vanishes and we obtain $N(f) = N(f') \cdot N(f_b)$ for each $b \in B$ from [3]. Now since $\pi_1(E)$ and $\pi_1(B)$ are abelian and E, B aspherical we have $T(E) = \pi_1(E)$ and $T(B) = \pi_1(B)$ [2]. Thus $i(f)N(f) = \pm i(f')N(f') \cdot N(f_b)$ for each $b \in B$ follows from [1], [4], [8] and [12]. Now if we divide this by $N(f) = N(f')N(f_b)$ we get $i(f) = \pm i(f')$.

REMARK. Of course, this result follows more or less trivially from the results of [3] and [6], but for the sake of having some insight into the theorem we gave a proof in full detail. Also note that this theorem follows directly from Theorem 7 and aspherically of the spaces involved.

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