INEQUALITIES FOR A COMPLEX MATRIX
WHOSE REAL PART IS POSITIVE DEFINITE

BY

CHARLES R. JOHNSON

ABSTRACT. Denote the real part of \( A \in M_n(C) \) by \( H(A) = \Re(A + A^*) \).
We provide dual inequalities relating \( H(A^{-1}) \) and \( H(A)^{-1} \) and an identity
between two functions of \( A \) when \( A \) satisfies \( H(A) > 0 \). As an application we
give an inequality (for matrices \( A \) satisfying \( H(A) > 0 \)) which generalizes Hadamard's
determinantal inequality for positive definite matrices.

0. Introduction. Denote the real part of an \( n \times n \) complex matrix \( A \) by

\[ H(A) = \Re(A + A^*) \]

and define \( \Pi_n = \{ A \in M_n(C) : H(A) > 0 \} \). If \( A \in \Pi_n \), then \( A \) is nonsingular and
\( A^{-1} \in \Pi_n \). It is our goal to present inequalities relating the positive definite
matrices \( H(A^{-1}) \) and \( H(A)^{-1} \) when \( A \in \Pi_n \). These results may then be com-
pared with different inequalities obtained in [4] for the same problem when \( A \) is,
in addition, restricted to have real entries. This leads to an identity linking two
functions of \( A \) when \( A \in \Pi_n \). As an application of these inequalities we also pre-
sent a result which generalizes Hadamard's determinantal inequality for positive
definite matrices. We also generalize the Ostrowski-Taussky inequality for matrices
in \( \Pi_n \).

1. Main result. Proofs of the following useful fact may be found in [1] or [2].

**Lemma.** If \( A \in \Pi_n \), then \( A^{-1}A^* \) is similar to a unitary matrix.

In order to facilitate the statement of results, we define

\[ M = M(A) \equiv \max \Re(\lambda) \quad \text{and} \quad m = m(A) \equiv \min \Re(\lambda) \]

where the maximum and minimum are taken over all eigenvalues \( \lambda \) of \( A^{-1}A^* \), \( A \)
nonsingular. In view of the lemma and the fact \( I + A^{-1}A^* = A^{-1}(A + A^*) \) is
invertible for \( A \in \Pi_n \), we necessarily have (for \( A \in \Pi_n \)) that 
\(-1 < m(A) < M(A) \leq 1\). Our main result is
Theorem 1. If $A \in \Pi_n$, then

(i) $cH(A^{-1}) - H(A)^{-1} > 0$ if and only if $c > 2/(m + 1)$ (equivalently $m > (2 - c)/c$), and

(ii) $dH(A)^{-1} - H(A^{-1}) > 0$ if and only if $d > (M + 1)/2$ (equivalently $M < 2d - 1$).

Proof. In the following calculation we let $\lambda(X)$ denote an arbitrary eigenvalue of the $n$ by $n$ matrix $X$.

\[ cH(A^{-1}) - H(A)^{-1} > 0 \]

\[ \iff \lambda([cH(A^{-1})]^{-1}(cH(A^{-1}) - H(A)^{-1})) > 0 \]

\[ \iff \lambda(I - [cH(A)H(A^{-1})]^{-1}) > 0 \iff \lambda(- [cH(A)H(A^{-1})]^{-1}) > -1 \]

\[ \iff \lambda([cH(A)H(A^{-1})]^{-1}) < 1 \iff \lambda(cH(A)H(A^{-1})) > 1 \]

\[ \iff \lambda(H(A)H(A^{-1})) > 1/c \iff \lambda((A + A^*)/2 \cdot (A^{-1} + (A^{-1})^*)/2) > 1/c \]

(i) \[ \iff \lambda((2 - c)/2c) > 1/c \]

\[ \iff \lambda(A^*A^{-1} + AA^{-1}^*)/4) > 1/c \]

\[ \iff \lambda(A^*A^{-1} + (A^*A^{-1})^{-1})/4) > 1/c - 1/2 = (2 - c)/2c \]

\[ \iff \lambda(A^*A^{-1} + (A^*A^{-1})^{-1}) > 2((2 - c)/c) \]

\[ \iff \text{Re } \lambda(A^*A^{-1}) = \text{Re } \lambda(A^{-1}A^*) > (2 - c)/c \]

\[ \iff m > (2 - c)/c \iff cm > 2 - c \iff cm + c > 2 \]

\[ \iff c(m + 1) > 2 \iff c > 2/(m + 1), \]

and (i) is complete.

Note. Each matrix mentioned, excepting $A^*A^{-1}$ and $A^{-1}A^*$, has had necessarily real roots. In particular, since the roots of $A^*A^{-1} + (A^*A^{-1})^{-1}$, or equivalently $A^{-1}A^* + (A^{-1}A^*)^{-1}$, are necessarily real, we have by this calculation alone that any complex roots of $A^{-1}A^*, A \in \Pi_n$, must be 1 in absolute value.

The proof of (ii) is similar.

\[ dH(A)^{-1} - H(A^{-1}) > 0 \]

\[ \iff \lambda([dH(A)^{-1}]^{-1}[dH(A)^{-1} - H(A^{-1})]) > 0 \]

\[ \iff \lambda(I - H(A)H(A^{-1})/d) > 0 \iff \lambda(- H(A)H(A^{-1})/d) > -1 \]

\[ \iff \lambda(- H(A)H(A^{-1})) > -d \iff \lambda(H(A)H(A^{-1})) < d \]

(ii) \[ \iff \lambda(I/2 + (A^*A^{-1} + AA^{-1}^*)/4) < d \]

\[ \iff \lambda(A^*A^{-1} + (A^*A^{-1})^{-1}) < (d - 1/2)4 \]

\[ \iff \text{Re } \lambda(A^*A^{-1}) = \text{Re } \lambda(A^{-1}A^*) < (d - 1/2)2 \]

\[ \iff M < 2d - 1 \iff M + 1 < 2d \iff d > (M + 1)/2 \]

and the proof is complete.
Corollary 1. \( A \in \Pi_n \) implies \( H(A)^{-1} \succeq H(A^{-1}) \).

Proof. Since \( M \) is at most 1 by the lemma, \( d = 1 \) must satisfy \( d \geq (M + 1)/2 \). The corollary then follows from part (ii) of Theorem 1.

From Corollary 1 the following fact about determinants immediately follows:

Corollary 2. \( A \in \Pi_n \) implies

\[
\det H(A)^{-1} \succeq \det H(A^{-1}) \quad \text{and} \quad \det H(A)H(A^{-1}) \leq 1.
\]

2. Comparison to real case. We define

\[
S(A) = \frac{1}{2}(A - A^*) \quad \text{and} \quad T = T(A) = \max \{ \pm t_j \}
\]

where \( \pm t_j \) are the eigenvalues of \( H(A)^{-1}S(A) \), for \( H(A) \) nonsingular. In [4] it is shown that

Theorem 2. If \( A \in \Pi_n \), \( A \) is real, and \( c \) is a real scalar, then \( cH(A^{-1}) - H(A)^{-1} > 0 \) if and only if \( c > 1 + T^2 \).

The validity of both Theorems 1 and 2 implies remarkably that

\[
m = \frac{1 - T^2}{1 + T^2} \quad \text{or} \quad T = \left( \frac{1 - m}{1 + m} \right)^{1/2}
\]

at least when \( A \in \Pi_n \) has real entries. In fact this identity holds also when \( A \in \Pi_n \) is complex.

Theorem 3. For \( A \in \Pi_n \),

\[
m = \frac{1 - T^2}{1 + T^2} \quad \text{or} \quad T = \left( \frac{1 - m}{1 + m} \right)^{1/2}
\]

Proof. The two assertions

\[
m = \frac{1 - T^2}{1 + T^2} \quad \text{and} \quad T = \left( \frac{1 - m}{1 + m} \right)^{1/2}
\]

are equivalent. We shall prove the former. Again let \( \lambda(X) \) denote an arbitrary eigenvalue of \( X \in M_n(C) \).

First we obtain an expression for \( T^2 \), assuming \( A \in \Pi_n \):

\[
T^2 = \max |\lambda(H(A)^{-1}S(A)H(A)^{-1}S(A))| = \max (-\lambda(H(A)^{-1}S(A)H(A)^{-1}S(A)))
\]

\[
= -\min (\lambda(H(A)^{-1}S(A)H(A)^{-1}S(A)))
\]

\[
= -\min (\lambda((A + A^*)^{-1}(A - A^*)(A + A^*)^{-1}(A - A^*)))
\]

\[
= -\min (\lambda((I + A^{-1}A^*)^{-1}(I - A^{-1}A^*)(I + A^{-1}A^*)^{-1}(I - A^{-1}A^*)))
\]

\[
= -\min (\lambda((I - A^{-1}A^*)^2(I + A^{-1}A^*)^{-2})).
\]

We therefore have

\[
\frac{1 - T^2}{1 + T^2} = \frac{1 + \min (\lambda((I - A^{-1}A^*)^2(I + A^{-1}A^*)^{-2}))}{1 - \min (\lambda((I - A^{-1}A^*)^2(I + A^{-1}A^*)^{-2}))}
\]

which we hope to show is equal to \( m = \min \text{Re}(\lambda(A^{-1}A^*)) \).
By the lemma of §1, the eigenvalues of $A^{-1}A^*$ are all of absolute value 1. We have also noted that none of them is equal to $-1$. If we let $\alpha_1, \cdots, \alpha_n$ be $n$ complex numbers of absolute value 1, none of which is $-1$, it then suffices to show that

$$\min_{1 \leq i \leq n} \Re \alpha_i = \frac{1 + \min_{1 \leq i \leq n} \left( (1 - \alpha_i)/(1 + \alpha_i) \right)^2}{1 - \min_{1 \leq i \leq n} \left( (1 - \alpha_i)/(1 + \alpha_i) \right)^2}.$$  

Let $\alpha_j = a_j + ib_j, a_j^2 + b_j^2 = 1, j = 1, \ldots, n$. First note that $((1 - \alpha_j)/(1 + \alpha_j))^2 = -b_j^2/(1 + a_j)$, a nonpositive real number, so that $\min_{1 \leq i \leq n} ((1 - \alpha_i)/(1 + \alpha_i))^2$ is well defined, nonpositive and is attained for some particular $\alpha_i$, call it $\alpha = a + bi$. Then the right-hand side of (*) evaluated at $\alpha$ is equal to

$$\frac{(1 + \alpha)^2 + (1 - \alpha)^2}{(1 + \alpha)^2 - (1 - \alpha)^2} = \frac{2 + 2\alpha^2}{4\alpha} = \frac{1 + (a + bi)^2}{2(a + bi)} = (1/(a + bi) + a + bi)/2 = (a - bi + a + bi)/2 = a = \Re(\alpha).$$

Now, since $(1 + t)/(1 - t)$ is an increasing function of $t$ when $t \leq 0$ is real, it follows that the right-hand side of (*) is smallest (over all $\alpha_j, 1 \leq j \leq n$) when evaluated at $\alpha$, the minimizing value of $((1 - \alpha_j)/(1 + \alpha_j))^2$. Therefore the left-hand side of (*) is also equal to $\Re(\alpha)$ and the proof is complete.

It is now clear that Theorem 2 is a corollary of Theorems 1 and 3. In fact we may simply relax the assumption that $A$ is real in Theorem 2.

**Corollary 3.** If $A \in \Pi_n$ and $c$ is a real scalar, then $c H(A^{-1}) - H(A)^{-1} > 0$ if and only if $c > 1 + T^2$.

We also give another corollary which will be used later.

**Corollary 4.** For $A \in \Pi_n$, we have $m(A^{-1}) = m(A), M(A^{-1}) = M(A)$ and $T(A^{-1}) = T(A)$.

**Proof.** The first two asserted equalities follow from the lemma of §1 and the definitions of $m$ and $M$. The third follows from the fact that $T$ may be expressed as a function of $m$.

3. Hadamard generalization. For a positive definite hermitian matrix $A = (a_{ij})$, Hadamard's inequality states that

$$\det A \leq \prod_{i=1}^{n} a_{ii}.$$  

We shall say that a general matrix $A = (a_{ij}) \in M_n(C)$ satisfies the $H$-inequality if

$$|\det A| \leq d(A) \equiv \left| \prod_{i=1}^{n} \Re(a_{ii}) \right|.$$  

Matrices in $\Pi_n$ do not necessarily satisfy the $H$-inequality, though, of course, the hermitian elements do. However, we may use Theorem 1 to obtain a generalization of Hadamard's inequality valid throughout $\Pi_n$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem 4. Suppose \( A \in \Pi_n \). Then \(|\det A| \leq kd(A)\) where\( k = |\det (I + B)^{-1}|, B = H(A)^{-1}S(A) \) and \( c = 1 + T^2 \).

Proof. Let \( k = k(A) \) be as defined and we first note that \( k(A^{-1}) = k(A) \). This is valid because of Corollary 4 and because

\[
((I + H(A^{-1})^{-1}S(A^{-1}))^{-1}^* = (AH(A^{-1}))^* = A^{-1}H(A) = (H(A)^{-1}A)^{-1},
\]

\[(I + H(A)^{-1}S(A))^{-1} = (I + B)^{-1}.
\]

The fact that \( A = H(A)[I + B] \) implies that

\[
|\det A^{-1}| = |\det (I + B)^{-1}\det H(A)^{-1}|
\]

which is \( \leq |\det (I + B)^{-1}\det cH(A^{-1})| \) because of Corollary 3. But

\[
|\det (I + B)^{-1}\det cH(A^{-1})| = |\det c(I + B)^{-1}\det H(A^{-1})| = k \det H(A^{-1})
\]

which is \( \leq kd(A^{-1}) \) because of the original \( H \)-inequality. We thus have

\[
|\det A^{-1}| \leq k(A)d(A^{-1}).
\]

However, because \( k(A^{-1}) = k(A) \) and since \( \Pi_n \) is closed under inversion, we may as well write \( |\det A| \leq k(A)d(A) \).

Remark 1. If \( A \in \Pi_n \) is hermitian, then \( k = 1 \) and we obtain the usual \( H \)-inequality as a special case.

Remark 2. Because of Theorem 3 we may replace “\( c = 1 + T^2 \)” in Theorem 4 by “\( c = 2/(m + 1) \)”. Also, in case \( A \) is real, the absolute value bars may be dropped throughout the previous argument.

Remark 3. \( 0 \leq |\det (I + B)^{-1}| \leq c \) and \( 1 \leq |\det c(I + B)^{-1}| \) in Theorem 4.

Example. Let \( A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \). Then \( A \in \Pi_2 \) and equality is attained in the inequality asserted by Theorem 4. In this case \( \det A = 3; d(A) = 2; c = 3/2; (I + B)^{-1} = (2/3)\begin{bmatrix} 1 & 1/2 \\ -1 & 1 \end{bmatrix} \) and \( k = \det c(I + B)^{-1} = 3/2. \) Thus \( \det A = 3 = (3/2)2 = kd(A) \).

4. The Ostrowski-Taussky inequality. In [5] it is shown that for \( A \in \Pi_n \)

\[
\det H(A) \leq |\det A|
\]

and equality holds if and only if the skew-hermitian part \( S(A) = 0 \). To some extent, the inequalities of this and other papers which have been cited are generalizations of the Ostrowski-Taussky inequality. We now give a direct generalization, the statement of which was suggested to us by M. Marcus.

Theorem 5. If \( A \in \Pi_n \), then

\[
|\det A|^{2/n} \geq (\det H(A))^{2/n} + |\det S(A)|^{2/n}.
\]
Equality holds if and only if each eigenvalue of $H(A)^{-1} S(A)$ has the same absolute value.

**Proof.** It suffices to assume $H(A) = I$ because since each component is positive

$$|\det A|^{2/n} \geq (\det H(A))^{2/n} + |\det S(A)|^{2/n}$$

if and only if

$$(\det H(A))^{-1/n} |\det A|^{2/n} (\det H(A))^{-1/n}$$

$$\geq (\det H(A))^{-1/n} [(\det H(A))^{2/n} + |(\det S(A))^{2/n}|] (\det H(A))^{1/n}$$

or equivalently

$$|\det(I + S)|^{2/n} \geq (\det I)^{2/n} + |\det S|^2/n$$

where $S$ is the skew-hermitian matrix $H(A)^{-1/2} S(A) H(A)^{-1/2}$.

Now let $t_1, \ldots, t_n$ be real numbers such that the eigenvalues of $S$ are $it_1, \ldots, it_n$. Then $|\det(I + S)|^2 = \prod_{j=1}^n (1 + t_j^2)$ and we thus must show that

$$\prod_{j=1}^n (1 + t_j^2)^{1/n} \geq 1 + \left( \prod_{j=1}^n a_j \right)^{1/n}$$

where the $a_j = t_j^2$ are arbitrary nonnegative numbers, $j = 1, \ldots, n$. But this latter statement is just Minkowski's well-known inequality in which equality is attained if and only if $a_1 = a_2 = \ldots = a_n$. Since the eigenvalues of $S$ are the same as those of $H(A)^{-1} S(A)$, this completes the proof of the theorem.

As a corollary it follows that another term may be added linearly to the Ostrowski-Taussky result.

**Corollary 5.** If $A \in \Pi_n$, then

$$|\det A| \geq \det H(A) + |\det S(A)|.$$