TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 212, 1975

INEQUALITIES FOR A COMPLEX MATRIX WHOSE REAL PART IS POSITIVE DEFINITE

BY

CHARLES R. JOHNSON[1]

ABSTRACT. Denote the real part of $A \in M_n(C)$ by $H(A) = \frac{1}{2}(A + A^*)$. We provide dual inequalities relating $H(A^{-1})$ and $H(A)^{-1}$ and an identity between two functions of A when A satisfies $H(A) > 0$. As an application we give an inequality (for matrices A satisfying $H(A) > 0$) which generalizes Hadamard's determinantal inequality for positive definite matrices.

0. Introduction. Denote the real part of an n by n complex matrix A by

$$H(A) \equiv \tfrac{1}{2}(A + A^*)$$

and define $\Pi_n = \{A \in M_n(C) \colon H(A) > 0\}$. If $A \in \Pi_n$, then A is nonsingular and $A^{-1} \in \Pi_n$. It is our goal to present inequalities relating the positive definite matrices $H(A^{-1})$ and $H(A)^{-1}$ when $A \in \Pi_n$. These results may then be compared with different inequalities obtained in [4] for the same problem when A is, in addition, restricted to have real entries. This leads to an identity linking two functions of A when $A \in \Pi_n$. As an application of these inequalities we also present a result which generalizes Hadamard's determinantal inequality for positive definite matrices. We also generalize the Ostrowski-Taussky inequality for matrices in Π_n.

1. Main result. Proofs of the following useful fact may be found in [1] or [2].

LEMMA. *If $A \in \Pi_n$, then $A^{-1}A^*$ is similar to a unitary matrix.*

In order to facilitate the statement of results, we define

$$M = M(A) \equiv \max \operatorname{Re}(\lambda) \quad \text{and} \quad m = m(A) \equiv \min \operatorname{Re}(\lambda)$$

where the maximum and minimum are taken over all eigenvalues λ of $A^{-1}A^*$, A nonsingular. In view of the lemma and the fact $I + A^{-1}A^* = A^{-1}(A + A^*)$ is invertible for $A \in \Pi_n$, we necessarily have (for $A \in \Pi_n$) that $-1 < m(A) \leq M(A) \leq 1$. Our main result is

---

Received by the editors July 2, 1974.

AMS (MOS) subject classifications (1970). Primary 15A09, 15A45, 15A57.

Key words and phrases. Eigenvalues, Hadamard inequality, real part, positive definite.

(1) A portion of this work is included in the author's doctoral dissertation written under the advisement of Professor Olga Taussky Todd at the California Institute of Technology.

THEOREM 1. *If $A \in \Pi_n$, then*

(i) $cH(A^{-1}) - H(A)^{-1} > 0$ *if and only if* $c > 2/(m+1)$ (*equivalently* $m > (2-c)/c$), *and*

(ii) $dH(A)^{-1} - H(A^{-1}) > 0$ *if and only if* $d > (M+1)/2$ (*equivalently* $M < 2d-1$).

PROOF. In the following calculation we let $\lambda(X)$ denote an arbitrary eigenvalue of the n by n matrix X.

$$
\begin{aligned}
&cH(A^{-1}) - H(A)^{-1} > 0\\
&\longleftrightarrow \lambda([cH(A^{-1})]^{-1}(cH(A^{-1}) - H(A)^{-1})) > 0\\
&\longleftrightarrow \lambda(I - [cH(A)H(A^{-1})]^{-1}) > 0 \longleftrightarrow \lambda(-[cH(A)H(A^{-1})]^{-1}) > -1\\
&\longleftrightarrow \lambda([cH(A)H(A^{-1})]^{-1}) < 1 \longleftrightarrow \lambda(cH(A)H(A^{-1})) > 1\\
&\longleftrightarrow \lambda(H(A)H(A^{-1})) > 1/c \longleftrightarrow \lambda((A+A^*)/2 \cdot (A^{-1} + (A^{-1})^*)/2) > 1/c\\
\text{(i)}\quad &\longleftrightarrow \lambda(I/2 + (A^*A^{-1} + AA^{-1*})/4) > 1/c\\
&\longleftrightarrow \lambda((A^*A^{-1} + (A^*A^{-1})^{-1})/4) > 1/c - 1/2 = (2-c)/2c\\
&\longleftrightarrow \lambda(A^*A^{-1} + (A^*A^{-1})^{-1}) > 2((2-c)/c)\\
&\longleftrightarrow \operatorname{Re} \lambda(A^*A^{-1}) = \operatorname{Re} \lambda(A^{-1}A^*) > (2-c)/c\\
&\longleftrightarrow m > (2-c)/c \longleftrightarrow cm > 2 - c \longleftrightarrow cm + c > 2\\
&\longleftrightarrow c(m+1) > 2 \longleftrightarrow c > 2/(m+1),
\end{aligned}
$$

and (i) is complete.

Note. Each matrix mentioned, excepting A^*A^{-1} and $A^{-1}A^*$, has had necessarily real roots. In particular, since the roots of $A^*A^{-1} + (A^*A^{-1})^{-1}$, or equivalently $A^{-1}A^* + (A^{-1}A^*)^{-1}$, are necessarily real, we have by this calculation alone that any complex roots of $A^{-1}A^*$, $A \in \Pi_n$, must be 1 in absolute value.

The proof of (ii) is similar.

$$
\begin{aligned}
&dH(A)^{-1} - H(A^{-1}) > 0\\
&\longleftrightarrow \lambda([dH(A)^{-1}]^{-1}[dH(A)^{-1} - H(A^{-1})]) > 0\\
&\longleftrightarrow \lambda(I - H(A)H(A^{-1})/d) > 0 \longleftrightarrow \lambda(-H(A)H(A^{-1})/d) > -1\\
\text{(ii)}\quad &\longleftrightarrow \lambda(-H(A)H(A^{-1})) > -d \longleftrightarrow \lambda(H(A)H(A^{-1})) < d\\
&\longleftrightarrow \lambda(I/2 + (A^*A^{-1} + (A^*A^{-1})^{-1})/4) < d\\
&\longleftrightarrow \lambda(A^*A^{-1} + (A^*A^{-1})^{-1}) < (d - 1/2)4\\
&\longleftrightarrow \operatorname{Re} \lambda(A^*A^{-1}) = \operatorname{Re} \lambda(A^{-1}A^*) < (d - 1/2)2\\
&\longleftrightarrow M < 2d - 1 \longleftrightarrow M + 1 < 2d \longleftrightarrow d > (M+1)/2
\end{aligned}
$$

and the proof is complete.

COROLLARY 1. *$A \in \Pi_n$ implies $H(A)^{-1} \geqslant H(A^{-1})$.*

PROOF. Since M is at most 1 by the lemma, $d = 1$ must satisfy $d \geqslant (M + 1)/2$. The corollary then follows from part (ii) of Theorem 1.

From Corollary 1 the following fact about determinants immediately follows:

COROLLARY 2. *$A \in \Pi_n$ implies*

$$(\det H(A))^{-1} \geqslant \det H(A^{-1}) \quad \textit{and} \quad \det H(A)H(A^{-1}) \leqslant 1.$$

2. **Comparison to real case.** We define

$$S(A) \equiv \tfrac{1}{2}(A - A^*) \quad \text{and} \quad T = T(A) \equiv \max_j\{|t_j|\}$$

where $\pm it_j$ are the eigenvalues of $H(A)^{-1}S(A)$, for $H(A)$ nonsingular. In [4] it is shown that

THEOREM 2. *If $A \in \Pi_n$, A is real, and c is a real scalar, then $cH(A^{-1}) - H(A)^{-1} > 0$ if and only if $c > 1 + T^2$.*

The validity of both Theorems 1 and 2 implies remarkably that

$$m = (1 - T^2)/(1 + T^2) \quad \text{or} \quad T = ((1 - m)/(1 + m))^{1/2}$$

at least when $A \in \Pi_n$ has real entries. In fact this identity holds also when $A \in \Pi_n$ is complex.

THEOREM 3. *For $A \in \Pi_n$,*

$$m = (1 - T^2)/(1 + T^2) \quad \textit{or} \quad T = ((1 - m)/(1 + m))^{1/2}.$$

PROOF. The two assertions

$$m = (1 - T^2)/(1 + T^2) \quad \text{and} \quad T = ((1 - m)/(1 + m))^{1/2}$$

are equivalent. We shall prove the former. Again let $\lambda(X)$ denote an arbitrary eigenvalue of $X \in M_n(C)$.

First we obtain an expression for T^2, assuming $A \in \Pi_n$:

$$\begin{aligned}
T^2 &= \max|\lambda(H(A)^{-1}S(A)H(A)^{-1}S(A))| = \max(-\lambda(H(A)^{-1}S(A)H(A)^{-1}S(A))) \\
&= -\min(\lambda(H(A)^{-1}S(A)H(A)^{-1}S(A))) \\
&= -\min(\lambda((A + A^*)^{-1}(A - A^*)(A + A^*)^{-1}(A - A^*))) \\
&= -\min(\lambda((I + A^{-1}A^*)^{-1}(I - A^{-1}A^*)(I + A^{-1}A^*)^{-1}(I - A^{-1}A^*))) \\
&= -\min(\lambda((I - A^{-1}A^*)^2(I + A^{-1}A^*)^{-2})).
\end{aligned}$$

We therefore have

$$\frac{1 - T^2}{1 + T^2} = \frac{1 + \min(\lambda((I - A^{-1}A^*)^2(I + A^{-1}A^*)^{-2}))}{1 - \min(\lambda((I - A^{-1}A^*)^2(I + A^{-1}A^*)^{-2}))}$$

which we hope to show is equal to $m \equiv \min \operatorname{Re}(\lambda(A^{-1}A^*))$.

By the lemma of §1, the eigenvalues of $A^{-1}A^*$ are all of absolute value 1. We have also noted that none of them is equal to -1. If we let $\alpha_1, \cdots, \alpha_n$ be n complex numbers of absolute value 1, none of which is -1, it then suffices to show that

$$(*) \qquad \min_{1\leqslant j\leqslant n} \operatorname{Re}\alpha_i = \frac{1+\min_{1\leqslant j\leqslant n}((1-\alpha_j)/(1+\alpha_j))^2}{1-\min_{1\leqslant j\leqslant n}((1-\alpha_j)/(1+\alpha_j))^2}.$$

Let $\alpha_j = a_j + ib_j$, $a_j^2 + b_j^2 = 1$, $j = 1, \ldots, n$. First note that $((1-\alpha_j)/(1+\alpha_j))^2 = -b_j^2/(1+a_j)$, a nonpositive real number, so that $\min_{1\leqslant j\leqslant n}((1-\alpha_j)/(1+\alpha_j))^2$ is well defined, nonpositive and is attained for some particular α_j, call it $\alpha = a + bi$. Then the right-hand side of $(*)$ evaluated at α is equal to

$$\frac{(1+\alpha)^2 + (1-\alpha)^2}{(1+\alpha)^2 - (1-\alpha)^2} = \frac{2+2\alpha^2}{4\alpha} = \frac{1+(a+bi)^2}{2(a+bi)}$$

$$= (1/(a+bi) + a + bi)/2 = (a - bi + a + bi)/2 = a = \operatorname{Re}(\alpha).$$

Now, since $(1+t)/(1-t)$ is an increasing function of t when $t \leqslant 0$ is real, it follows that the right-hand side of $(*)$ is smallest (over all α_j, $1 \leqslant j \leqslant n$) when evaluated at α, the minimizing value of $((1-\alpha_j)/(1+\alpha_j))^2$. Therefore the left-hand side of $(*)$ is also equal to $\operatorname{Re}(\alpha)$ and the proof is complete.

It is now clear that Theorem 2 is a corollary of Theorems 1 and 3. In fact we may simply relax the assumption that A is real in Theorem 2.

Corollary 3. *If $A \in \Pi_n$ and c is a real scalar, then $cH(A^{-1}) - H(A)^{-1} > 0$ if and only if $c > 1 + T^2$.*

We also give another corollary which will be used later.

Corollary 4. For $A \in \Pi_n$, *we have $m(A^{-1}) = m(A)$, $M(A^{-1}) = M(A)$ and $T(A^{-1}) = T(A)$.*

Proof. The first two asserted equalities follow from the lemma of §1 and the definitions of m and M. The third follows from the fact that T may be expressed as a function of m.

3. **Hadamard generalization.** For a positive definite hermitian matrix $A = (a_{ij})$, Hadamard's inequality states that

$$\det A \leqslant \prod_{i=1}^{n} a_{ii}.$$

We shall say that a general matrix $A = (a_{ij}) \in M_n(C)$ satisfies the H-inequality if

$$|\det A| \leqslant d(A) \equiv \left|\prod_{i=1}^{n} \operatorname{Re}(a_{ii})\right|.$$

Matrices in Π_n do not necessarily satisfy the H-inequality, though, of course, the hermitian elements do. However, we may use Theorem 1 to obtain a generalization of Hadamard's inequality valid throughout Π_n.

THEOREM 4. *Suppose* $A \in \Pi_n$. *Then* $|\det A| \leqslant kd(A)$ *where* $k = |\det c(I+B)^{-1}|$, $B = H(A)^{-1}S(A)$ *and* $c = 1 + T^2$.

PROOF. Let $k = k(A)$ be as defined and we first note that $k(A^{-1}) = k(A)$. This is valid because of Corollary 4 and because

$$((I + H(A^{-1})^{-1}S(A^{-1}))^{-1})^* = (AH(A^{-1}))^* = A^{-1}H(A) = (H(A)^{-1}A)^{-1},$$

$$(I + H(A)^{-1}S(A))^{-1} = (I+B)^{-1}.$$

The fact that $A = H(A)[I + B]$ implies that

$$|\det A^{-1}| = |\det(I+B)^{-1}\det H(A)^{-1}|$$

which is $\leqslant |\det(I+B)^{-1}\det cH(A^{-1})|$ because of Corollary 3. But

$$|\det(I+B)^{-1}\det cH(A^{-1})| = |\det c(I+B)^{-1}\det H(A^{-1})| = k\det H(A^{-1})$$

which is $\leqslant kd(A^{-1})$ because of the original H-inequality. We thus have

$$|\det A^{-1}| \leqslant k(A)d(A^{-1}).$$

However, because $k(A^{-1}) = k(A)$ and since Π_n is closed under inversion, we may as well write $|\det A| \leqslant k(A)d(A)$.

REMARK 1. If $A \in \Pi_n$ is hermitian, then $k = 1$ and we obtain the usual H-inequality as a special case.

REMARK 2. Because of Theorem 3 we may replace "$c = 1 + T^2$" in Theorem 4 by "$c = 2/(m+1)$". Also, in case A is real, the absolute value bars may be dropped throughout the previous argument.

REMARK 3. $0 \leqslant |\det(I+B)^{-1}| \leqslant 1 \leqslant c$ and $1 \leqslant |\det c(I+B)^{-1}|$ in Theorem 4.

EXAMPLE. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. Then $A \in \Pi_2$ and equality is attained in the inequality asserted by Theorem 4. In this case $\det A = 3$; $d(A) = 2$; $c = 3/2$; $(I+B)^{-1} = (2/3)\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & 1 \end{bmatrix}$ and $k = \det c(I+B)^{-1} = 3/2$. Thus $\det A = 3 = (3/2)2 = kd(A)$.

4. The Ostrowski-Taussky inequality. In [5] it is shown that for $A \in \Pi_n$

$$\det H(A) \leqslant |\det A|$$

and equality holds if and only if the skew-hermitian part $S(A) = 0$. To some extent, the inequalities of this and other papers which have been cited are generalizations of the Ostrowski-Taussky inequality. We now give a direct generalization, the statement of which was suggested to us by M. Marcus.

THEOREM 5. *If* $A \in \Pi_n$, *then*

$$|\det A|^{2/n} \geqslant (\det H(A))^{2/n} + |\det S(A)|^{2/n}.$$

Equality holds if and only if each eigenvalue of $H(A)^{-1}S(A)$ has the same absolute value.

PROOF. It suffices to assume $H(A) = I$ because since each component is positive

$$|\det A|^{2/n} \geq (\det H(A))^{2/n} + |\det S(A)|^{2/n}$$

if and only if

$$(\det H(A))^{-1/n}|\det A|^{2/n}(\det H(A))^{-1/n} \geq (\det H(A))^{-1/n}[(\det H(A))^{2/n} + |(\det S(A))|^{2/n}](\det H(A))^{1/n}$$

or equivalently

$$|\det(I + S)|^{2/n} \geq (\det I)^{2/n} + |\det S|^{2/n}$$

where S is the skew-hermitian matrix $H(A)^{-1/2}S(A)H(A)^{-1/2}$.

Now let $t_1, \ldots, t_n$ be real numbers such that the eigenvalues of S are $it_1, \ldots, it_n$. Then $|\det(I + S)|^2 = \Pi_{j=1}^n(1 + t_j^2)$ and we thus must show that

$$\prod_{j=1}^n (1 + a_j)^{1/n} \geq 1 + \left(\prod_{j=1}^n a_j\right)^{1/n}$$

where the $a_j = t_j^2$ are arbitrary nonnegative numbers, $j = 1, \ldots, n$. But this latter statement is just Minkowski's well-known inequality in which equality is attained if and only if $a_1 = a_2 = \ldots = a_n$. Since the eigenvalues of S are the same as those of $H(A)^{-1}S(A)$, this completes the proof of the theorem.

As a corollary it follows that another term may be added linearly to the Ostrowski-Taussky result.

COROLLARY 5. *If $A \in \Pi_n$, then*

$$|\det A| \geq \det H(A) + |\det S(A)|.$$

REFERENCES

1. C. R. DePrima and C. R. Johnson, *The range of $A^{-1}A^*$ in $GL(n, C)$*, Linear Algebra Appl. 9 (1974), 209–222.

2. Ky Fan, *Generalized Cayley transforms and strictly dissipative matrices*, Linear Algebra Appl. 5 (1972), 155–172. MR 45 #5145.

3. ———, *On real matrices with positive definite symmetric component*, Linear Multilinear Algebra 1 (1973), 1–4.

4. C. R. Johnson, *An inequality for matrices whose symmetric part is positive definite*, Linear Algebra Appl. 6 (1973), 13–18. MR 47 #251.

5. A. M. Ostrowski and O. Taussky, *On the variation of the determinant of a positive definite matrix*, Nederl. Akad. Wetensch. Proc. Ser. A 54 = Indag. Math. 13 (1951), 383–385. MR 13, 900.

INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742