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EXTENSION OF FOURIER $L^p - L^q$ MULTIPLIERS[1]

BY

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ABSTRACT. By $M_p^q(\Gamma)$ we denote the space of Fourier $L^p - L^q$ multipliers on the LCA group Γ. K. de Leeuw [4] (for $\Gamma = R^a$), N. Lohoué [16] and S. Saeki [19] have shown that if Γ_0 is a closed subgroup of Γ, and ϕ is a continuous function in $M_p^p(\Gamma)$, then the restriction ϕ_0 of ϕ to Γ_0 is in $M_p^p(\Gamma_0)$, and $\|\phi_0\|_{M_p^p} \leqslant \|\phi\|_{M_p^p}$. We answer here a natural question arising from this result: we show that every continuous function ψ in $M_p^p(\Gamma)$ is the restriction to Γ_0 of a continuous $M_p^p(\Gamma)$ function whose norm is the same as that of ψ. A Figà-Talamanca and G. I. Gaudry [8] proved this with the extra condition that Γ_0 be discrete: our technique develops their ideas. An extension theorem for $M_p^q(\Gamma_0)$ is obtained: this complements work of Gaudry [11] on restrictions of $M_p^q(\Gamma)$-functions to Γ_0.

1. **Introduction.** By G and Γ we denote dual LCA (locally compact abelian hausdorff topological) groups, written additively, whose Haar measures dx and $d\gamma$ are normalized so that the inversion theorem holds. For an extended positive real number p, satisfying the conditions $1 \leqslant p \leqslant \infty$, $L^p(G)$ denotes the usual Lebesgue space of functions (function classes, strictly speaking) on G, and p' denotes the conjugate index defined by the rule: $p'^{-1} + p^{-1} = 1$. The space of bounded regular complex Borel measures on G is denoted $M(G)$: $M(G)$ can be viewed as a subspace of the space $M_{loc}(G)$ of Radon measures on G. By $C_{loc}(G)$ we denote the space of all continuous functions on G, with the topology of locally uniform convergence; $C_c(G)$ is the subspace of $C_{loc}(G)$ of functions f whose support supp(f) is compact, and $C_0(G)$ is the closure of $C_c(G)$ in the uniform topology. The dual space of $C_c(G)$ is the space $M_{loc}(G)$; that of $C_0(G)$ is $M(G)$.

The Fourier transformation F is defined on $L^1(G)$ by the formula

$$Ff(\gamma) = \int_G dx\overline{\gamma(x)}f(x) \qquad \forall \gamma \in \Gamma;$$

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we shall often write $\hat{f}$ for the Fourier transform of f. The space $A(\Gamma)$ is the space of Fourier transforms of functions in $L^1(G)$ with the inherited norm: the Riemann-Lebesgue lemma shows that a function in $A(\Gamma)$ is continuous and vanishes at infinity. The inversion theorem states that, for any f in $L^1 \cap A(G)$,

$$(\hat{f})\hat{}(x) = f_\vee(x) \qquad \forall x \in G,$$

$f_\vee$ being the reflection of f: $f_\vee(x) = f(-x)$.

The Fourier transformation extends naturally to a mapping of $M(G)$:

$$\hat{\mu}(\gamma) = \int_G d\mu(x)\overline{\gamma(x)} \qquad \forall \gamma \in \Gamma.$$

For any μ in $M(G)$, $\hat{\mu}$ is a bounded continuous function on Γ: $B(\Gamma)$ is the image of $M(G)$ under the Fourier transformation. The space $B(\Gamma)$ has the inherited norm:

$$\|\hat{\mu}\|_B = \|\mu\|_M = \int_G |d\mu|.$$

The Fourier transformation maps $L^1 \cap L^2(G)$ into $L^2(\Gamma)$, and

$$\|\hat{f}\|_2 = \|f\|_2 \qquad \forall f \in L^1 \cap L^2(G)$$

(provided the Haar measures of G and Γ are adjusted so that the inversion theorem holds) and so extends uniquely to an isometry of $L^2(G)$ onto $L^2(\Gamma)$. One can interpolate between the spaces $L^1(G)$ and $L^2(G)$ [respectively $C_0(\Gamma)$ and $L^2(\Gamma)$] and extend the Fourier transformation to a mapping of $L^p(G)$ into $L^{p'}(\Gamma)$ ($1 \leq p \leq 2$); but unless G is compact, to extend the Fourier transformation to $L^p(G)$ ($p > 2$) one must use distributional methods. We shall use the notion of "quasi-measures", introduced by Gaudry [9], [10]; we shall give a different, though equivalent, definition.

The space $A_c(G)$ is the subspace of $A(G)$ of all functions with compact supports. A linear functional Φ on $A_c(G)$ is called a quasimeasure if, for each compact subset K of G,

$$|\langle \Phi, u \rangle| \leq c(K)\|u\|_A \qquad \forall u \in A_K(G),$$

$A_K(G)$ being the Banach space of $A(G)$-functions which vanish off K. The convolution $\Phi * u$ of a quasimeasure Φ and an $A_c(G)$-function u is defined by the rule

$$\Phi * u(x) = \langle \Phi, T_{-x}u \rangle \qquad \forall x \in G,$$

$T_x u$ being the translate by x of u: $T_x u(y) = u(y - x)$. If we imbed the space of locally integrable functions $L^1_{\mathrm{loc}}(G)$ into the space $Q(G)$ of quasimeasure on G by the formula

$$\langle f, u \rangle = f * u(0) \qquad \forall f \in L^1_{\mathrm{loc}}(G), \forall u \in A_c(G),$$

where the convolution of two functions f and g on G is defined as usual:

$$f * g(x) = \int_G dy f(x-y)g(y) \qquad \forall x \in G,$$

then the convolution of a quasimeasure and an $A_c(G)$-function coincides with the usual convolution if the quasimeasure "is" a locally integrable function. All this is proved in some detail in [3].

By $L_p^q(G)$, we denote the space of quasimeasure Φ on G such that $\|\Phi * u\|_q \leqslant C\|u\|_p \; \forall u \in A_c(G)$. If $p < \infty$, $L_p^q(G)$ can be identified with the space of continuous linear operators from $L^p(G)$ to $L^q(G)$ which commute with translations; if $p = \infty$, a similar result holds if $L^p(G)$ is replaced by $C_0(G)$. The least value of C in the inequality above, which we write $\|\Phi\|_{L_p^q}$, is just the operator norm of the associated linear mapping from $L^p(G)$ ($C_0(G)$ if $p = \infty$) to $L^q(G)$.

Figà-Talamanca [6] ($p = q$) and Figà-Talamanca and Gaudry [7] proved that $L_p^q(G)$ can be identified with the dual space of a space $A_p^q(G)$ of locally integrable functions on G. If $1 \leqslant p, q \leqslant \infty$, set

$$s = [\max\{p^{-1} - q^{-1}, 0\}]^{-1};$$

naturally, $\infty = 0^{-1}$. Write $E^p(G)$ for the norm closure of $C_c(G)$ in $L^p(G)$. If either G is compact and $1 \leqslant q < p \leqslant \infty$ or G is arbitrary and $1 \leqslant p \leqslant q \leqslant \infty$, then $A_p^q(G)$ is the image of the completed projective tensor product $E^p(G) \hat{\otimes} E^{q'}(G)$ under the continuous linear mapping P: $E^p(G) \hat{\otimes} E^{q'}(G) \longrightarrow E^s(G)$ which carries $f \otimes g$ to $f * g$; $A_p^q(G)$ has the induced norm. For such p, q and G, $A_p^q(G)$ is a nontrivial subspace of $E^s(G)$. For other p, q and G (i.e. noncompact G, and p and q satisfying the inequalities $1 \leqslant q < p \leqslant \infty$), $A_p^q(G)$ is defined to be the trivial space containing only the zero function.

For those p, q and G for which $A_p^q(G)$ is nontrivial, it is easy to see that a function u on G belongs to $A_p^q(G)$ if and only if it can be represented:

$$u = \sum_1^\infty f_n * g_n \quad \text{in } L^1_{\mathrm{loc}}(G),$$

where f_n and g_n are in $E^p(G)$ and $E^{q'}(G)$ respectively for each n, and

$$\sum_1^\infty \|f_n\|_p \|g_n\|_{q'} < \infty.$$

The infimum of all sums $\Sigma_1^\infty \|f_n\|_p \|g_n\|_{q'}$ such that $u = \Sigma_1^\infty f_n * g_n$ is the $A_p^q(G)$-norm of u, written $\|u\|_{A_p^q}$. For all these p, q and G, $A_c(G)$ is a dense subspace of $A_p^q(G)$ [3].

For certain values of p and q, $A_p^q(G)$ is equivalent to more extrinsically defined spaces: $A_2^2(G) = A(G)$, $A_1^1(G) = C_0(G)$, and $A_1^q(G) = L^{q'}(G)$ $(q > 1)$. These characterizations, proved by Figà-Talamanca and Gaudry [7], lead to dual characterizations of certain spaces $L_p^q(G)$: $L_1^1(G) = M(G)$, and $L_1^p(G) = L^p(G)$ if $p > 1$.

We are now in a position to define a distributional Fourier transformation (also denoted $\mathcal{F}$). It is an easy exercise to show that $\mathcal{F}A_c(G)$ is a dense subspace of $A_p^q(G)$ (see [2]): for a quasimeasure Φ in $L_p^q(G)$ (in particular, for $L^q(G)$-functions with $q > 2$), we define the Fourier transform $\hat{\Phi}$ to be the unique quasimeasure on Γ such that $\langle \hat{\Phi}, u \rangle = \langle \Phi, \hat{u} \rangle \; \forall u \in A_c(\Gamma)$. This extended Plancherel formula uses the $A_p^q(G) - L_p^q(G)$ duality; unfortunately, no really satisfactory definition of the Fourier transform of an arbitrary quasimeasure seems possible.

Write $M_p^q(\Gamma)$ for the space of Fourier transforms of quasimeasure in $L_p^q(G)$. Unless G is compact, the Fourier transform of an element of $L_p^q(G)$ ($p < q$) is not necessarily a Radon measure. We shall restrict our attention to the subspace $\mathcal{M}_p^q(\Gamma)$ of $M_p^q(\Gamma)$ of those elements which "are" locally integrable functions. This is not so limiting as it might appear: if $p = q$, or if $1 \leqslant p \leqslant q \leqslant 2$, or if $2 \leqslant p \leqslant q \leqslant \infty$, then we are excluding nothing from consideration; $\mathcal{M}_p^q(\Gamma)$ is all of $M_p^q(\Gamma)$ in these cases [14]. Also, when G is compact, $\mathcal{F}Q(\Gamma) = L^\infty(\Gamma)$.

We conclude this section with a little group theory and an introductory theorem. Suppose that G_0 is a closed subgroup of the LCA group G. Let Γ_0 be the annihilator of G_0 in the dual group Γ of G. Then the dual group of G/G_0 can be identified with Γ_0. The canonical projection π of G onto G/G_0 is dual to the injection $\hat{\pi}$ of Γ_0 into Γ, in that

$$\pi x(\gamma_0) = x(\hat{\pi}\gamma_0) \qquad \forall x \in G, \forall \gamma_0 \in \Gamma_0.$$

For these facts, see [13, §24]. If the Haar measures dx, dx_0 and $d\dot{x}$ on G, G_0 and G/G_0 are normalized so that

$$\int_G dx f(x) = \int_G d\dot{x} \int_{G_0} dx_0 \, f(x + x_0) \qquad \forall f \in C_c(G)$$

[13, 28.54], then our standing assumption about the normalizations of the Haar measures $d\gamma$, $d\dot{\gamma}$ and $d\gamma_0$ on the dual groups Γ, Γ/Γ_0 and Γ_0 imply that

$$\int_\Gamma d\gamma g(\gamma) = \int_{\Gamma/\Gamma_0} d\dot{\gamma} \int_{\Gamma_0} d\gamma_0 g(\gamma + \gamma_0) \qquad \forall g \in C_c(\Gamma).$$

In particular, if G_0 is compact and has total mass one (the standard normalization for the Haar measure on a compact group), then Γ_0 is open in Γ and the Haar measure on Γ_0 is that on Γ restricted to Γ_0.

THEOREM 1. *Suppose that G_0 is a compact subgroup of the LCA group G, and that the total mass of G_0 is one. Then the "periodification" mapping π^*: $f \longrightarrow f \circ \pi$, induced by the canonical projection π of G onto G/G_0, is an isometry of $A_p^q(G/G_0)$ onto the subspace of $A_p^q(G)$ of functions constant on cosets of G_0 in G. Dually, the restriction mapping $\hat{\pi}^*$: $\phi \longrightarrow \phi \circ \hat{\pi}$ induced by the injection of Γ_0 into Γ maps $\mathcal{M}_p^q(\Gamma)$ onto $\mathcal{M}_p^q(\Gamma_0)$ without increasing norms; restricted to the subspace of $\mathcal{M}_p^q(\Gamma)$ of functions which vanish off Γ_0, $\hat{\pi}^*$ is an isometry onto $\mathcal{M}_p^q(\Gamma_0)$.*

PROOF. The proof of this theorem is quite simple. Cowling [2] gives complete details; most readers will prefer to reconstruct the proof from the sketch given here.

If $f \in L^p(G/G_0)$ and $g \in L^{q'}(G/G_0)$, then $f \circ \pi \in L^p(G)$, $g \circ \pi \in L^{q'}(G)$, and $(f * g) \circ \pi = (f \circ \pi) * (g \circ \pi)$; moreover

$$\|(f*g) \circ \pi\|_{A_p^q} \leq \|f \circ \pi\|_p \|g \circ \pi\|_{q'} = \|f\|_p \|g\|_{q'}.$$

It follows that π^* maps $A_p^q(G/G_0)$ into $A_p^q(G)$ without increasing norms.

Let μ_{G_0} be the idempotent measure on G defined by the rule

$$\mu_{G_0} * f(0) = \int_{G_0} dx_0 f(x_0) \qquad \forall f \in C_c(G).$$

If $f \in L^p(G)$ and $g \in L^{q'}(G)$, then

$$f * g * \mu = (f * \mu) * (g * \mu) = (h \circ \pi) * (k \circ \pi)$$

for appropriate h and k in $L^p(G/G_0)$ and $L^{q'}(G/G_0)$ respectively. Thus

$$f * g * \mu = (h * k) \circ \pi$$

and

$$\|h * k\|_{A_p^q} \leq \|h\|_p \|k\|_{q'} = \|f * \mu\|_p \|g * \mu\|_{q'} \leq \|f\|_p \|g\|_{q'}.$$

If u belongs to $A_p^q(G)$ and is constant on cosets of G_0 in G, then $u = u * \mu$. It follows that $u = v \circ \pi$ for some v in $A_p^q(G/G_0)$ of norm no greater than that of u.

We have shown that π^* is a homomorphism of $A_p^q(G/G_0)$ into $A_p^q(G)$ which does not increase norms, and that the mapping M_μ: $u \longrightarrow \pi^{*-1}(u * \mu)$ is a left inverse of π^* (i.e. $M_\mu \cdot \pi^*$ is the identity map on $A_p^q(G/G_0)$), which also does not increase norms. The first part of the theorem is proved.

It is easy to show that the adjoint maps of π^* and M_μ, which we write as $\hat{\pi}^*$ and $\hat{M}_\mu$, are given by the formula

$$\hat{\pi}^*\Phi = \Phi|_{\Gamma_0} \qquad \forall \Phi \in Q(\Gamma)$$

and

$$\langle \hat{M}_\mu \Phi, u \rangle = \langle \Phi, u|_{\Gamma_0} \rangle \qquad \forall \Phi \in Q(\Gamma_0), \forall u \in A_c(\Gamma);$$

here we have identified the dual spaces of $FA_c(G)$ and $FA_c(G/G_0)$ with the spaces $Q(\Gamma)$ and $Q(\Gamma_0)$ respectively via the Fourier transformation. From this identification and the continuity properties of π^* and M_μ on the A_p^q-spaces, the last part of the theorem follows immediately.

As this theorem disposes of the periodification action on A_p^q-spaces induced by the canonical projection π of G onto G/G_0 for compact groups G, we can assume hereafter that G is not compact. The only indices p and q of interest in a discussion of noncompact groups are those satisfying the inequalities $1 \leq p \leq q$

$\leqslant \infty$. For notational convenience, we shall write S for the set $\{(p, q)\colon 1 \leqslant p \leqslant q \leqslant \infty\}$.

2. **The main theorem. Discussion.** As we stated in the abstract of this paper, if Γ_0 is a closed subgroup of the LCA group Γ, then the restriction ϕ_0 of a continuous function in $M_p^p(\Gamma)$ belongs to $M_p^p(\Gamma_0)$, and $\|\phi_0\|_{M_p^p} \leqslant \|\phi\|_{M_p^p}$. In Theorem 1, we showed that any ϕ_0 in $M_p^q(\Gamma_0)$ can be extended to some ϕ in $M_p^q(\Gamma)$ without increasing norms, provided that Γ_0 is open in Γ. Figà-Talamanca and Gaudry [8] proved that if Γ_0 is discrete then any ϕ_0 in $M_p^q(\Gamma_0)$ can be extended to a continuous function ϕ in $M_p^q(\Gamma)$ without increasing norms. Our main theorem, whose proof is an extension of Figà-Talamanca and Gaudry's ideas, states

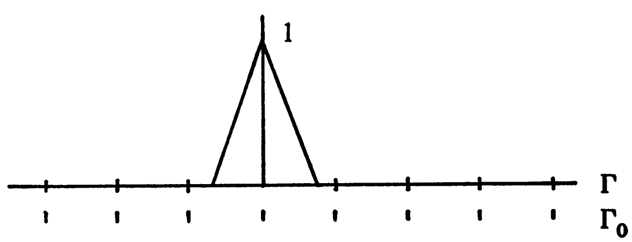
THEOREM 2. *Suppose that Γ_0 is a closed but not open subgroup of the LCA group Γ. There exist a linear operator L from $C_{loc}(\Gamma_0)$ to $C_{loc}(\Gamma)$ and a compact subset K of Γ such that, for any continuous function ψ on Γ_0,*

$$\|L\psi\|_{M_p^q} \leqslant \|\psi\|_{M_p^q} \quad \forall (p, q) \in S, \qquad L\psi(\gamma_0) = \psi(\gamma_0) \quad \forall \gamma_0 \in \Gamma_0,$$

$$\operatorname{supp}(L\psi) \subseteq \operatorname{supp}(\psi) + K.$$

The proof of Theorem 2 occupies the rest of this paper. The remainder of §2 reviews the proof of Figà-Talamanca and Gaudry's theorem, and sketches the proof of our own. §3 contains three technical theorems, whose proof can be profitably omitted at first reading: Theorem 3 is a structure theorem of independent interest, Theorem 4 is a lemma of Figà-Talamanca and Gaudry [8] and Theorem 5 is a useful group-theoretic construction. §4 ties together Theorems 3, 4 and 5 to prove Theorem 2.

We now review the extension theorem of Figà-Talamanca and Gaudry. Suppose that Γ_0 is a discrete subgroup of the LCA group Γ. Let Δ be a "triangular" function on Γ such that $\Delta(0) = 1$ and $\operatorname{supp}(\Delta) \cap \Gamma_0 = \{0\}$; we picture Δ thus:



A natural extension Φ of the function ϕ on Γ_0 is

$$\Phi(\gamma) = \sum_{\gamma_0 \in \Gamma_0} \Delta(\gamma - \gamma_0)\phi(\gamma_0) \qquad \forall \gamma \in \Gamma. \tag{2.1}$$

For technical reasons, we take Δ^2 instead of Δ. The theorem that, with suitable conditions on Δ, Φ is in $M_p^q(\Gamma)$ if ϕ is in $M_p^q(\Gamma_0)$, can be proved by showing that the mapping $f \longrightarrow F$,

$$F(\dot{x}) = \int_{G_0} dx_0 f(x + x_0)\hat{\Delta} * \hat{\Delta}(x + x_0) \qquad \forall x \in G,$$

maps $A_p^q(G)$ into $A_p^q(G/G_0)$ continuously (cf. Theorem 4) and that its adjoint is the mapping $\phi \longrightarrow \Phi$.

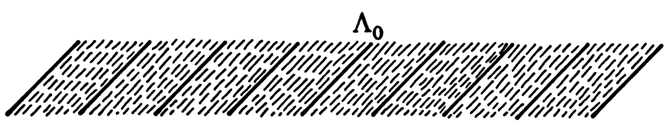
If Γ_0 is any closed subgroup of Γ and ϕ on Γ_0 is a continuous function, then the formula

$$\Phi(\gamma) = \int_{\Gamma_0} d\gamma_0 \Delta^2(\gamma - \gamma_0)\phi(\gamma_0) \qquad \forall \gamma \in \Gamma,$$

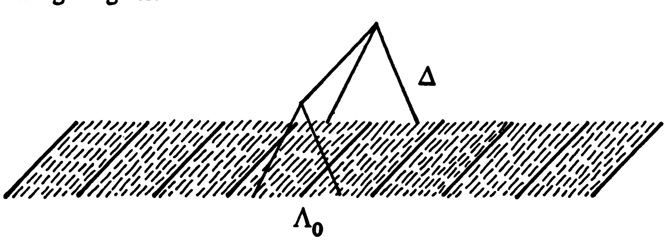
which is a natural generalization of (2.1), still makes sense for any compactly supported Δ on G. However, there is no obvious choice of Δ so that the desired restriction property—$\phi = \Phi|_{\Gamma_0}$—still holds.

The first step of our proofs is to simplify the situation by using structure theory. The subgroup Γ_0 can be written as $R^a \oplus \Delta_0$, where R is the group of real numbers, a is a nonnegative integer and Δ_0 has a compact open subgroup Λ_0 [13, 24.30]; R^a is an absolute direct summand (Theorem 3), so that Γ can be written as $R^a \oplus \Delta$, where Δ_0 is a closed subgroup of Δ. We discuss the case where $a = 0$, i.e. where Γ_0 has a compact open subgroup Λ_0. Only notational changes are needed to restore the R^a-factor to the proofs, but these modifications do complicate the notation quite substantially.

Next, assuming that Λ_0 is compact and open in Δ_0, we observe that the quotient group Γ_0/Λ_0 is a closed discrete subgroup of Γ/Λ_0. We picture this as follows:



(Λ_0 is as shown, Γ_0 is the union of the solid lines, and the union of the solid and dotted lines is Γ.) On Γ/Λ_0, there is a natural "triangular" function, k, say. We extend k by periodicity over Λ_0, and obtain a function Δ which is indicated in the following diagram:



Consider the formula

$$\Phi(\gamma) = \int_{\Gamma_0} d\gamma_0 \Delta^2(\gamma - \gamma_0)\phi(\gamma_0) \qquad \forall \gamma \in \Gamma.$$

A Φ given by this formula is constant on cosets of Λ_0 in Γ, and only if ϕ is constant on cosets of Λ_0 in Γ_0 is ϕ the restriction of Φ to Γ_0. However, if in place of Δ^2 we have $\Delta^2 \cdot F_\alpha$, where F_α behaves like a Fejér kernel on Λ_0, the integral becomes $\Phi_\alpha(\gamma)$:

$$\begin{aligned}\Phi_\alpha(\gamma) &= \int_{\Gamma_0} d\gamma_0 \Delta^2(\gamma - \gamma_0) F_\alpha(\gamma - \gamma_0)\phi(\gamma_0) \\ &= \int_{\Gamma_0/\Lambda_0} d\dot{\gamma}_0 \int_{\Lambda_0} d\lambda_0 \Delta^2(\gamma - \gamma_0 - \lambda_0) F_\alpha(\gamma - \gamma_0 - \lambda_0)\phi(\gamma_0 + \lambda_0) \\ &= \int_{\Gamma_0/\Lambda_0} d\dot{\gamma}_0 k^2(\dot{\gamma} - \dot{\gamma}_0) \int_{\Lambda_0} d\lambda_0 F_\alpha(\gamma - \gamma_0 - \lambda_0)\phi(\gamma_0 + \lambda_0),\end{aligned}$$

since Δ is the periodic extension of k over Λ_0. But k can be chosen to vanish on all nonzero points of Γ_0/Λ_0, and therefore

$$\Phi_\alpha(\gamma) = \int_{\Lambda_0} d\lambda_0 F_\alpha(-\lambda_0)\phi(\gamma + \lambda_0)$$

for any γ in Γ_0. Because $(F_\alpha)_{\alpha \in A}$ is a Fejér-type net on Λ_0, if ϕ is continuous, the last expression will converge to $\phi(\gamma)$ as "α increases".

Consequently, the essence of the proof is now seen to be the construction of a net $(F_\alpha)_{\alpha \in A}$ of functions on Γ which is "Fejér on Λ_0" and with the property that if $\Phi_\alpha(\gamma) = \int_{\Gamma_0} d\gamma_0 \Delta^2(\gamma - \gamma_0) F_\alpha(\gamma - \gamma_0)\phi(\gamma_0)$, then

$$\|\Phi_\alpha\|_{M_p^q} \leqslant C\|\phi\|_{M_p^q}$$

and Φ_α converges at least locally uniformly on all of Γ. For then the net $(\Phi_\alpha)_{\alpha \in A}$ would have a weak-star limit Φ in $M_p^q(\Gamma)$, which would be the pointwise limit

$$\Phi(\gamma) = \lim_\alpha \Phi_\alpha(\gamma) \qquad \forall \gamma \in \Gamma,$$

and since $\Phi_\alpha(\gamma_0)$ converges to $\phi(\gamma_0)$ for γ_0 in Γ_0, it would follow that $\Phi|_{\Gamma_0} = \phi$. To have Φ_α converging locally uniformly, we need to ensure that $(F_\alpha)_{\alpha \in A}$, or $(\hat{F}_\alpha)_{\alpha \in A}$, behaves nicely. In our proof, we require $(\hat{F}_\alpha)_{\alpha \in A}$ to increase monotonely.

The natural way to construct a Fejér-type net on a compact group K is to consider, for each finite subset α of the discrete dual D, the function $\hat{F}_\alpha$:

$$\hat{F}_\alpha = |\alpha|^{-1}\chi_\alpha * \chi_\alpha.$$

If K is the circle group T, and hence D is the group of integers Z, the functions $\hat{F}_N = (2N + 1)^{-1}\chi_{[-N,N]}$, $N = 1, 2, \ldots$, will do the trick. However, even if

$D = Z$, it is not generally true that $|\alpha|^{-1}\chi_\alpha * \chi_\alpha \leq |\beta|^{-1}\chi_\beta * \chi_\beta$ when α and β are finite subsets of D and $\alpha \subseteq \beta$. The construction of the net $(F_\alpha)_{\alpha\in A}$ therefore requires some work in the general case.

The idea of the construction is to take the dual group G/H_0 of Λ_0, and to consider the discrete measures m_α on G:

$$m_\alpha = |\alpha|^{-1} \sum_{x\in\alpha} \epsilon_x * \sum_{x\in\alpha} \epsilon_{-x},$$

where α is a finite subset of G containing at most one element of each coset of H_0 in G, and ϵ_x is the unit mass at the point x. As remarked earlier, this procedure does not necessarily give the required monotonicity property, but the following technique does. Select x_1 from a nonzero coset $x_1 + H_0$. Either the iterates $0, x_1, 2x_1, \ldots$ belong to distinct cosets of H_0 in G or nx_1 is in H_0 for some positive integer n. In other words, $x_1 + H_0$ is of infinite or finite order in G/H_0. We take the measures m_α, where $\alpha = \{0\}$ or $\{0, x_1, 2x_1, \ldots, (n-1)x_1\}$ if $x_1 + H_0$ is of finite order n in G/H_0. If $x_1 + H_0$ is of infinite order in G/H_0, we let α range over all the sets α_n, where $\alpha_n = \{0, \pm x_1, \pm 2x_1, \ldots, \pm nx_1\}$, and n is any nonnegative integer. If H_1:

$$H_1 = gp(\{x_1\}) + H_0$$

is equal to G, we are done. Otherwise, we consider the nonzero coset $x_2 + H_1$ of H_1 in G, and take only the sets $\beta = \{0\}$ or $\{0, x_2, 2x_2, \ldots, (n-1)x_2\}$ if $x_2 + H_1$ is of finite order n in G/H_1, but allow β to range over all the sets β_n, where $\beta_n = \{0, \pm x_2, \pm 2x_2, \ldots, \pm nx_2\}$ and n ranges over the nonnegative integers, if $x_2 + H_1$ is of infinite order in G/H_1. We then take the measures $m_{\alpha+\beta}$ on G:

$$m_{\alpha+\beta} = |\alpha + \beta|^{-1} \sum_{x\in\alpha+\beta} \epsilon_x * \sum_{x\in\alpha+\beta} \epsilon_{-x},$$

as α and β run over their respective ranges.

The inductive process that is used if G/H_0 is countable is now quite clear. If G/H_0 is uncountably infinite, a transfinite induction process is effected, mimicking the ideas above. In any case, we end up with a net $(m_\alpha)_{\alpha\in A}$ of finitely supported discrete measures on G, where A is a subnet of the net of all finite subsets of a subset S of G containing one element of each coset of H_0 in G. We take $F_\alpha = \hat{m}_\alpha$.

This completes our outline of the proof; the details follow in the next sections.

3. **Three theorems.** The first theorem in this section is a structure theorem of some interest *per se*, which develops already known results [8, 24.30 and 25.31] and [1]. The result may not be original, but if not, it seems to be

fairly inaccessible. We need the notion of an absolute direct summand.

Let G be an LCA group. We say that G is an *absolute direct summand* if, whenever H is an LCA group, G is (topologically and algebraically isomorphic to) a closed subgroup of H, and H_0 is a closed subgroup of H such that $G \cap H_0 = \{0\}$ and $G + H_0$ is closed in H; then there exists a closed subgroup H_1 of H containing H_0 such that $H = G \oplus H_1$, the direct sum being interpreted both algebraically and topologically.

It is well known that divisible groups are absolute direct summands algebraically [13, A.8], but whether they are also topological summands is another question. Both T^a (a an arbitrarily large cardinal number) and R^b (b a nonnegative integer) are divisible groups. We prove

THEOREM 3. *T^a (a an arbitrary cardinal number) and R^b (b a positive integer) are absolute direct summands.*

PROOF. Suppose that T^a and H_0 are closed subgroups of the LCA group H, and that $T^a \cap H_0 = \{0\}$. Because T^a is compact, $T^a + H_0$ is closed in H automatically. Let Δ and Δ_0 be the annihilators of T^a and H_0 respectively in the dual group Γ of H.

The dual group Z^a (the discrete incomplete direct sum) of T^a can be identified with Γ/Δ, so Δ is open in Γ. Since $T^a \cap H_0 = \{0\}$, $\Delta + \Delta_0$ is dense in Γ; since also Δ is open in Γ, $\Delta + \Delta_0 = \Gamma$.

Let $\{\gamma_i + \Delta \colon \gamma_i \in \Gamma, i \in A\}$ be a generating set for the (free abelian) group Z^a (i.e. a is the cardinality $|A|$ of the set A). From each coset $\gamma_i + \Delta$, select δ_i in Δ_0 (this is possible because $\Delta + \Delta_0 = \Gamma$). Denote by Δ_1 the discrete (free abelian) group generated by $\{\delta_i \colon i \in A\}$. Then $\Delta_1 \subseteq \Delta_0$, and $\Delta_1 \oplus \Delta = \Gamma$, the sum being interpreted both algebraically and topologically. Let H_1 be the annihilator of Δ_1 in H; evidently $H_0 \subseteq H_1$ and $T^a \oplus H_1 = H$ both topologically and algebraically, as required.

The proof that R^b is an absolute direct summand is rather more difficult, as it involves structure theory, elementary vector space ideas, and group theory. Before starting this proof, we claim that: *if G_1 and G_2 are closed subgroups of G with trivial intersection, if G_1 is σ-compact, and if $G_1 + G_2$ is closed in G, then $G_1 + G_2$, with the relative topology as a subgroup of G, is topologically and algebraically isomorphic to the direct sum $G_1 \oplus G_2$.* For $G_1 + G_2$ is an algebraic direct sum, and if $x_\alpha \longrightarrow x$ in G_1 and $y_\beta \longrightarrow y$ in G_2, then $x_\alpha + y_\beta \longrightarrow x + y$ in G. Conversely, if $x_\alpha + y_\alpha$ converges in G (x_α in G_1, y_α in G_2), then $x_\alpha + y_\alpha$ converges to some element $x + y$ in $G_1 + G_2$, because $G_1 + G_2$ is closed. By a standard isomorphism theorem [13, 5.33], since G_1 is σ-compact, $G_1 + G_2/G_2$ is isomorphic to G_1, under the natural isomorphism taking $(x + y)$ to x. Thus $x_\alpha \longrightarrow x$ in G_1. Consequently, $y_\alpha \longrightarrow y$ in G_2, which suffices to prove our claim.

Now suppose that R^b and H_0 are closed subgroups of G with trivial intersection, and that $R^b + H_0$ is closed in G. Immediately, we see that $R^b + H_0$ "is" the direct sum $R^b \oplus H_0$ both algebraically and topologically. The problem is to "expand" H_0 to a closed subgroup H of G such that $G = R^b \oplus H$. We appeal to a structure theorem [18, 2.4.1] which states that G has an *open* subgroup of the form $R^a \oplus K$, where a is a positive integer ($a \geqslant b$!) and K is compact. It is natural first to expand H_0 to $H_0 + K$, which is a closed group since H_0 is closed and K is compact. The essence of the proof is then to find a subgroup R^{a-b} of R^a such that

$$R^b + (R^{a-b} + H_0 + K) \supseteq R^a \oplus K,$$

and is therefore open in G, and also such that

$$R^b + (R^{a-b} + H_0 + K) = R^b \oplus (R^{a-b} + H_0 + K).$$

Then since R^b is divisible, the group $R^{a-b} + H_0 + K$ can be extended to a group H such that $G = R^b + H$, the sum being an algebraic direct sum. However, because $R^b \oplus (R^{a-b} + H_0 + K)$ is open in G, it follows that the direct sum is topological as well as algebraic, i.e., $G = R^b \oplus H$. Here now is a detailed realization of the proof just outlined.

The first step of the proof is to "expand" H_0 to the group $H_0 + K$, and check that $R^b + H_0 + K$ is in fact the direct sum $R^b \oplus (H_0 + K)$. Since K is compact, $H_0 + K$ and $R^b + H_0 + K$ are both closed in G (recall that one of the hypotheses of the theorem is that $R^b + H_0$ is closed in G); it is sufficient to show that $R^b \cap (H_0 + K) = \{0\}$.

If $R^b \cap (H_0 + K)$ is nontrivial, there is a nonzero element r of R^b of the form $r = x + k$, $x \in H_0$, $k \in K$. The element $r - x$ of $R^b \oplus H_0$ must generate a discrete group, since its "first component" r generates a discrete group. Thus, the group generated by k is a discrete subgroup of the compact group K, whence $mk = 0$ for some positive integer m. But then $mr = mx$, contradicting the assumption that $R^b \cap H_0 = \{0\}$. We conclude that $R^b + H_0 + K$ can indeed be written in the form $R^b \oplus (H_0 + K)$.

The key step, constructing R^{a-b}, is carried out in the following way. The subgroup $R^a \oplus K$ of G is open, and R^b is connected. It follows that R^b is contained in $R^a \oplus K$, through not necessarily in R^a. First, we show that the projection, S say, of R^b onto R^a is isomorphic to R^b. We also project $(H_0 + K) \cap (R^a \oplus K)$ onto its set of R^a-components, say U_0; U_0 is isomorphic to $R^c \oplus Z^d$. We then show that $S + U_0$ is actually the direct sum $S \oplus U_0$. Next, in case $d > 0$, we "fill out" Z^d to its R-linear span R^d, and show that $U - U = R^c \oplus R^d$ also has the requisite intersection property $S \cap U = \{0\}$. Finally, in case $b + c + d < a$, we extend $S \oplus R^c \oplus R^d$ to all of R^a by using a subgroup V of R^a, isomorphic to $R^{a-b-c-d}$, for which $V \cap (S \oplus R^c \oplus R^d) = \{0\}$. Once the subgroup

$U \oplus V$ of R^a, isomorphic to R^{a-b}, has been constructed, it is necessary to check that the sum $R^b + (R^{a-b} + H_0 + K)$ is actually direct. This we shall do after constructing R^{a-b}.

Take a basis $\{x_1, \dots, x_b\}$ for R^b over R. We write $x_j = r_j + k_j$, $r_j \in R^a$, $k_j \in K$, $j = 1, \dots, b$. For any λ in R, we represent λx_j uniquely:

$$\lambda x_j = r_j(\lambda) + k_j(\lambda), \qquad r_j(\lambda) \in R^a,\ k_j(\lambda) \in K.$$

An easy argument shows that $r_j(\lambda) = \lambda r_j$ for rational λ, hence for all real λ by continuity. Accordingly, $k_j(\lambda)$ is the element of K such that $\lambda x_j - \lambda r_j = k_j(\lambda)$. It follows that $\{r_1, \dots, r_b\}$ is a linearly independent set in the R-module R^a and that the projection S of R^b onto R^a is isomorphic to R^b.

The projection (S) of R^b onto R^a is $(R^b + K) \cap R^a$, while the projection (U_0) of $(H_0 + K) \cap (R^a \oplus K)$ onto R^a is $(H_0 + K) \cap R^a$. However,

$$(R^b + K) \cap R^a \cap (H_0 + K) = \{[R^b \cap (H_0 + K)] + K\} \cap R^a = \{0\}$$

since, as we have already seen, $R^b \cap (H_0 + K) = \{0\}$. Moreover,

$$[(R^b + K) \cap R^a] + [(H_0 + K) \cap R^a] = (R^b + H_0 + K) \cap R^a,$$

which is closed since $R^b + K + H_0$ is closed. Thus, because R^b and $H_0 + K$ are direct summands, one with the other, the projections S and U_0 are direct summands, one with the other. Being a closed subgroup of R^a, U_0 is of the form $R^c \oplus Z^d$ [13, 9.11].

We now propose to fill-out the group $R^c \oplus Z^d$ by taking its R-linear span,

$$U = R^c \oplus R^d,$$

and showing that U also has the direct summand property $S \cap U = \{0\}$. To establish this, it is clear enough to show that the R-linear span of Z^d has only 0 in common with $S \oplus R^c$. If the contrary is the case, and we denote by $\{z_1, \dots, z_d\}$ a set of independent generators of the group Z^d, then there exist real scalars $\lambda_1, \dots, \lambda_d$ and an element r of $S \oplus R^c$ such that $r = \Sigma_1^d \lambda_j z_j$. To obtain a contradiction, choose for each positive integer n those integers $l_1^n, \dots, l_d^n$ for which $l_j^n/n \leq \lambda_j < (l_j^n + 1)/n$. Denoting by $\|\cdot\|$ the Euclidean norm on R^a, we see that

$$\left\| r - \sum_1^d \frac{l_j^n}{n} z_j \right\| \leq \frac{1}{n} \sum_1^d \|z_j\|.$$

This implies that the sequence $(nr - \Sigma_1^d l_j^n z_j)_{n=1}^\infty$ of points of $(S \oplus R^c) \oplus Z^d$ is bounded in R^a, and hence has a limit point of $(S \oplus R^c) \oplus Z^d$. Clearly this can happen only if some subsequence of each coefficient sequence $(l_j^n)_{n=1}^\infty$ is bounded. Since $l_j^n \leq n\lambda_j < l_j^n + 1$, this entails that $\lambda_j = 0$ for each j.

We have now shown that the R-linear span U of U_0 forms a direct sum in R^a with S. The space U is isomorphic to $R^c \oplus R^d$, though the actual dimension is quite irrelevant.

If the dimension of $S \oplus U$ is less than a, choose a subgroup R^e of R^a such that $(S \oplus U) \oplus R^e = R^a$. Note that, since S is isomorphic to R^b, $U + R^e$ is isomorphic to R^{a-b}.

We have thus established the existence of a subgroup R^{a-b} of R^a such that

$$R^b + R^{a-b} + K + H_0 = (R^a \oplus K) + H_0 \supseteq R^a \oplus K,$$

and such that

$$(H_0 + K) \cap R^a \subseteq R^{a-b}.$$

We now check that the sum $R^b + (R^{b-a} + K + H_0)$ is actually direct. The group R^b is closed; the group $R^{b-a} + K + H_0$ is the union of cosets of $(R^{b-a} + K + H_0) \cap (R^a \oplus K)$, where $R^a \oplus K$ is open in G. Since $(R^{b-a} + K + H_0) \cap (R^a \oplus K) = R^{b-a} \oplus K$, the group $R^{b-a} + K + H_0$ is also closed in G. Further, $R^b + (R^{b-a} + K + H_0)$ is open in G, so it suffices to show that $R^b \cap (R^{a-b} + K + H_0)$ is trivial. However,

$$\begin{aligned} R^b \cap (R^{a-b} + K + H_0) &= R^b \cap (R^a \oplus K) \cap (R^{a-b} + K + H_0) \\ &= R^b \cap [R^{a-b} + K + \{H_0 \cap (R^a \oplus K)\}] \\ &= R^b \cap [R^{a-b} + \{(H_0 + K) \cap (R^a \oplus K)\}] \\ &= R^b \cap [R^{a-b} + \{(H_0 + K) \cap R^a\} + K] \\ &= R^b \cap [R^{a-b} + K] = \{0\}, \end{aligned}$$

and so

$$R^b + (R^{a-b} + K + H_0) = R^b \oplus (R^{a-b} + K + H_0)$$

as required.

Finally, R^b is divisible. Consequently [13, A.8], there exists a subgroup H of G such that $G = R^b + H$, this sum being an algebraic direct sum, and

$$R^{a-b} + K + H_0 \subseteq H.$$

To show that $G = R^b \oplus H$ topologically, we need only prove that H is closed. But

$$\begin{aligned} H \cap [R^b \oplus (R^{a-b} + K + H_0)] &= (R^b \cap H) + (R^{a-b} + K + H_0) \\ &= R^{a-b} + K + H_0 \end{aligned}$$

which is closed, and $R^b \oplus (R^{a-b} + K + H_0)$ is open in G, so H is closed, and $G = R^b \oplus H$, algebraically and topologically, as required.

REMARKS. If R^b and H_0 are closed subgroups of G with trivial intersection, there need not be a closed subgroup H of G containing H_0 such that $G = R^b \oplus H$ unless $R^b + H_0$ is closed. For instance, let G be the group $R \oplus \bar{R}$ ($\bar{R}$ is the Bohr compactification of R), and take

$$R_0 = \{(r, 0): r \in R\} \quad \text{and} \quad H_0 = \{(r, r): r \in R\}.$$

Both R_0 and H_0 are closed in G and isomorphic (topologically and algebraically) to R. However, G cannot be written as $R_0 \oplus H$, where H_0 is a closed subgroup of H. Thus, the first stage of our proof of Theorem 3 (for R^b), i.e. showing that $R^b + (H_0 + K)$ is actually the direct sum $R^b \oplus (H_0 + K)$ (where K is a compact subgroup of G such that $R^a \oplus K$ is open in G), requires that $R^b + H_0$ be closed in G. The second stage of the proof, i.e. constructing a subgroup R^{a-b} of R^a such that $R^b + (R^{a-b} + K + H_0)$ is open in G and is the direct sum $R^b \oplus (R^{a-b} + K + H_0)$, also breaks down if $R^b + H_0$ is not closed in G. For example, let G be the group $R \oplus R$, and take

$$R_0 = \{(r, rs): r \in R\} \quad \text{and} \quad H_0 = \{(m, n): m, n \in Z\}.$$

If s is irrational, $R_0 \cap H_0 = \{0\}$, but G cannot be expressed in the form $R_0 \oplus H$, where H_0 is a subgroup of H.

On purely topological grounds, if $G = R^b \oplus H$, and H_0 is a closed subgroup of H, then $R^b + H_0$ is closed in G. Therefore there is no hope of extending a closed subgroup H_0 of G such that $R^b \cap H_0 = \{0\}$ to a subgroup H of G such that $G = R^b \oplus H$ unless $R^b + H_0$ is closed in G.

COROLLARY 3a. *Suppose that Γ_0 is a closed subgroup of the LCA group Γ. There exist closed subgroups R^a, Δ_0 and Δ of Γ such that Δ_0 has a compact open subgroup, $\Delta_0 \subseteq \Delta$, and*

$$\Gamma_0 = R^a \oplus \Delta_0, \qquad \Gamma = R^a \oplus \Delta.$$

PROOF. Recall that any group Γ_0 can be expressed as $R^a \oplus \Delta_0$, where Δ_0 has a compact open subgroup [13, 24.30]. The corollary follows immediately.

With our next theorem, due to Figà-Talamanca and Gaudry [8], we show that operators of a certain type map $A_p^q(G)$ to $A_p^q(G/G_0)$ continuously; the adjoint of such an operator maps $M_p^q(\Gamma_0)$ to $M_p^q(\Gamma)$.

THEOREM 4. *Suppose that G_0 is a closed subgroup of the LCA group G. If h and k are bounded integrable functions on G such that*

$$\int_{G_0} dx_0 |h(x + x_0)| \leqslant C \qquad \forall x \in G$$

and

$$\int_{G_0} dx_0 |k(x + x_0)| \leqslant C \qquad \forall x \in G,$$

then the operator J, defined by the formula

$$Jf(\dot{x}) = \int_{G_0} dx_0 f(x+x_0) h * k(x+x_0) \qquad \forall x \in G,$$

initially as a mapping from $C_c(G)$ *into* $C_c(G/G_0)$, *maps* $A_c(G)$ *into* $A_c(G/G_0)$ *and, for each* (p, q) *in* S,

$$\|Jf\|_{A_p^q} \leq C^{s'^{-1}} \|h\|_\infty^{s^{-1}} \|h\|_1^{q'^{-1}} \|k\|_1^{q'^{-1}} \|f\|_{A_p^q}$$

for each f *in* $A_c(G)$ $(s^{-1} = p^{-1} - q^{-1})$.

REMARK. The condition

$$\int_{G_0} dx_0 |h(x+x_0)| \leq C \tag{3.1}$$

holds for all x in G if it holds for almost all x in G (see the proof of (4.1) in §4), provided h is continuous.

PROOF. We assume that the Haar measures of G, G/G_0 and G_0 are normalized so that

$$\int_G dx f(x) = \int_{G/G_0} d\dot{x} \int_{G_0} dx_0 f(x+x_0) \qquad \forall f \in C_c(G).$$

Because h and k are integrable and bounded, $h * k$ is a bounded continuous function. Therefore, if f is in $C_c(G)$, so is $f \cdot h * k$, and hence Jf is continuous, and

$$\operatorname{supp}(Jf) \subseteq \pi(\operatorname{supp}(f)) \qquad \forall f \in C_c(G),$$

where π is the canonical projection of G onto G/G_0.

Suppose that f and g are in $C_c(G)$. For x in G,

$$\begin{aligned}
J(f * g)(\dot{x}) &= \int_{G_0} dx_0 \left[\int_G dy f(x+x_0-y) g(y)\right]\left[\int_G dz h(x+x_0-z) k(z)\right] \\
&= \int_{G_0} dx_0 \int_G dy \int_G dz f(x+x_0-y) g(y) h(x+x_0-y-z) k(y+z) \\
&= \int_G dz \int_G dy \int_{G_0} dx_0 f(x+x_0-y) h(x+x_0-y-z) g(y) k(y+z) \\
&= \int_G dz \int_{G/G_0} d\dot{y} \int_{G_0} dy_0 \int_{G_0} dx_0 (f \cdot T_z h)(x+x_0-y-y_0) \\
&\qquad \cdot (g \cdot T_{-z} k)(y+y_0) \qquad (3.2) \\
&= \int_G dz \int_{G/G_0} d\dot{y} \left[\int_{G_0} dx_0 f \cdot T_z h(x+x_0-y)\right] \\
&\qquad \cdot \left[\int_{G_0} dy_0 g \cdot T_{-z} k\,(y+y_0)\right] \\
&= \int_G dz F_z * G_z(\dot{x}),
\end{aligned}$$

where F_z and G_z are the bounded measurable compactly supported functions on G/G_0 defined by the formulae:

$$F_z(\dot{x}) = \int_{G_0} dx_0 (f \cdot T_z h)(x + x_0) \qquad \forall x \in G,$$

$$G_z(\dot{x}) = \int_{G_0} dx_0 (g \cdot T_{-z} k)(x + x_0) \qquad \forall x \in G.$$

The $A_p^q(G/G_0)$-valued function on G taking z to $F_z * G_z$ is continuous. We show that

$$\int_G dx \|F_z * G_z\|_{A_p^q} \leqslant C^{s'^{-1}} \|h\|_\infty^{s^{-1}} \|h\|_1^{q^{-1}} \|k\|_1^{q'^{-1}} \|f\|_p \|g\|_{q'}; \tag{3.3}$$

it follows that the Bochner integral $\int_G dz F_z * G_z$ converges in $A_p^q(G/G_0)$ and has $A_p^q(G/G_0)$-norm no greater than $C^{s'^{-1}} \|h\|_\infty^{s^{-1}} \|h\|_1^{q^{-1}} \|k\|_1^{q'^{-1}} \|f\|_p \|g\|_{q'}$. From (3.2), we can deduce by a variation of the argument of C. Herz [12] that $J(f * g)$ is equal to $\int_G dz F_z * G_z$; the inequality

$$\|Jf\|_{A_p^q} \leqslant C^{s'^{-1}} \|h\|_\infty^{s^{-1}} \|h\|_1^{q^{-1}} \|k\|_1^{q'^{-1}} \|f\|_{A_p^q}$$

follows from the definition of $A_p^q(G)$, the linearity of J, and the completeness of $A_p^q(G/G_0)$.

To prove (3.3), we observe first that, from Hölder's inequality,

$$\begin{aligned} \int_G dz \|F_z * G_z\|_{A_p^q} &\leqslant \int_G dz \|F_z\|_p \|G_z\|_{q'} \\ &\leqslant \left[\int_G dz \|F_z\|_p^q\right]^{q^{-1}} \left[\int_G dz \|G_z\|_{q'}^{q'}\right]^{q'^{-1}} \end{aligned} \tag{3.4}$$

if $1 < q < \infty$; the corresponding inequality holds if q is either 1 or ∞. We employ an interpolation theorem to further our estimation. On the one hand,

$$\begin{aligned} \sup\{\|F_z\|_\infty : z \in G\} &= \sup\left\{\left|\int_{G_0} dx_0 f(x + x_0) h(x + x_0 - z)\right| : x, z \in G\right\} \\ &\leqslant \sup\left\{\|f\|_\infty \int_{G_0} dx_0 |h(x + x_0 - z)| : x, z \in G\right\} \leqslant C\|f\|_\infty \end{aligned}$$

from the inequality (3.1). On the other hand, if $1 \leqslant r < \infty$,

$$\begin{aligned} \left[\int_G dz \|F_z\|_1^r\right]^{r^{-1}} &= \left[\int_G dz \left\{\int_{G/G_0} d\dot{x} \left|\int_{G_0} dx_0 f(x + x_0) h(x + x_0 - z)\right|\right\}^r\right]^{r^{-1}} \\ &\leqslant \left[\int_G dz \left\{\int_G dx \left|f(x) h(x - z)\right|\right\}^r\right]^{r^{-1}} \leqslant \|h\|_r \|f\|_1 \end{aligned}$$

by Minkowski's inequality; if $r = \infty$, the analogous inequality holds.

The linear mapping taking f to F therefore maps $L^1(G)$ into $L^r(G; L^1(G/G_0))$ and $C_0(G)$ into $L^\infty(G; L^\infty(G/G_0))$ respectively. By a vector-valued Riesz-Thorin interpolation theorem, proved by J.-L. Lions and J. Peetre [15] using their real method of interpolation, this mapping also maps $L^p(G)$ into $L^{pr}(G; L^p(G/G_0))$, and if p and r are both finite,

$$\left| \int_G dz \|F_z\|_p^{pr} \right|^{(pr)^{-1}} \leqslant \|f\|_p \|h\|_r^{p^{-1}} C^{p'^{-1}},$$

while if p or r is infinite,

$$\sup\{\|F_z\|_p : z \in G\} \leqslant \|f\|_p \|h\|_r^{p^{-1}} C^{p'^{-1}}.$$

Similar inequalities can be proved likewise when f, F, h and p are replaced by g, G, k and q'. Taking r equal to qp^{-1} in the F-inequalities and 1 in the G-inequalities, we deduce from (3.4) that

$$\begin{aligned}\int_G dz \|F_z * G_z\|_{A_p^q} &\leqslant \|f\|_p \|h\|_{qp^{-1}}^{p^{-1}} C^{p'^{-1}} \|g\|_{q'} \|k\|_1^{q'^{-1}} C^{q^{-1}} \\ &\leqslant \|f\|_p \|h\|_1^{q^{-1}} \|h\|_\infty^{s^{-1}} C^{p'^{-1}} \|g\|_{q'} \|k\|_1^{q'^{-1}} C^{q^{-1}} \\ &= C^{s'^{-1}} \|h\|_\infty^{s^{-1}} \|h\|_1^{q^{-1}} \|k\|_1^{q'^{-1}} \|f\|_p \|g\|_q,\end{aligned}$$

proving the inequality (2.3) and thereby the theorem.

The following result is of some use in our proof of Theorem 4.1, but is of interest in itself. It is quite well known (see [13, 40.12], [17, p. 121]). Our proof follows Lohoué [16].

COROLLARY 4a. *Let G_0 be a closed subgroup of the LCA group G. For any compact neighborhood K in G, the mapping taking k to $\dot{k}$,*

$$\dot{k}(\dot{x}) = \int_{G_0} dx_0 k(x + x_0) \qquad \forall x \in G,$$

maps $A_K(G)$ into $A_{\pi(K)}(G/G_0)$ (where π is the canonical projection of G onto G/G_0), and

$$\|\dot{k}\|_A \leqslant C\|k\|_A \qquad \forall k \in A_k(G).$$

The constant C depends only on K.

PROOF. Let U be a compact neighborhood of 0 in G. The function $\|\chi_U\|_1^{-1}\chi_{K-U} * \chi_U$ takes the value one on K and so, for any k in $A_K(G)$,

$$\dot{k}(\dot{x}) = \int_{G_0} dx_0 k(x + x_0) \|\chi_U\|_1^{-1} \chi_{K-U} * \chi_U(x + x_0)$$

for every x in G. From Theorem 4, $\|\dot{k}\|_A \leqslant C\|k\|_A$, where C depends only on K. The corollary is proved.

Theorem 5, the last of this section, is the construction of a Fejér-type net. We take an open subgroup H_0 of the LCA group H and produce a monotone increasing net $(m_\alpha)_{\alpha\in A}$ of finitely supported (discrete) measures on H, whose Fourier transforms are an approximate identity on Λ_0, the compact annihilator of H_0 in the dual group of H.

THEOREM 5. *Let H_0 be an open subgroup of the LCA group H. There exist a subset S of H containing exactly one element of each coset of H_0 in H and a subnet A of the net of all finite subsets of S such that $(m_\alpha)_{\alpha\in A}$,*

$$m_\alpha = |\alpha|^{-1}\left(\sum_{x\in\alpha}\epsilon_x\right) * \left(\sum_{x\in\alpha}\epsilon_{-x}\right) \qquad \forall\alpha\in A,$$

is a monotone increasing net of measures on H and, further, for each coset $y + H_0$ of H_0 in H, $\|\chi_{y+H_0}\cdot m_\alpha\|_M$ increases monotonely to one.

PROOF. Let $(z_\omega + H_0)_{\omega\in\Omega}$ be a well-ordering of the elements of H/H_0, or, more accurately, let $(z_\omega)_{\omega\in\Omega}$ be a collection of elements of H, containing one member of each coset of H_0 in H, indexed by the well-ordered set Ω. The existence of such a set is a consequence of the axiom of choice and Zermelo's well-ordering theorem.

Define the subgroups H_ω and H_ω^0 of H:

$$H_\omega = gp(\{z_\sigma, h\colon \sigma \leqslant \omega, h \in H_0\}),$$

$$H_\omega^0 = gp(\{z_\sigma, h\colon \sigma < \omega, h \in H_0\}).$$

Now H_ω/H_ω^0 is a group with one generator, i.e. a cyclic group; denote by O_ω the order of this group. Define S to be the set of all (finite) sums $m_1 z_{\omega_1} + \cdots + m_k z_{\omega_k}$, where the finitely many ω_j ($j = 1, 2, \ldots, k$) are all different elements of Ω and each m_j is an integer, satisfying the inequalities $0 \leqslant m_j < O_{\omega_j}$ if O_{ω_j} is finite. We now show that S contains exactly one element of each coset of H_0 in H. The proof is inductive.

For each ω in Ω, let S_ω be the subset of S consisting of all elements of the form $m_1 z_{\omega_1} + \cdots + m_k z_{\omega_k}$, where $\omega_1, \ldots, \omega_k$ are distinct elements of Ω, each no greater than ω, m_j is an integer and $0 \leqslant m_j < O_{\omega_j}$ if O_{ω_j} is finite. Clearly, $S = \bigcup_{\omega\in\Omega} S_\omega$. Furthermore, $H = \bigcup_{\omega\in\Omega} H_\omega$. In order to prove that S contains precisely one element of each coset of H_0 in H, it will therefore suffice to establish the following inductive assertion:

P_ω: *For each ω in Ω, S_ω contains exactly one element of each coset of H_0 in H_ω.*

If η is the least element of Ω (Ω is well-ordered), P_η is true. For H_η/H_0 is a cyclic group, generated by $\dot{z}_n$, of order O_η. Therefore

$$H_\eta = \bigcup_{m\in I_\eta} mz_\eta + H_0,$$

where $I_\eta = Z$ if O_η is infinite and $I_\eta = [0, 0_\eta)$ otherwise. The sets $mz_\eta + H_0$, as m ranges over I_η, are disjoint, from the definitions of O_η and I_η.

If P_σ is true for all σ less than ω, then P_ω is true. For, as above,

$$H_\omega = \bigcup_{m \in I_\omega} mz_\omega + H_\omega^0$$

where $I_\omega = Z$ if O_ω is infinite, and $I_\omega = [0, O_\omega)$ otherwise; and the sets $mz_\omega + H_\omega^0$ (m in I_ω) are pairwise disjoint. The set S_ω is clearly a subset of H. It remains to show that if x is in H_ω, the set $x + H_0$ contains precisely one element of S_ω. We can express x in the form $x = mz_\omega + h$ where m is in I_ω and h is in H_ω^0. But if h is in H_ω^0, then h is in H_σ for some σ less than ω. The inductive hypothesis P_σ implies that $h = s_\sigma + h_0$, where s_σ is in S_σ and h_0 is in H_σ. Thus

$$\begin{aligned} x &= mz_\omega + s_\sigma + h_0 \\ &\in S_\omega + H_0. \end{aligned}$$

It remains to show that $x + H_0$ contains only the element $mz_\omega + s_\sigma$ of S_ω. If $nz_\omega + s_\tau$ ($\tau < \omega$, $n \in I_\omega$) also lies in the coset $x + H_0$ of H_0 in H_ω, then certainly $mz_\omega + s_\sigma$ and $nz_\omega + s_\tau$ both lie in the same coset of H_ω^0 in H_ω, since $H_0 \subseteq H_\omega^0$. Consequently $m = n$, and s_σ and s_τ therefore both lie in the same coset of H_0 in H_σ. We conclude that S_ω contains exactly one element of each coset of H_0 in H_ω, as required.

We recall the definition of m_α:

$$m_\alpha = |\alpha|^{-1}\Big(\sum_{x \in \alpha} \epsilon_x\Big) * \Big(\sum_{x \in \alpha} \epsilon_x\Big).$$

A *monotone increasing net* $(m_\alpha)_{\alpha \in A}$ *is to be constructed* from an appropriate net A of finite subsets of S.

Before we actually construct the sets α, we observe that, if z is in S, there is a *unique* finite subset $\{\omega_1, \ldots, \omega_k\}$ of Ω and a *unique* finite set $\{n_1, \ldots, n_k\}$ of nonzero integers such that

$$z = n_1 z_{\omega_1} + \cdots + n_k z_{\omega_k},$$

and $1 \leqslant n_j < O_{\omega_j}$ if O_{ω_j} is finite. This assertion can be readily verified using the total order on Ω and the definition of O_ω.

The subsets α of S are constructed as follows. We select a finite subset $\{\omega_1, \ldots, \omega_k\}$ of Ω and a set $\{n_1, \ldots, n_k\}$ of positive integers subject to the condition that $n_j = O_{\omega_j} - 1$ if O_{ω_j} is finite. This condition excludes elements ω of Ω such that $O_\omega = 1$ (i.e. $z_\omega \in H_\omega^0$). The set α consists of all elements x of S of the form:

$$x = m_1 z_{\omega_1} + \cdots + m_k z_{\omega_k},$$

where $|m_j| \leqslant n_j$ if O_{ω_j} is infinite and $0 \leqslant m_j \leqslant n_j$ if O_{ω_j} is finite. The set of all such α is denoted A, and is ordered by set inclusion. Notice that if F is any finite subset of S, there is a subset β, constructed as above, such that $F \subseteq \beta$. Consequently the net A is a cofinal subfamily [5, p. 8] of the power set of S ordered by inclusion.

We wish to prove that the net $(m_\alpha)_{\alpha\in A}$, where $m_\alpha = |\alpha|^{-1}(\Sigma_{x\in\alpha}\epsilon_x) * (\Sigma_{x\in\alpha}\epsilon_{-x})$ is monotone increasing. With the finite set β, we can associate the unique sets $\{\omega_1, \ldots, \omega_k\}$ of elements of Ω and $\{n_1, \ldots, n_k\}$ of positive integers such that

(i) $O_{\omega_j} > 1, j = 1, 2, \ldots, k$;

(ii) $n_j = O_{\omega_j} - 1, j$: $O_{\omega_j} < \infty$;

(iii) β consists of the elements x of S of the form $x = m_1 z_{\omega_1} + \cdots + m_k z_{\omega_k}$ where $|m_j| \leq n_j$ if O_{ω_j} is infinite and $0 \leq m_j \leq n_j$ if O_{ω_j} is finite. We write this association symbolically:

$$\beta \sim \{\omega_1, \ldots, \omega_k; n_1, \ldots, n_k\}.$$

It is now easily seen that, in order to prove that $m_\alpha \leq m_\beta$ when $\alpha \leq \beta$, it suffices to prove the inequality in each of the following three basic cases:

(a) $\alpha \sim \{\omega_1, \ldots, \omega_k, \omega; n_1, \ldots, n_k, n\}$, $O_\omega = \infty$ and $\beta \sim \{\omega_1, \ldots, \omega_k, \omega; n_1, \ldots, n_k, n+1\}$,

(b) $\alpha \sim \{\omega_1, \ldots, \omega_k; n_1, \ldots, n_k\}$, $O_\omega = \infty$ and $\beta \sim \{\omega_1, \ldots, \omega_k, \omega; n_1, \ldots, n_k, 1\}$,

(c) $\alpha \sim \{\omega_1, \ldots, \omega_k; n_1, \ldots, n_k\}$, $O_\omega < \infty$ and $\beta \sim \{\omega_1, \ldots, \omega_k, \omega; n_1, \ldots, n_k, O_\omega - 1\}$.

The argument is really very similar to an argument about finite sums of cyclic groups. Rather than bore the reader with the details of each case, we treat only the first case. The proof of case (b) is essentially the same, with n taken to be 0, and case (c) is only different in that sums from 0 to n rather than from $-n$ to $+n$ are involved.

Suppose that $\alpha \sim \{\omega_1, \ldots, \omega_k, \omega; n_1, \ldots, n_k, n\}$, $O_\omega = \infty$, and $\beta \sim \{\omega_1, \ldots, \omega_k, \omega; n_1, \ldots, n_k, n+1\}$. Observe that each element x of α has a *unique* representation:

$$x = m_1 z_{\omega_1} + \cdots + m_k z_{\omega_k} + m z_\omega,$$

such that $0 \leq m_j \leq n_j$ if O_{ω_j} is finite, $|m_j| \leq n_j$ if O_{ω_j} is infinite, and $|m| \leq n$. Each element of β can be similarly represented (except that we stipulate that $|m| \leq n+1$). Let γ be the set in A:

$$\gamma \sim \{\omega_1, \ldots, \omega_k; n_1, \ldots, n_k\}.$$

Then

$$|\alpha| = (2n+1)|\gamma| \quad \text{and} \quad |\beta| = (2n+3)|\gamma|;$$

further

$$\sum_{x\in\alpha} \epsilon_x = \Big(\sum_{|m|\leq n} \epsilon_{mz_\omega}\Big) * \Big(\sum_{x\in\gamma} \epsilon_x\Big) \quad \text{and}$$

$$\sum_{x\in\beta} \epsilon_x = \Big(\sum_{|m|\leq n+1} \epsilon_{mz_\omega}\Big) * \Big(\sum_{x\in\gamma} \epsilon_x\Big).$$

Consequently,

$$\begin{aligned} m_\alpha &= |\alpha|^{-1}\Big(\sum_{x\in\alpha} \epsilon_x\Big) * \Big(\sum_{x\in\alpha} \epsilon_{-x}\Big) \\ &= (2n+1)^{-1}|\gamma|^{-1}\Big(\sum_{|m|\leq n} \epsilon_{mz_\omega}\Big) * \Big(\sum_{|m|\leq n} \epsilon_{-mz_\omega}\Big) \\ &\quad * \Big(\sum_{x\in\gamma} \epsilon_x\Big) * \Big(\sum_{x\in\gamma} \epsilon_{-x}\Big) \\ &= (2n+1)^{-1}\Big[\sum_{|m|\leq 2n} (2n+1-|m|)\epsilon_{mz_\omega}\Big] * m_\gamma. \end{aligned}$$

Similarly,

$$m_\beta = (2n+3)^{-1}\Big[\sum_{|m|\leq 2n+2} (2n+3-|m|)\epsilon_{mz_\omega}\Big] * m_\gamma.$$

Since m_γ is positive and

$$(2n+1)^{-1} \sum_{|m|\leq 2n} (2n+1-|m|)\epsilon_{mz_\omega} \leq (2n+3)^{-1} \sum_{|m|\leq 2n+2} (2n+3-|m|)\epsilon_{mz_\omega},$$

$m_\alpha \leq m_\beta$, and the monotonicity assertion is established.

The last stage of the proof is to show that, for any coset $y + H_0$ of H_0 in H, $\|\chi_{y+H_0} \cdot m_\alpha\|_M$ increases monotonely to one. The quantity $\|\chi_{y+H_0} \cdot m_\alpha\|_M$ is just the amount of mass the measure m_α has in a particular coset $y + H_0$ of H_0 in H. The key observation here is that computing $\|\chi_{y+H_0} \cdot m_\alpha\|_M$ is exactly the same as regarding the convolution $|\alpha|^{-1}(\Sigma_{x\in\alpha}\epsilon_x) * (\Sigma_{x\in\alpha}\epsilon_{-x})$ as a convolution on the quotient group H/H_0 (with counting measure) $|\alpha|^{-1}(\Sigma_{x\in\alpha}\epsilon_{\dot{x}}) * (\Sigma_{x\in\alpha}\epsilon_{-\dot{x}})$ and computing the mass at the point $\dot{y}$; symbolically,

$$\|\chi_{y+H_0} \cdot m_\alpha\|_M = |\alpha|^{-1}\Big(\sum_{x\in\alpha} \epsilon_{\dot{x}}\Big) * \Big(\sum_{x\in\alpha} \epsilon_{-\dot{x}}\Big)(\dot{y}),$$

for any y in H. This observation follows immediately from the fact that

$$\|\chi_{y+H_0}(\epsilon_x * \epsilon_{-z})\|_M = \chi_{x-z+H_0}(y) = \epsilon_{\dot{x}} * \epsilon_{-\dot{z}}(y).$$

We deduce first that

$$\|\chi_{y+H_0}\cdot m_\alpha\|_M = |\alpha|^{-1}\left|\left(\sum_{x\in\alpha}\epsilon_{\dot x}\right) * \left(\sum_{x\in\alpha}\epsilon_{-\dot x}\right)(\dot y)\right|$$
$$\leqslant |\alpha|^{-1}\left\|\sum_{x\in\alpha}\epsilon_{\dot x}\right\|_1\left\|\sum_{x\in\alpha}\epsilon_{-\dot x}\right\|_\infty = 1.$$

Next, since m_α is monotone increasing, $\|\chi_{y+H_0}\cdot m_\alpha\|_M$ is monotone increasing. Therefore, to complete the proof, we have only to show the validity, for all ω in Ω, of the following inductive assertion:

Q_ω: *For any y in H_ω and any small positive quantity ϵ there exists a finite set α in* A, $\alpha \sim \{\omega_1, \ldots, \omega_k; n_1, \ldots, n_k\}$, *such that*

$$\omega_j \leqslant \omega,\quad j = 1, \ldots, k, \quad \text{and} \quad |\alpha|^{-1}\left(\sum_{x\in\alpha}\epsilon_{\dot x}\right) * \left(\sum_{x\in\alpha}\epsilon_{-\dot x}\right)(\dot y) > 1 - \epsilon.$$

If η is the least element of Ω, then Q_η is true. For if $O_\eta = \infty$, then H_η/H_0 is isomorphic to the group of integers, Z, and we are therefore required to show that, if m is any integer, there is a positive integer N such that

$$(2N+1)^{-1}\left(\sum_{-N}^{N}\epsilon_k\right) * \left(\sum_{-N}^{N}\epsilon_k\right)(m) > 1 - \epsilon.$$

This is of course a well-known and elementary fact. The proof for the case where $O_\eta < \infty$ is simpler, if anything.

If Q_σ is true for every σ less than ω, then Q_ω is true. For clearly we may assume that the element z_ω does not belong to H_ω^0, since, in that case, y would belong to a subgroup H_σ, for some σ less than ω. The element y can therefore be uniquely expressed as $y = mz_\omega + h_\sigma$, where h_σ is in H_σ for some σ less than ω and m is an integer satisfying the condition $0 < m < O_\omega$ if O_ω is finite. For the remainder of this proof, we shall deal with the case where O_ω is finite. The other case can be resolved along similar, but easier, lines.

Because O_ω is finite, $O_\omega z_\omega + h_\sigma$ lies in some subgroup H_τ of H_ω^0, with τ less than ω. By our inductive hypothesis, there exist sets β and γ in A,

$$\beta \sim \{\sigma_1, \ldots, \sigma_s; n_1, \ldots, n_s\},$$
$$\gamma \sim \{\tau_1, \ldots, \tau_t; p_1, \ldots, p_t\},$$

such that

$$\sigma_j \leqslant \sigma < \omega,\ j = 1, \ldots, s,$$
$$\tau_j \leqslant \tau < \omega,\ j = 1, \ldots, t,$$
$$|\beta|^{-1}\left(\sum_{x\in\beta}\epsilon_{\dot x}\right) * \left(\sum_{x\in\beta}\epsilon_{-\dot x}\right)(\dot h_\sigma) > 1 - \epsilon,$$

and

$$|\gamma|^{-1}\left(\sum_{x\in\gamma}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\gamma}\epsilon_{-\dot{x}}\right)(O_\omega\dot{z}_\omega+\dot{h}_\sigma)>1-\epsilon.$$

Let δ be the smallest set in A which contains both β and γ. The set δ is given by the correspondence

$$\delta \sim \{\rho_1, \ldots, \rho_r; q_1, \ldots, q_r\}$$

where $\{\rho_1, \ldots, \rho_r\} = \{\sigma_1, \ldots, \sigma_s\} \cup \{\tau_1, \ldots, \tau_t\}$, so $\rho_j < \omega$ $(j = 1, \ldots, r)$, and $q_j = \max\{n_h, p_k\colon \rho_j = \sigma_h \text{ or } \tau_k\}$. From the monotonicity property of the convolution measures m_α,

$$|\alpha|^{-1}\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)(\dot{h}_\sigma)>1-\epsilon$$

and

$$|\alpha|^{-1}\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)(O_\omega\dot{z}_\omega+\dot{h}_\sigma)>1-\epsilon.$$

Finally, take α to be the set δ "augmented by powers of z_ω", i.e. α is the set described by the correspondence $\alpha \sim \{\rho_1, \ldots, \rho_r, \omega; q_1, \ldots, q_r, n\}$ where $n = O_\omega - 1$. Then

$$\begin{aligned}
&|\alpha|^{-1}\left(\sum_{x\in\alpha}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\alpha}\epsilon_{-\dot{x}}\right)(\dot{y})\\
&\quad=(n+1)^{-1}|\delta|^{-1}\left(\sum_0^n\epsilon_{j\dot{z}_\omega}\right)*\left(\sum_0^n\epsilon_{-j\dot{z}_\omega}\right)*\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)(\dot{y})\\
&\quad=(n+1)^{-1}|\delta|^{-1}\left(\sum_{-n}^n(n+1-|j|)\epsilon_{j\dot{z}_\omega}\right)\\
&\qquad*\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)(m\dot{z}_\omega+\dot{h}_\sigma)\\
&\quad=(n+1)^{-1}\sum_{-n}^n(n+1-|j|)\left[|\delta|^{-1}\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)\right]\\
&\qquad\qquad\cdot(m\dot{z}_\omega-j\dot{z}_\omega+\dot{h}_\sigma)\\
&\quad\geqslant(n+1)^{-1}\left\{(n+1-m)\left[|\delta|^{-1}\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)\right](\dot{h}_\sigma)\right.\\
&\qquad+(n+1-|m-(n+1)|)\left[|\delta|^{-1}\left(\sum_{x\in\delta}\epsilon_{\dot{x}}\right)*\left(\sum_{x\in\delta}\epsilon_{-\dot{x}}\right)\right]\\
&\qquad\qquad\left.\cdot((n+1)\dot{z}_\omega+\dot{h}_\sigma)\right\}\\
&\quad>(n+1)^{-1}\{(n+1-m)(1-\epsilon)+m(1-\epsilon)\}=1-\epsilon,
\end{aligned}$$

proving Q_ω, with the assumption that O_ω is finite. When O_ω is infinite, the proof is easier, because we do not have to take "$O_\omega z_\omega$" into account. This concludes our proof of Theorem 5.

While the net of measures $(m_\alpha)_{\alpha\in A}$ constructed in Theorem 5 is monotone increasing and locally convergent in $M(H)$, much stronger conditions can be deduced about the net $(m_\alpha * f)_{\alpha\in A}$ where f is in $A_c(H)$. The following corollary bears this out.

COROLLARY 5a. *Let A be as in Theorem 5. If l in $L^1 \cap L^2(H)$ is nonnegative and supported in H_0, then $m_\alpha * l * l_\vee$:*

$$m_\alpha * l * l_\vee = |\alpha|^{-1}\left(\sum_{x\in\alpha}\epsilon_x * l\right) * \left(\sum_{x\in\alpha}\epsilon_x * l\right)_\vee$$

is in $A(H)$ and $\|m_\alpha * l * l_\vee\|_A \leqslant \|l\|_2^2$ for each α in A. The net $(m_\alpha * l * l_\vee)_{\alpha\in A}$ is monotone increasing. If j denotes the pointwise limit of the net, j is in $B(H)$ and $\|j\|_B \leqslant \|l\|_2^2$; further, for any f in $A(H)$,

$$(m_\alpha * l * l_\vee)\cdot f \longrightarrow j\cdot f \quad \text{in } A(H)$$

as α "tends to infinity". Finally

$$\int_{H_0} dx_0 (m_\alpha * l * l_\vee)(x + x_0) \longrightarrow \|l\|_1^2 \qquad \forall x \in H$$

as α "tends to infinity".

PROOF. Since α contains at most one element of each coset of H_0 in H, and l is supported in H_0,

$$\left\|\left(\sum_{x\in\alpha}\epsilon_{-x}\right) * l\right\|_2 = |\alpha|^{1/2}\|l\|_2.$$

Consequently, $\|m_\alpha * l * l_\vee\|_A \leqslant \|l\|_2^2$. Because l is nonnegative, $l * l_\vee$ is nonnegative; $(m_\alpha)_{\alpha\in A}$ is monotone increasing, so $(m_\alpha * l * l_\vee)_{\alpha\in A}$ is monotone increasing. The net $(m_\alpha * l * l_\vee)_{\alpha\in A}$ therefore converges pointwise to a bounded function j on H. In fact, this convergence is locally uniform: for any compact subset of H is contained in a union of a finite number of cosets of the open subgroup H_0. Since $l * l_\vee$ is supported in H_0,

$$(m_\alpha * l * l_\vee)\cdot \chi_{y+H_0} = (m_\alpha \cdot \chi_{y+H_0}) * l * l_\vee$$

for any coset $y + H_0$ of H_0 in H. It follows immediately from Theorem 5 that $((m_\alpha * l * l_\vee)\cdot\chi_{y+H_0})_{\alpha\in A}$ converges uniformly, and so $m_\alpha * l * l_\vee \longrightarrow j$ in $C_{\rm loc}(H)$ as α "tends to infinity".

Now, since $m_\alpha * l * l_\vee$ is in $A(H)$,

$$\left|\int_H dx(m_\alpha * l * l_\vee)\cdot\phi(x)\right| \leqslant \|l\|_2^2\|\hat{\phi}\|_\infty \qquad \forall\phi\in A_c(H).$$

Further, because $(m_\alpha * l * l_\vee)_{\alpha \in A}$ converges locally uniformly to j,

$$\left| \int_H dxj \cdot \phi(x) \right| \leqslant \|l\|_2^2 \|\hat{\phi}\|_\infty \qquad \forall \phi \in A_c(H).$$

Therefore j is in $B(H)$ and $\|j\|_B \leqslant \|l\|_2^2$, as claimed.

To show that, for f in $A(H)$, $(m_\alpha * l * l_\vee) \cdot f \longrightarrow j \cdot f$ in $A(H)$, it suffices to consider only arbitrary f in $A_c(H)$, since $\|f \cdot g\|_A \leqslant \|f\|_A \|g\|_A$ and $\|m_\alpha * l * l_\vee\|_A$ is uniformly bounded. Any f in $A_c(H)$ vanishes off a finite union of cosets of H_0 in H, whence it is enough to show that

$$(m_\alpha * l * l_\vee) \cdot \chi_{y+H_0} \longrightarrow j \cdot \chi_{y+H_0} \quad \text{in } A(H).$$

But $(m_\alpha * l * l_\vee) \cdot \chi_{y+H_0} = (m_\alpha \cdot \chi_{y+H_0}) * l * l_\vee$ and $(m_\alpha \cdot \chi_{y+H_0})_{\alpha \in A}$ converges in $M(H)$, from which the result follows.

Finally, because m_α and l are nonnegative, and l is supported in H_0,

$$\begin{aligned} \int_{H_0} dx_0 m_\alpha * l * l_\vee(x + x_0) &= \|\chi_{x+H_0} \cdot (m_\alpha * l * l_\vee)\|_1 \\ &= \|\chi_{x+H_0} \cdot m_\alpha\|_M \|l\|_1^2, \end{aligned}$$

which tends to $\|l\|_1^2$ from Theorem 5.

4. **Proof of the extension theorem.** In this section, we prove the extension theorem (Theorem 2), under the assumption that Γ_0 has a compact open subgroup Λ_0, i.e. that $a = 0$ in Corollary 3a. This assumption makes the proof much easier to read. At the end of the section, we indicate the modifications needed to prove the theorem unrestrictedly. The reader is referred to the diagrams in §2, which may be helpful in keeping track of the ideas of the proof.

Suppose, then, that Γ_0 is a closed subgroup of the LCA group Γ, and that Γ_0 has a compact open subgroup Λ_0. We make the standard assumptions about the Haar measures of these groups: we assume that the Haar measure on Λ_0 is the restriction to Λ_0 of that on Γ_0, that the total mass of Λ_0 is one, and that the Haar measures on the quotient groups are all normalized so that the "fold-up" formulae hold:

$$\int_{\Gamma_0} = \int_{\Gamma_0/\Lambda_0} \int_{\Lambda_0}, \quad \int_\Gamma = \int_{\Gamma/\Lambda_0} \int_{\Lambda_0} \quad \text{and} \quad \int_\Gamma = \int_{\Gamma/\Gamma_0} \int_{\Gamma_0}.$$

We have supposed also that

$$\int_{\Lambda_0} d\lambda_0 = \int_{\Gamma_0} d\gamma_0 \chi_{\Lambda_0}(\gamma_0) = 1.$$

Now Γ_0/Λ_0 is a discrete subgroup of Γ/Λ_0; let k be a positive definite $A_c(\Gamma/\Lambda_0)$-function such that

$$k(0) = 1, \quad \|k\|_1 \leqslant 1 \quad \text{and} \quad \operatorname{supp}(k) \cap \Gamma_0/\Lambda_0 = \{\dot{0}\}.$$

For any positive definite A-function, in particular k, $\|k\|_A = k(0)$; in this case, $\|k\|_A = 1$.

Let G be the dual group of Γ. The annihilators of Λ_0 and Γ_0 in G are denoted by H_0 and G_0 respectively. Since Λ_0 is compact, H_0 is open in G; H_0/G_0, being identified with the dual group of Γ_0/Λ_0, is compact. We summarize:

$$\{0\} \subseteq G_0 \subseteq H_0 \subseteq G, \qquad \Gamma \supseteq \Gamma_0 \supseteq \Lambda_0 \supseteq \{0\}.$$

Our assumptions on the Haar measures of Γ, Γ_0, etc., imply corresponding conditions on the Haar measures of G, G_0, etc., which do not bear repeating.

Define l to be the $FA_c(G)$-function, supported in the open subgroup H_0 of G such that $l|_{H_0} = \hat{k}$. Then l is nonnegative, continuous, and integrable, so the $L^1(G/G_0)$-function $\dot{l}$,

$$\dot{l}(\dot{x}) = \int_{G_0} dx_0 l(x + x_0) \qquad \forall x \in G,$$

is nonnegative and lower-semicontinuous. The Fourier transform of $\dot{l}$ restricted to H_0/G_0 is just $k_\vee$ restricted to Γ_0/Λ_0, so $\dot{l}$ is equal to one almost everywhere by our choice of k. Because $\dot{l}$ is lower-semicontinuous, the set of points $\dot{x}$ where $\dot{l}(\dot{x}) > 1$ is open, hence empty. It follows that

$$0 \leqslant \int_{G_0} dx_0 l(x + x_0) \leqslant 1 \qquad \forall x \in G \tag{4.1}$$

and

$$\int_{G_0} dx_0 l(x + x_0) = \chi_{H_0/G_0}(\dot{x}) \qquad \text{a.e. } \dot{x} \text{ in } G/G_0. \tag{4.2}$$

Also,

$$\|l\|_1 = \|k\|_A = 1, \tag{4.3}$$

$$\|l\|_\infty \leqslant \|k\|_1 \leqslant 1 \tag{4.4}$$

and so

$$\|l\|_2 \leqslant \|l\|_1^{1/2} \|l\|_\infty^{1/2} \leqslant 1. \tag{4.5}$$

Let S be a subset of G containing exactly one element of each coset of the open subgroup H_0 of G in G, as in Theorem 5, and let A be the net of finite subsets of S constructed there. Define l_α and j_α by the formulae

$$l_\alpha = \sum_{x \in \alpha} T_x l = \sum_{x \in \alpha} \epsilon_x * l \quad \text{and} \quad j_\alpha = |\alpha|^{-1} l_\alpha * (l_\alpha)_\vee.$$

From (4.1),

$$0 \leqslant \int_{G_0} dx_0 l_\alpha(x + x_0) \leqslant 1 \qquad \forall x \in G;$$

evidently, $\|l_\alpha\|_1 = |\alpha|$ and $\|l_\alpha\|_\infty \leqslant 1$, from (4.3) and (4.4).

Define, for each α in A, the linear operator J_α: $C_c(G) \longrightarrow C_c(G/G_0)$, by the formula

$$J_\alpha f(\dot{x}) = \int_{G_0} dx_0 f(x + x_0) j_\alpha(x + x_0) \qquad \forall x \in G,$$

for each f in $C_c(G)$. Note that

$$J_\alpha f(\dot{x}) = |\alpha|^{-1} \int_{G_0} dx_0 f(x + x_0) l_\alpha * (l_\alpha)_\vee(x + x_0) \qquad \forall x \in G.$$

From Theorem 4, J_α maps $A_c(G)$ into $A_c(G/G_0)$ and, for every (p, q) in S,

$$\|J_\alpha f\|_{A_p^q} \leqslant \|f\|_{A_p^q} \qquad \forall f \in A_c(G). \tag{4.6}$$

From Corollary 5a and (4.5), each j_α is in $A(G)$, $\|j_\alpha\|_A \leqslant 1$, and the net $(j_\alpha)_{\alpha \in A}$ increases monotonely to the function j in $B(G)$. Set

$$Jf(\dot{x}) = \int_{G_0} dx_0 f(x + x_0) j(x + x_0) \qquad \forall x \in G.$$

For any f in $A_c(G)$, $f \cdot j_\alpha$ converges in $A(G)$ to $f \cdot j$ (Corollary 5a) and, of course, $\operatorname{supp}(f \cdot j_\alpha) \subseteq \operatorname{supp}(f)$. It follows from Corollary 4a that $J_\alpha f$ converges to Jf in $A_c(G/G_0)$, and so in $A_p^q(G/G_0)$ for any (p, q) in S. We conclude that Jf is in $A_c(G/G_0)$ if f is in $A_c(G)$, and

$$\|Jf\|_{A_p^q} \leqslant \|f\|_{A_p^q} \qquad \forall f \in A_c(G) \tag{4.7}$$

for every (p, q) in S.

Let μ be the measure on Γ such that $\hat{\mu} = j$. Since j_α converges locally uniformly and boundedly to j (Corollary 5a) and

$$\operatorname{supp}(\hat{j}_\alpha) = \operatorname{supp}(\hat{l}_\alpha) = \operatorname{supp}(\hat{l}),$$

which is compact, μ is in fact a measure with compact support, K say:

$$\operatorname{supp}(\mu) = K.$$

We now identify the adjoint of J. The linear operator J maps $A_c(G)$ continuously into $A_c(G/G_0)$, and so its adjoint J^* maps $Q(G/G_0)$ continuously into $Q(G)$. Furthermore, for any (p, q) in S,

$$\|J^*\Phi\|_{L_p^q} \leqslant \|\Phi\|_{L_p^q} \qquad \forall \Phi \in L_p^q(G/G_0), \tag{4.8}$$

from (4.7). For any u in $A_c(\Gamma)$, $\hat{u}$, and so also $J\hat{u}$, is integrable. If γ_0 is in Γ_0,

$$(J\hat{u})^\wedge(\gamma_0) = \int_{G/G_0} d\dot{x}\overline{\gamma_0(\dot{x})} \int_{G_0} dx_0 \hat{u}(x + x_0)\hat{\mu}(x + x_0)$$

$$= \int_G dx\overline{\gamma_0(x)}\hat{u}(x)\hat{\mu}(x) = \int_G dx\overline{\gamma_0(x)}(u * \mu)^\wedge(x) = (u * \mu)_\vee(\gamma_0),$$

since $u * \mu$ is in $L^1 \cap A(\Gamma)$, so the inversion theorem holds. In fact, because u

and μ are compactly supported, $u * \mu$ is in $A_c(\Gamma)$, whence

$$(4.9)\qquad \begin{aligned}((J^*\Phi)\hat{}, u) &= (J^*\Phi, \hat{u}) = (\Phi, J\hat{u})\\ &= (\hat{\Phi}, (J\hat{u})_\vee) = (\hat{\Phi}, (u * \mu)|_{\Gamma_0}),\end{aligned}$$

for any Φ in any $L_p^q(G/G_0)$ with (p, q) in S.

Let L be the linear operator from $C_{\mathrm{loc}}(\Gamma_0)$ to $M_{\mathrm{loc}}(\Gamma)$ (the space of Radon measures on Γ) defined by the rule

$$L\phi = (\phi \cdot m_0) * \mu$$

where $(\phi \cdot m_0, u) = (\phi, u|_{\Gamma_0})\ \forall u \in C_c(\Gamma)$ for any continuous function ϕ on Γ_0. From (4.8) and (4.9), we deduce that, if ϕ is in $M_p^q \cap C_{\mathrm{loc}}(\Gamma_0)$, then $L\phi$ is in $M_p^q(\Gamma)$ and

$$(4.10)\qquad \|L\phi\|_{M_p^q} \leqslant \|\phi\|_{M_p^q};$$

this applies for any (p, q) in S. Now μ is supported in a compact set K; it follows from the definition of L that

$$(4.11)\qquad \mathrm{supp}(L\phi) \subseteq \mathrm{supp}(\phi) + K.$$

It remains only to show that L maps $C_{\mathrm{loc}}(\Gamma_0)$ into $C_{\mathrm{loc}}(\Gamma)$ and that $L\phi|_{\Gamma_0} = \phi$, for all ϕ in $C_{\mathrm{loc}}(\Gamma_0)$.

To show that $L\phi$ is a continuous function whenever ϕ is a continuous function, we apply the inequality (4.10) for particular cases of (p, q) in S, and use (4.11). First, if ϕ is in $A_c(\Gamma_0)$, then $L\phi$ is in $B(\Gamma)$ ($B(\Gamma) = M_1^1(\Gamma)$) from (4.10) with (p, q) taken to be $(1, 1)$. However, $\mathrm{supp}(L\phi)$ is compact, from (4.11). Thus L maps $A_c(\Gamma_0)$ into $A_c(\Gamma)$. Next, if ϕ is continuous and compactly supported, we can write $\phi = \Sigma_1^\infty \phi_n$ where each ϕ_n is in $A_c(\Gamma)$ and $\Sigma_1^\infty \|\phi_n\|_\infty < \infty$.

From (4.10), with (p, q) taken to be $(2, 2)$, we note that $\Sigma_1^\infty \|L\phi_n\|_\infty < \infty$, since $M_2^2(\Gamma) = L^\infty(\Gamma)$, and then deduce that $L\phi = \Sigma_1^\infty L\phi_n$, where the right-hand side is a uniformly convergent series of $A_c(\Gamma)$-functions. Hence $L\phi$ is (identifiable with) a continuous function. From (4.11), $L\phi$ is actually in $C_c(\Gamma)$. Observe further that

$$L\phi|_{\Gamma_0} = \sum_1^\infty (L\phi_n)|_{\Gamma_0},$$

so if the restriction property $L\phi|_{\Gamma_0} = \phi$ holds for every ϕ in $A_c(\Gamma_0)$, then it holds for each ϕ in $C_c(\Gamma)$.

Suppose finally that ϕ is any continuous function on Γ_0. Even if ϕ is in some space $M_p^q(\Gamma_0)$, we can say very little about $L\phi$ at the outset. For example, if $p \leqslant 2 \leqslant q$ and Γ is not discrete, $L\phi$ need not be *a priori* a function class. However, let $(\eta_\beta)_{\beta \in B}$ be an approximate identity for multiplication of $C_c(\Gamma_0)$-functions which is also a local unit. It follows that, in a distributional topology ($\sigma(M_{\mathrm{loc}}, C_c)$),

$$L\phi = L \lim_{\beta \in B} (\eta_\beta \cdot \phi) = \lim_{\beta \in B} L(\eta_\beta \cdot \phi),$$

from (4.11). Each $L(\eta_\beta \cdot \phi)$ is continuous, and the net $(L(\eta_\beta \cdot \phi))_{\beta \in B}$ is eventually constant on any given, arbitrarily large, compact subset of Γ. Thus $L\phi$ is a continuous function on Γ. Moreover,

$$L\phi|_{\Gamma_0} = \lim_{\beta \in B} L(\eta_\beta \cdot \phi)|_{\Gamma_0},$$

and so if the restriction property $L\phi|_{\Gamma_0} = \phi$ holds for all continuous compactly supported functions on Γ_0, then it holds also for all continuous functions on Γ_0.

We have established that the linear mapping L maps continuous functions on Γ_0 to continuous functions on Γ, and further, that the restriction property $L\phi|_{\Gamma_0} = \phi$ holds for all continuous functions ϕ on Γ_0 if it holds for $A_c(\Gamma_0)$-functions. To complete the proof of the theorem we must therefore prove that, for any ϕ in $A_c(\Gamma_0)$, $L\phi|_{\Gamma_0} = \phi$. This we do by showing that

$$\int_{G_0} dx_0 j(x + x_0) = 1 \qquad \forall x \in G. \tag{4.12}$$

The motivation for this may be loosely expressed as follows: The requirement that $L\phi|_{\Gamma_0}$ be equal to ϕ is, in dual form, that when u is a function on G/G_0 and $u \circ \pi$ is its "periodification", then u should be recovered from $u \circ \pi$ by application of J.

The vital step in proving (4.12) is showing that

$$\int_{G_0} dx_0 l * (l_\vee)(x + x_0) = \begin{cases} 1, & x \in H_0, \\ 0, & \text{elsewhere.} \end{cases}$$

Recall (4.2) that

$$\dot{l}(\dot{x}) = \int_{G_0} dx_0 l(x + x_0) = \chi_{H_0/G_0}(\dot{x}) \quad \text{a.e., } \dot{x} \text{ in } G/G_0.$$

Then, because H_0/G_0 is a subgroup of G/G_0 of mass one,

$$\begin{aligned} \int_{G_0} dx_0 l * (l_\vee)(x + x_0) &= \int_{G_0} dx_0 \int_G dy l(x + x_0 + y) l(y) \\ &= \int_{G/G_0} d\dot{y} \int_{G_0} dy_0 \int_{G_0} dx_0 l(x + x_0 + y + y_0) l(y + y_0) = \dot{l} * \dot{l}_\vee(\dot{x}) \\ &= \begin{cases} 1, & x \in H_0, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{4.13}$$

Recall that $j = \lim_{\alpha \in A} m_\alpha * l * l_\vee$, where $(m_\alpha)_{\alpha \in A}$ is the net of measures constructed in Theorem 5. If we denote by m_x the measure in the coset $x + H_0$ to which $\chi_{x+H_0} \cdot m_\alpha$ converges, then m_x is a discrete, nonnegative measure of total mass one. We write $m_x = \Sigma_1^\infty b_n \epsilon_{x_n}$, where $\Sigma_1^\infty b_n = 1$ and the points x_n all belong to the coset $x + H_0$. Since l is supported in H_0, it follows that

$$j(x + x_0) = m_x * l * l_\vee(x + x_0) = \sum_1^\infty b_n l * l_\vee(x + x_0 - x_n)$$

and consequently,

$$\int_{G_0} dx_0 j(x + x_0) = \sum_1^\infty b_n \int_{G_0} dx_0 l * l_\vee(x + x_0 - x_n)$$

$$= \sum_1^\infty b_n = 1,$$

by (4.13). Since x is arbitrary, (4.12) is proved.

Finally, recall the definition of L:

$$L\phi = (\phi \dot{\cdot} m_0) * \mu.$$

Then, for every x in G,

$$(L\phi)\hat{}(x) = (\phi \cdot m_0)\hat{}(x)\hat{\mu}(x) = \hat{\phi} \circ \pi(x) j(x),$$

where π is the canonical projection of G onto G/G_0. Let $[(L\phi)\hat{}]^\cdot$ be the function on G/G_0 defined by the rule

$$[(L\phi)\hat{}]^\cdot(\dot{x}) = \int_{G_0} dx_0 (L\phi)\hat{}(x + x_0) \qquad \forall x \in G.$$

From (4.12), $[(L\phi)\hat{}]^\cdot = \hat{\phi}$. It is easily checked that $[(L\phi)|_{\Gamma_0}]\hat{} = [(L\phi)\hat{}]^\cdot$ for any $L\phi$ in $A_c(\Gamma)$, whence $(L\phi)|_{\Gamma_0} = \phi$, as required to complete the proof of Theorem 2, under the condition that Γ_0 have a compact open subgroup Λ_0.

In the general case, Γ_0 may not have a compact open subgroup Λ_0. However, Γ_0 can be written in the form $R^a \oplus \Delta_0$, where Δ_0 has a compact open subgroup Λ_0, and then write Γ as $R^a \oplus \Delta$, where Δ_0 is a closed subgroup of Δ (Corollary 3a). In effect, one produces an extension operator L from $C_{\text{loc}}(\Delta_0)$ to $C_{\text{loc}}(\Delta)$, and then this operator gives rise naturally to an operator L' from the space of continuous functions on $R^a \oplus \Delta_0$ to a space of functions on $R^a \oplus \Delta$: for any ϕ in $C_{\text{loc}}(R^a \oplus \Delta_0)$ and ρ in R^a, $L'\phi(\rho, \cdot)$ is just the function on Δ obtained by extending the continuous function $\phi(\rho, \cdot)$ on Δ_0 using L. However, the details of the proof are worth outlining.

Let H be the dual group of Δ. Then $R^a \oplus H$ is the dual group of $R^a \oplus \Delta$. Let H_0 and G_0 be the annihilators of Λ_0 and Δ_0 in H. Symbolically

$$\{0\} \subseteq \Lambda_0 \subseteq \Delta_0 \subseteq \Delta, \qquad H \supseteq H_0 \supseteq G_0 \supseteq \{0\}.$$

The function l, to be used in the definition of operators J_α and J, is a function on H, not on all of G, such that

$$0 \leqslant \int_{G_0} dx_0 l(x + x_0) \leqslant 1 \qquad \forall x \in H,$$

$$\int_{G_0} dx_0 l(x + x_0) = \chi_{H_0/G_0}(\dot{x}) \quad \text{a.e., } \dot{x} \text{ in } H,$$

$$\|l\|_1 = 1, \qquad \|l\|_\infty \leqslant 1;$$

l is the Fourier transform of an $A_c(\Delta)$-function.

The net A, constructed in Theorem 5, is now a set of subsets of S, say, where S contains exactly one element of each coset of the open subgroup H_0 of H (rather than G). The functions l_α and j_α on H are defined by the formulae:

$$l_\alpha = \left(\sum_{x \in \alpha} \epsilon_x \right) * l \quad \text{and} \quad j_\alpha = |\alpha|^{-1} l_\alpha * (l_\alpha)_\vee.$$

Theorem 4 must be modified slightly to prove that the operators J_α: $A_c(R^a \oplus H) \longrightarrow A_c(R^a \oplus H/G_0)$, defined

$$J_\alpha f(u + \dot{x}) = \int_{G_0} dx_0 f(u + x + x_0) j_\alpha(x + x_0) \qquad \forall u \in R^a \;\; \forall x \in H,$$

map $A_p^q(R^a \oplus H)$ into $A_p^q(R^a \oplus H/G_0)$ without increasing norms. The new feature of the proof is that the operators J_α "fold up" only in the H-component. One shows first that

$$|\alpha| J_\alpha(f * g)(u + \dot{x}) = \int_H dz F_z * G_z(u + \dot{z}) \qquad \forall u \in R^a, \forall x \in H$$

(compare with 4.2), where

$$F_z(u + \dot{x}) = \int_{G_0} dx_0 f(u + x + x_0) T_z l_\alpha(x + x_0) \qquad \forall u \in R^a, \forall x \in H$$

and

$$G_z(u + \dot{x}) = \int_{G_0} dx_0 g(u + x + x_0)(T_z l_\alpha)_\vee(x + x_0) \qquad \forall u \in R^a, \forall x \in H,$$

and then estimates $\int_H dz \|F_z\|_p \|G_z\|_{q'}$ much as before.

In showing that the operators J_α converge in the strong operator topology (qua operators from $A_c(R^a \oplus H)$ to $A_c(R^a \oplus H/G_0)$) to the operator J:

$$Jf(u + \dot{x}) = \int_{G_0} dx_0 f(u + x + x_0) j(x + x_0) \qquad \forall u \in R^a, \forall x \in H,$$

a small point arises. It is necessary to multiply the integrands by a function g in $A_c(H)$ which takes the value one on the compact projection of supp(f) onto H in order to apply Corollary 4a unchanged. Having done so, one observes that

$$\|J_\alpha f - Jf\|_A = \|[f(u,x)g(x)(j_\alpha(x) - j(x))]\|_A$$
$$\leqslant C\|f\|_A \|g(j_\alpha - j)\|_A$$

which tends to zero as α increases.

The measure μ on $R^a \oplus \Delta$ is defined by the rule

$$\hat{\mu}(u + x) = j(x) \qquad \forall u \in R^a, \forall x \in H;$$

μ is supported in the closed subgroup Δ of $R^a \oplus \Delta$. Immediately

$$\int_{G_0} dx \hat{\mu}(u + x + x_0) = \int_{G_0} dx_0 j(x + x_0) = 1.$$

The operator L: $C_{\text{loc}}(\Gamma_0) \longrightarrow M_{\text{loc}}(\Gamma)$, defined by the formula $L\phi = \phi \cdot m_0 * \mu$, is shown to have the requisite properties in the same way as in the particular case treated fully.

REMARK I. Let $B_p(G)$ be the space of (pointwise) multipliers of $A_p^p(G)$, with the operator norm. The formula

$$Jf(\dot{x}) = \int_{G_0} dx_0 f(x + x_0)\hat{\mu}(x + x_0) \qquad \forall x \in G$$

clearly makes sense whenever f is continuous and bounded, not just when f has compact support. It is easy to show that Jf is continuous for any f in $C(G)$ and furthermore that J maps $B_p(G)$ into $B_p(G/G_0)$ without increasing norms. This latter fact is obtained by natural extension from the continuity of J from $A_p^p(G)$ to $A_p^p(G/G_0)$. On the other hand, it has been known for some time (see, e.g. [16]) that if u is in $B_p(G/G_0)$ and π denotes the canonical projection of G onto G/G_0, then $u \circ \pi$ is in $B_p(G)$ and $\|u \circ \pi\|_{B_p} \leqslant \|u\|_{B_p}$. However, it is clear that $J(u \circ \pi) = u$, so that $\|u\|_{B_p} \leqslant \|u \circ \pi\|_{B_p}$ for any continuous function u such that $u \circ \pi$ is in $B_p(G)$. We conclude that the induced mapping π^*, taking u on G/G_0 to $u \circ \pi$ on G, is an isometry of $B_p(G/G_0)$ onto the subspace of $B_p(G)$ of functions constant on cosets of G_0 in G.

REMARK II. By judicious choice of the function k (made in the early part of this section), we can show that if $\epsilon > 0$ there is an extension operator J_ϵ: $C_{\text{loc}}(\Gamma_0) \longrightarrow C_{\text{loc}}(\Gamma)$ such that

$$\|J_\epsilon \phi\|_{M_p^q} \leqslant \epsilon^{s^{-1}} \|\phi\|_{M_p^q} \qquad \forall (p,q) \in S, \forall \phi \in M_p^q(\Gamma_0)$$

($s^{-1} = p^{-1} - q^{-1}$); one has merely to pick k such that $\|k\|_1 < \epsilon$. Thus

$$\inf\{\|\psi\|_{M_p^q}: \psi \in M_p^q \cap C(\Gamma), \psi|_{\Gamma_0} = \phi\} = 0$$

for every ϕ in $M_p^q \cap C(\Gamma_0)$. This complements results of Gaudry [11] about the restriction of $L^p - L^q$ multipliers ($p < q$) to closed subgroups.

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