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THE HYPERSPACE OF THE CLOSED UNIT INTERVAL IS A HILBERT CUBE[1]

BY

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ABSTRACT. Let X be a compact metric space and let 2^X be the space of all nonvoid closed subsets of X topologized with the Hausdorff metric. For the closed unit interval I the authors prove that 2^I is homeomorphic to the Hilbert cube I^∞, settling a conjecture of Wojdyslawski that was posed in 1938. The proof utilizes inverse limits and near-homeomorphisms, and uses (and developes) several techniques and theorems in infinite-dimensional topology.

1. **Introduction.** Let X be a compact metric space and let 2^X be the space of all nonvoid closed subsets of X topologized with the Hausdorff metric. In the Bull. Amer. Math. Soc. [9], we announced that 2^I is homeomorphic to ($\approx$) the Hilbert cube Q and gave an outline of our proof. §3 of this paper is essentially an expanded version of [9]; The bulk of this paper, §§4–8 consists of the proofs of the alleged but unproved claims in [9]. In §5 we prove that a certain finite dimensional subspace A_n of 2^I is a Q-factor, that is, $A_n \times Q \approx Q$, and the rest of the paper is devoted to proving that two specific maps, $f = f_n \times \mathrm{id}$ and r_n (defined in §3), are near-homeomorphisms, i.e., uniform limits of homeomorphisms.

Our original and unpublished proof that f and r_n are near-homeomorphisms was based on a theory of reduced mapping cylinders. See [13] for a corresponding account of mapping cylinders. Since then we have obtained shorter and easier proofs that these maps are near-homeomorphisms using Q-factor decompositions, a notion introduced by D. W. Curtis in [3]. In this paper we present in §§6–8 these more recently obtained proofs.

Very recently T. A. Chapman announced a theorem that characterizes near-homeomorphisms between Hilbert cubes as being those continuous surjections with the property that the inverse image of each point has trivial shape. This result is a consequence of his paper [*Cell-like mappings of Hilbert cube manifolds: Applications to simple-homotopy theory*, Bull. Amer. Math. Soc. 79 (1973), 1286–

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1291], and relies on a great deal of apparatus including algebraic K-theory and computations of Whitehead groups. The use of this characterization of near-homeomorphisms of the Hilbert cube would shorten this paper. However, because of the extensive background needed for Chapman's proof, use of his result would significantly lessen the accessibility of the proof of the main result of this paper.

Brief history of the problem. Probably the first result in the direction of 2^I was by L. Vietoris when he proved in [10] that if X is a Peano continuum, then so is 2^X. In [11], Wazewski proved the converse. Mazurkiewicz [7] showed that if X is compact and connected, then 2^X is a continuous image of the Cantor star and Wojdyslawski [14] showed that if X is a Peano continuum, then 2^X is contractible and locally contractible and later [15] that 2^X is an absolute retract if and only if X is a Peano continuum. In his earlier paper Wojdyslawski specifically asked if $2^I \approx Q$ and, more generally, he asked if $2^X \approx Q$ where X is any nondegenerate Peano continuum. Professor Kuratowski has informed us that the conjecture that $2^I \approx Q$ was well known to the Polish topologists in the 1920's.

It was well known that Q is contractible, locally contractible, an absolute retract, and that 2^I and Q each contain homeomorphic copies of each other. These were some of the obvious reasons that Wojdyslawski's conjecture seemed reasonable. It was known that Q is homogeneous [6], but it was not previously known that 2^I is. From the homogeneity of Q it easily follows that each point of Q is unstable and in [5] Neil Gray proved the corresponding property for 2^X, where X is any nondegenerate Peano space.

Other efforts in the direction of 2^X were focused on the study of the subspace of 2^X consisting of all nonvoid subsets of X containing less than or equal to n points ($n \geqslant 1$), denoted by $X(n)$ and called the *n-fold symmetric product of X*. See the introduction of [8] for an account of these results.

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2. **Definitions and preliminaries.** If X is a compact metric space, then the Hausdorff metric D on 2^X can be defined by

$$D(A, B) = \inf\{\epsilon > 0 : A \subset U(B, \epsilon) \text{ and } B \subset U(A, \epsilon)\}$$

where, for $C \subset X$, $U(C, \epsilon)$ is the open ϵ-neighborhood of C in X.

Let I be the closed unit interval $[0, 1]$ and for $S \subset I$, let $H(S)$ denote the subspace of 2^I consisting of all closed subsets of I that contain S. If $t_1, \ldots, t_n$ are points of I, denote $H(\{t_1, \ldots, t_n\})$ by $H(t_1, \ldots, t_n)$. In related papers $H(0, 1)$ has often been denoted by 2^I_{01}.

Let Q denote the countable infinite product of copies of I and define a

Hilbert cube as any space homeomorphic to Q. A space X is a *Q-factor* if $X \times Q \approx Q$. This is equivalent to saying that there exists a space Y such that $X \times Y \approx Q$ since if the later is true, then $Q \approx (X \times Y)^\omega \approx X \times (X \times Y)^\omega \approx X \times Q$.

A *map* is a continuous function and homeomorphisms are always onto. If X and Y are homeomorphic compact metric spaces, then a map $f: X \longrightarrow Y$ is a *near-homeomorphism* if for each $\epsilon > 0$ there is a homeomorphism $h: X \longrightarrow Y$ such that $d(h, f) < \epsilon$. It follows easily that the composition of two near-homeomorphisms is a near-homeomorphism and that the cartesian product of two near-homeomorphisms is a near-homeomorphism. We say that $f: X \longrightarrow Y$ *stabilizes to a near-homeomorphism* if $f \times \mathrm{id}: X \times Q \longrightarrow Y \times Q$ is a near-homeomorphism.

An *inverse sequence* (X_n, f_n) is a sequence of spaces X_n and maps f_n such that for each $n \geq 1, f_n: X_{n+1} \longrightarrow X_n$. The *inverse limit* of (X_n, f_n), denoted by $\lim (X_n, f_n)$, is the subspace of the product of the X_n consisting of all points $(x_n) \in \Pi_{n=1}^\infty X_n$ such that for each $n \geq 1, f_n(x_{n+1}) = x_n$.

We quote the following three results as they will be referred to in this paper.

THEOREM 2.1 (MORTON BROWN [2, THEOREM 4, p. 482]). *Let $S = \lim (X_n, f_n)$ where the X_n are all homeomorphic to a compact metric space X and each f_n is a near-homeomorphism. Then S is homeomorphic to X.*

THEOREM 2.2 [9, THEOREM 5.2, p. 405]. *Let $S = \lim (X_n, f_n)$ and $T = \lim (Y_n, g_n)$ where all the spaces are compact and for each n, let $h_n: X_n \longrightarrow Y_n$ be a map such that $g_n h_{n+1} = h_n f_n$. If for each n, both f_n and h_n are (stabilize to) near-homeomorphisms, then the induced map $h: S \longrightarrow T$ defined by $h(x_n) = (h_n(x_n))$ is a (stabilizes to a) near-homeomorphism.*

The following lemma was first observed by Fort and Segal in [4]. It provides a useful method for recognizing certain inverse limits.

LEMMA 2.3 [4, LEMMA 4, p. 132]. *Let X be a compact metric space, let $X_1, X_2, \ldots$ be closed subsets of X, and for each n let φ_n be a map of X onto X_n and let f_n be a map of X_{n+1} onto X_n such that $\varphi_n = f_n \circ \varphi_{n+1}$ and $\varphi_1, \varphi_2, \ldots$ converges uniformly to the identity map on X. Then the function φ on X defined by $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots)$ is a homeomorphism of X onto* $\lim(X_n, f_n)$.

3. **The reduction of the proof.** This section is a more explicit and detailed version of §3 of [9]. The basic idea of the proof that $2^I \approx Q$ is to identify a nested sequence of Hilbert cubes, $Y_1 \subset Y_2 \subset \ldots$, contained in $H(0, 1)$, whose union is dense in $H(0, 1)$, and to define retractions $r_n: Y_{n+1} \longrightarrow Y_n$ such that each r_n is a near-homeomorphism and such that $H(0, 1)$ is homeomorphic to $\lim (Y_n, r_n)$. Then (Morton Brown's) Theorem 2.1 implies that $H(0, 1)$ is a Hilbert cube. Then Proposition 3.1, below, says that if $H(0, 1)$ is a Hilbert cube,

then so is 2^I. This inverse sequence (Y_n, r_n) will be called the *principal sequence.* We will reduce this program to considering certain finite-dimensional subspaces A_n and $A_{n,t}$ of 2^I and maps $f_n : A_{n+1} \longrightarrow A_n$ and $h_n : A_n \longrightarrow A_{n,t}$ and in this section we will do everything needed except for proving that each A_n is a Q-factor and that f_n and h_n stabilize to near-homeomorphisms.

PROPOSITION 3.1. *If $H(0, 1)$ is a Hilbert cube, then so is 2^I.*

PROOF. In [8] it is shown that 2^I is homeomorphic to $CCH(0, 1)$, where CX denotes the cone over X. (The formula $(A, s, t) \longrightarrow \{(1-t)(1-s)a + t : a \in A\}$ defines a map from $H(0, 1) \times I \times I$ to 2^I producing the same identifications as the coning operations.) O. H. Keller proved in [6] that all infinite-dimensional, convex compacta of Hilbert space are Hilbert cubes, and since CQ has a geometric realization as such a subset of Hilbert space, then CQ and hence CCQ is a Hilbert cube and the result follows.

Each Y_n is a Hilbert cube. Before defining the Y_n we will prove that $H(0, 1)$ is a Q-factor. This result will be the main tool in proving that each Y_n is a Hilbert cube. We proceed as follows. For each $n \geqslant 1$, let $F_n : H(0, 1) \longrightarrow H(0, 1)$ be the function assigning to an element A in $H(0, 1)$ its closed $1/n$-neighborhood in I, i.e., $F_n(A) = \{s \in I : |s - a| \leqslant 1/n \text{ for some } a \in A\}$. These functions are continuous. Let $A_n = F_n(H(0, 1))$ and let $f_n : A_{n+1} \longrightarrow A_n$ be defined by $f_n = F_{n(n+1)}|A_{n+1}$. Since $1/n = 1/n + 1 + 1/n(n + 1)$, it follows that $F_n = f_n \circ F_{n+1}$. Thus, $f_n(A_{n+1}) = A_n$ and we call the inverse sequence (A_n, f_n) the *auxiliary sequence.*

PROPOSITION 3.2. *The inverse limit of (A_n, f_n) is homeomorphic to $H(0, 1)$.*

PROOF. For each n, $F_n = f_n \circ F_{n+1} : H(0, 1) \longrightarrow A_n$ and $F_1, F_2, \ldots$ converges uniformly to the identity map on $H(0, 1)$. Thus, by 2.3, the function $F : H(0, 1) \longrightarrow \lim (A_n, f_n)$ defined by $F(A) = (F_1(A), F_2(A), \ldots)$ is a homeomorphism.

The proof of the next proposition is given in §5.

PROPOSITION 3.3. *Each A_n is a Q-factor.*

A proof of the next proposition is given in §7.

PROPOSITION 3.4. *Each $f_n : A_{n+1} \longrightarrow A_n$ stabilizes to a near-homeomorphism.*

THEOREM 3.5. *$H(0, 1)$ is a Q-factor.*

PROOF. The proof is an immediate consequence of Theorem 2.1, the preceding three propositions, and the fact that $\lim (A_n, f_n) \times Q$ is homeomorphic to $\lim (A_n \times Q, f_n \times \mathrm{id})$.

For each $n \geqslant 1$, let $\sigma(n) = \{0, 1, 1/n, 1/n+1, \ldots\}$ and let $Y_n = H(\sigma(n))$.

COROLLARY 3.6. *Each Y_n is a Hilbert cube.*

PROOF. For a fixed $n \geqslant 1$, let J_m denote the mth subinterval from the right determined by $\sigma(n)$, i.e., $J_1 = [1/n, 1]$, $J_2 = [1/n+1, 1/n]$, etc., and let $H_m = \{A \in 2^{J_m} : A$ contains the endpoints of $J_m\}$. Also let $\alpha_m : J_m \longrightarrow I$ be the order-preserving linear homeomorphism, let $\alpha_m^* : H_m \longrightarrow H(0, 1)$ be the induced homeomorphism, and define β_m on 2^I to be the intersection with J_m map, i.e., $\beta_m(A) = A \cap J_m$. Then $\alpha : Y_n \longrightarrow \Pi_{m=1}^{\infty} H(0, 1)_m$, where $H(0, 1)_m = H(0, 1)$, defined by $\alpha = (\alpha_1^* \circ \beta_1, \alpha_2^* \circ \beta_2, \ldots)$ is a homeomorphism. Since $H(0, 1)$ is a Q-factor, it follows that Y_n is a Hilbert cube since it is known by [12, Theorem 6.2, p. 21] that the countable infinite product of nondegenerate Q-factors is a Hilbert cube.

The principal inverse sequence. We define the maps $r_n : Y_{n+1} \longrightarrow Y_n$ as follows. For $A \in Y_{n+1}$, let $u = \max\{x \in A : x \leqslant 1/n\}$ and $v = \min\{x \in A : x \geqslant 1/n\}$, and let $\alpha = \min\{d : A \cup [u, u+d] \cup [v-d, v] \in Y_n\}$. Then $r_n(A) = A \cup [u, u+\alpha] \cup [v-\alpha, v]$. Note that $\alpha = \min\{1/n - u, v - 1/n\}$. It is easy to see that each r_n is continuous and is a retraction.(2) The inverse sequence (Y_n, r_n) is called the *principal sequence.*

PROPOSITION 3.7. *The inverse limit of (Y_n, r_n) is homeomorphic to $H(0, 1)$.*

PROOF. For $x \in I$ and $B \subset I$. Let $d(x, B) = \inf\{|x-b| : b \in B\}$ and for $U \subset I$ and $n \geqslant 1$, let $\xi(U, n) = \max\{d(x, I \backslash U) : x \in \sigma(n)\}$. Define $R_n : H(0, 1) \longrightarrow Y_n$ by letting $R_n(A)$ be the union of A and

$$\bigcup\{[u, u+\xi(U, n)] \cup [v-\xi(U, n), v] : U = (u, v) \text{ is a component of } I \backslash A\}.$$

It easily follows that $R_n = r_n \circ R_{n+1}$ by observing what happens, for $A \in H(0, 1)$, on the component of $I \backslash A$ that contains $1/n$, if it exists. Thus, we can define $R : H(0, 1) \longrightarrow \lim(Y_n, r_n)$ by $R(A) = (R_1(A), R_2(A), \ldots)$ and this is a homeomorphism by 2.3 since $R_1, R_2, \ldots$ converges uniformly to the identity map on $H(0, 1)$.

The hardest part of our program is showing that each r_n is a near-homeomorphism. We start with the following reduction of r_n. For each $t \in (0, 1)$, let $h_t : H(0, 1) \longrightarrow H(0, t, 1)$ be the retraction analogous to r_n, that is, $h_t(A) = A \cup [u, u+\alpha] \cup [v-\alpha, v]$ where u is the maximal point of A less than or equal to

(2)Our definition was motivated by the following. If $s_n : Y_{n+1} \rightarrow Y_n$ is defined by $s_n(A) = A \cup \{1/n\}$, then $H(0, 1) \approx \lim(Y_n, s_n)$, but s_n patently cannot be a near-homeomorphism since if $A \in Y_n$, where $1/n$ is an isolated point of A, then $s_n^{-1}(A) = \{A, A \backslash \{1/n\}\}$ is not connected and it is easy to prove that if $r : Q \rightarrow Q$ is any near-homeomorphism, then the inverse images of points are connected. The authors are indebted to A. Verbeek for suggesting the s_n map which represented a step in the evolution of the argument.

t, v is the minimal point of A greater than or equal to t, and α is the minimum of $t - u$ and $v - t$.

We say the map $f: X \longrightarrow Y$ is *topologically equivalent* to $g: W \longrightarrow Z$ if there exist homeomorphisms $\beta: X \longrightarrow W$ and $\gamma: Y \longrightarrow Z$ such that $\gamma \circ f = g \circ \beta$.

PROPOSITION 3.8. *Each map $r_n: Y_{n+1} \longrightarrow Y_n$ is topologically equivalent to $h_t \times \mathrm{id}: H(0, 1) \times Q \longrightarrow H(0, t, 1) \times Q$, for some $t \in (0, 1)$.*

PROOF. Adopt all of the notation from the proof of 3.6, let $J_0 = J_1 \cup J_2 = [1/n + 1, 1]$, let $\alpha_0: J_0 \longrightarrow I$ be the order-preserving homeomorphism and let β_0 be the intersection with J_0 map, and let $t = \alpha_0(1/n)$. Define $\gamma: Y_{n+1} \longrightarrow H(0, 1)$ by $\gamma = \alpha_0^* \circ \beta_0$ and $\gamma_t: Y_n \longrightarrow H(0, t, 1)$ by $\gamma_t = \alpha_0^* \circ \beta_0$, let $\psi: Y_{n+1} \longrightarrow \Pi_{m=3}^{\infty} H(0, 1)_m$ be defined by $(\alpha_3^* \circ \beta_3, \alpha_4^* \circ \beta_4, \ldots)$ and let $\varphi: \Pi_{m=3}^{\infty} H(0, 1)_m \longrightarrow Q$ be any homeomorphism. Then $(\gamma, \varphi \circ \psi): Y_{n+1} \longrightarrow H(0, 1) \times Q$ and $(\gamma_t, \varphi \circ \psi): Y_n \longrightarrow H(0, t, 1) \times Q$ are homeomorphisms and $(\gamma_t, \varphi \circ \psi) \circ r_n = (h_t \times \mathrm{id}) \circ (\gamma, \varphi \circ \psi)$ and the proposition is proved.

The above proposition means that in order to prove that each $r_n: Y_{n+1} \longrightarrow Y_n$ is a near-homeomorphism it is sufficient to prove that $h_t: H(0, 1) \longrightarrow H(0, t, 1)$ stabilizes to a near-homeomorphism. We proceed as follows, again making use of the auxiliary system. For $n \geqslant 1$, let $A_{n,t} = F_n(H(0, t, 1))$ and let $f_{n,t}: A_{n+1,t} \longrightarrow A_{n,t}$ be the restriction of f_n to $A_{n+1,t}$.

PROPOSITION 3.9. *The inverse limit of $(A_{n,t}, f_{n,t})$ is homeomorphic to $H(0, t, 1)$.*

PROOF. As in the proof of 3.2, the map $F_t: H(0, t, 1) \longrightarrow \lim(A_{n,t}, f_{n,t})$ defined by $F_t(A) = (F_1(A), F_2(A), \ldots)$ is a homeomorphism.

We now will represent h_t as an inverse limit of maps. Let $F_{n,t}$ denote the restriction of F_n to $H(0, t, 1)$. There is a natural map $h_n: A_n \longrightarrow A_{n,t}$ such that $h_n \circ F_n = F_{n,t} \circ h_t$. We describe h_n as follows. Let $s_n: A_n \longrightarrow H(0, 1)$ be the natural (but discontinuous) section of F_n (i.e., $F_n \circ s_n = \mathrm{id}$) that cuts off an interval of length $1/n$ from each free end other than 0 or 1 of each component of A. That is, if $A \in A_n$ is of the form $A = [0, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, 1]$ $(a_i < b_i)$, then $s_n(A) = [0, a_1 - 1/n] \cup [b_1 + 1/n, a_2 - 1/n] \cup \cdots \cup [b_k + 1/n, 1]$. Then $h_n: A_n \longrightarrow A_{n,t}$ is defined by $h_n = F_{n,t} \circ h_t \circ s_n$. Thus we have

$$(*) \qquad h_n \circ F_n = F_{n,t} \circ h_t.$$

In the following, let $A_\infty = \lim(A_n, f_n)$ and $A_{\infty,t} = \lim(A_{n,t}, f_{n,t})$.

PROPOSITION 3.10. *The maps (h_n) satisfy the condition $h_n \circ f_n = f_{n,t} \circ h_{n+1}$, for each $n \geqslant 1$, and thus induce a map $h_\infty: A_\infty \longrightarrow A_{\infty,t}$ defined by $h_\infty(x_n) = (h_n(x_n))$. Moreover, $h_t: H(0, 1) \longrightarrow H(0, t, 1)$ is topologically equivalent to h_∞.*

PROOF. Since $F_n = f_n \circ F_{n+1}$ and $F_{n,t} = f_{n,t} \circ F_{n+1,t}$ we have from (*) that $h_n \circ f_n \circ F_{n+1} = f_{n,t} \circ F_{n+1,t} \circ h_t$ and applying (*) for $n + 1$ we have $h_n \circ f_n \circ F_{n+1} = f_{n,t} \circ h_{n+1} \circ F_{n+1}$. Since F_{n+1} is onto it follows that $h_n \circ f_n = f_{n,t} \circ h_{n+1}$ and thus we have the induced map $h_\infty : A_\infty \longrightarrow A_{\infty,t}$ defined by $h_\infty(x_n) = (h_n(x_n))$, for $(x_n) \in A_\infty$.

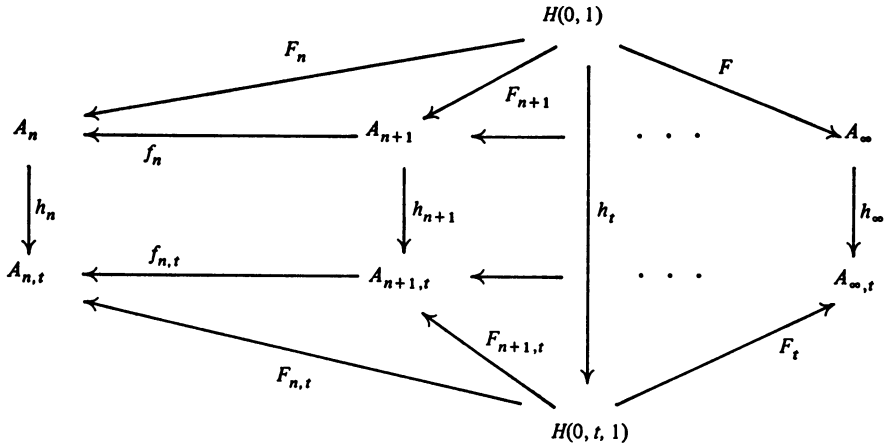

FIGURE 1

The homeomorphism $F : H(0, 1) \longrightarrow A_\infty$ of the proof of 3.2 and defined by $F(A) = (F_1(A), F_2(A), \ldots)$ has the property that for each $n \geq 1$, $F_n = \pi_n \circ F$ where $\pi_n : A_\infty \longrightarrow A_n$ is the projection map. We have the corresponding homeomorphism $F_t : H(0, t, 1) \longrightarrow A_{\infty,t}$ and the corresponding property $F_{n,t} = \pi_{n,t} \circ F_t$. Since $h_n \circ F_n = F_{n,t} \circ h_t$ it follows that $h_n \circ \pi_n \circ F = \pi_{n,t} \circ F_t \circ h_t$, and since h_∞ is induced by the h_n, we have that $h_n \circ \pi_n = \pi_{n,t} \circ h_\infty$. Thus, $\pi_{n,t} \circ h_\infty \circ F = \pi_{n,t} \circ F_t \circ h_t$, for all $n \geq 1$, and it immediately follows that $h_\infty \circ F = F_t \circ h_t$ which completes the proof.

A proof of the following proposition, which comprises the bulk of this paper, is found in §8.

PROPOSITION 3.11. *Each $h_n : A_n \longrightarrow A_{n,t}$ stabilizes to a near-homeomorphism.*

We finish this section with the main result.

THEOREM 3.12. *2^I is a Hilbert cube.*

PROOF. If f and g are topologically equivalent maps and f is a near-homeomorphism then so is g. This follows since the finite composition of near-homeomorphisms is a near-homeomorphism.

Since each h_n stabilizes to a near-homeomorphism, it follows by 2.2 that so does h_∞. Thus, since the maps h_∞ and h_t are topologically equivalent by 3.10 and the maps $h_t \times \mathrm{id}$ and r_n are topologically equivalent by 3.8, it follows that r_n

is a near-homeomorphism. Since $H(0, 1)$ is homeomorphic to $\lim(Y_n, r_n)$ by 3.7, and each Y_n is a Hilbert cube by 3.6, it follows by 2.1 that $H(0, 1)$ is a Hilbert cube. Finally, 3.1 completes the proof.

4. **The attaching theorem for Q-factors.** In this section and in the rest of the paper, the notion of Z-set, introduced by R. D. Anderson in [1], is crucial. A closed subset K of a topological space X is a *Z-set* in X if for every nonempty homotopically trivial (n-connected for all $n \geqslant 0$) open set U in X, $U \backslash K$ is nonempty and homotopically trivial. A useful characterization of Z-sets in a special class of spaces, which includes polyhedra and the Hilbert cube, is proved in [3, Lemma 2.1] and is stated as follows. A closed subset U of a metric ANR X is a Z-set in X if for each $\epsilon > 0$ there exists a map $f: X \longrightarrow X\backslash K$ with $d(f, \mathrm{id}) < \epsilon$. For example, the Z-sets in an n-cell are precisely the closed subsets of the boundary. One of the fundamental theorems in infinite-dimensional topology and proved by Anderson in [1] is the

HOMEOMORPHISM EXTENSION THEOREM. *If K_1, K_2 are Z-sets in Hilbert cubes Q_1, Q_2, respectively, and $f: K_1 \longrightarrow K_2$ is a homeomorphism, then f can be extended to a homeomorphism of Q_1 onto Q_2.*

A corollary of this is the

FIRST SUM THEOREM. *If each of Q_1, Q_2, and $Q_1 \cap Q_2$ is a Hilbert cube and $Q_1 \cap Q_2$ is a Z-set in each of Q_1 and Q_2, then $Q_1 \cup Q_2$ is a Hilbert cube.*

If X and Y are disjoint compact metric spaces, A a closed subset of X, and $f: A \longrightarrow Y$ a map, then the *adjunction space* of f, denoted by $X \cup_f Y$, is $(X \cup Y)/\sim$, where $\sim$ is the equivalence relation on $X \cup Y$ generated by $a \sim f(a)$, for all $a \in A$. We say X is *attached* to Y by f. If $g: X \longrightarrow Y$ is a map, then the *mapping cylinder* of g, denoted M_g, is the adjunction space $(X \times I) \cup_{g'} Y$ where $g': X \times \{0\} \longrightarrow Y$ is defined by $g'(x, 0) = g(x)$.

The following theorem is one of the basic theorems in the theory of Q-factors.

THEOREM 4.1 [13, THEOREM 1, p. 114]. *Let X and Y be Q-factors and let $g: X \longrightarrow Y$ be a map of X into Y, then the mapping cylinder of g is also a Q-factor.*

The following corollary is one of the basic tools of this paper.

COROLLARY 4.2 (THE ATTACHING THEOREM). *Let X and Y be disjoint Q-factors and let A be a closed subset of X that is a Q-factor and a Z-set in X. If $f: A \longrightarrow Y$ is any map, then the adjunction space $X \cup_f Y$ is a Q-factor.*

PROOF. Define $g: A \times Q \longrightarrow A \times \underline{Q} \times \{0\}$ by $g(x, y) = (x, y, 0)$. Since

A is a Z-set in X, it follows that $A \times Q$ is a Z-set in $X \times Q$, and since they are both Hilbert cubes, we can extend g, by the Homeomorphism Extension Theorem, to a homeomorphism $g_1 : X \times Q \to A \times Q \times I$. If $f_1 : A \times Q \times \{0\} \to Y \times Q$ is defined by $f_1(x, y, 0) = (f(x), y)$, then $f_1 \circ g = f \times \mathrm{id}$ and hence $(X \times Q) \cup_{f\times \mathrm{id}} (Y \times Q)$ is homeomorphic to $(A \times Q \times I) \cup_{f_1} (Y \times Q)$ which is the mapping cylinder of $f \times \mathrm{id} : A \times Q \to Y \times Q$ and hence is a Q-factor by 4.1. Since $(X \cup_f Y) \times Q \approx (X \times Q) \cup_{f\times \mathrm{id}} (Y \times Q)$, then $(X \cup_f Y) \times Q$ is a Q-factor and since $Q \times Q \approx Q$, then $(X \cup_f Y) \times Q \approx Q$ and hence $X \cup_f Y$ is a Q-factor.

5. **Each A_n is a Q-factor.** We now set up some notation for analysis of A_n which will be useful throughout the paper. Since A_n is the image of $H(0, 1)$ under F_n, it is $\{[0, u_1] \cup [v_1, u_2] \cup \cdots \cup [v_m, 1] \in 2^I : 0 \leqslant m < n/2, u_k < v_k$ for $1 \leqslant k \leqslant m$, $1/n \leqslant u_1, v_m \leqslant 1 - 1/n$, and $v_k + 2/n \leqslant u_{k+1}$ for $1 \leqslant k \leqslant m - 1\}$, where if $m = 0$ we mean $[0, 1]$. We let $A_n^0 = \{I\}$ and for $1 \leqslant m < n/2$, $A_n^m = \{A \in A_n : A$ has less than or equal to $(m + 1)$ components$\}$ and $B_n^m = \{A \in A_n : A$ has exactly $m + 1$ components$\}$. Each B_n^m corresponds naturally to a subset $\Delta(B_n^m)$ of E^{2m} under the function φ which sends $A = [0, u_1] \cup [v_1, u_2] \cup \cdots \cup [v_m, 1]$ to the point $\varphi(A) = (u_1, v_1, \ldots, u_m, v_m)$. It is easily checked that $\varphi : B_n^m \to \Delta(B_n^m)$ is a homeomorphism. The set $\Delta(B_n^m)$ is a convex set in E^{2m} and its closure, $\Delta_m = \{(u_1, v_1, \ldots, u_m, v_m) \in E^{2m} : u_k \leqslant v_k$ for $1 \leqslant k \leqslant m$, $1/n \leqslant u_1, v_m \leqslant 1 - 1/n$, and $v_k + 2/n \leqslant u_{k+1}\}$. This, for $n \geqslant 3$, is a $2m$ simplex in E^{2m} which can be obtained from the standard simplex $\sigma = \{(x_1, x_2, \ldots, x_{2m}) : 0 \leqslant x_i \leqslant x_{i+1} \leqslant 1\}$ by pushing every other face of σ towards its opposite vertex by either $1/n$ or $2/n$. For example, the face $\sigma_3 = \{x \in \sigma : x_2 = x_3\}$ is shifted over to become the face $\sigma_3' = \{x \in \sigma : x_2 + 2/n = x_3\}$. We define the map $\delta : \Delta_m \to A_n^m$ by $\delta(u_1, v_1, \ldots, u_m, v_m) = [0, u_1] \cup \cdots \cup [v_m, 1]$ and call it the *evaluation* map. Note that the restriction of δ to $\Delta(B_n^m)$ is φ^{-1}.

Let $\Delta_m' = \{(u_1, v_1, \ldots, u_m, v_m) \in \Delta_m$: for some i, $u_i = v_i\}$. Then $\Delta_m' = \Delta_m \backslash \Delta(B_n^m)$ and is the union of every second face of Δ_m, the faces of σ that were not shifted over to obtain Δ_m, and is therefore, as easily seen by an inductive argument, topologically a $(2m - 1)$-cell in the boundary of Δ_m.

PROPOSITION 5.1. *Each A_n is a Q-factor.*

PROOF. We build up A_n inductively by attaching Q-factors as follows: A_n^0 is one point, namely $\{I\}$, and if $m \geqslant 1$, and δ' is the restriction of δ to Δ_m', then A_n^m is naturally homeomorphic to $\Delta_m \cup_{\delta'} A_n^{m-1}$ in the following sense. If $i : A_n^{m-1} \to A_n^m$ is the injection map, then $(\delta \cup i) : \Delta_m \cup A_n^{m-1} \to A_n^m$ induces the same equivalence relation $\sim$ on $\Delta_m \cup A_n^{m-1}$ as $\delta' : \Delta_m' \to A_n^m$ and hence if $p : \Delta_m \cup A_n^{m-1} \to \Delta_m \cup_{\delta'} A_n^{m-1}$ is the quotient map, then $h : \Delta_m \cup_{\delta'} A_n^{m-1} \to A_n^m$ defined by $h(p(x)) = (\delta \cup i)(x)$ is a homeomorphism. It is in this sense that we say the spaces are naturally homeomorphic. Thus, by an inductive use of the Attaching Theorem, A_n is a Q-factor, since Δ_m and Δ'_m are Q-factors and since any closed subset of the boundary of Δ_m, namely Δ'_m, is a Z-set in Δ_m.

6. **Q-factor decompositions.** The concept of a Q-factor decomposition was introduced by D. W. Curtis in [3] and is a generalization of a simplicial complex where the Q-factors correspond to simplexes. Here we will generalize this notion to correspond to CW-complexes where the distinction is that, in the analogy with complexes, instead of insisting that the intersection of two simplexes be a face of each of them, we allow such an intersection to be a union of faces. Thus, we say that a collection of Q-factors has the *intersection property* if the nonempty intersection of two members of the collection is a union of members of the collection. The proper face relationship of simplexes corresponds to the following. A collection $\mathcal{D}$ of Q-factors has the *Z-set property* provided that if $D_1, D_2 \in \mathcal{D}$ where D_1 is a proper subset of D_2, then D_1 is a Z-set in D_2.

DEFINITION. A *Q-factor decomposition* of a compact metric space X is a finite cover of X by Q-factors that has the intersection and Z-set properties.

The next theorem follows from a special case of [3, Theorem 2.4] by D. W. Curtis. We include the proof here for completeness. The *mesh* of a collection of subsets of a metric space is the maximum of the diameters of the members of the collection. For each i, let $I_i = I$ and if $n < m \leqslant \infty$, let $I_n^m = \Pi_{i=n}^m I_i$.

THEOREM 6.1. *Let X and Y be compact metric spaces and let $f: X \longrightarrow Y$ be a map. If for each $\epsilon > 0$ there exists a Q-factor decomposition $\mathcal{D}$ of Y with mesh less than ϵ such that $f^{-1}(\mathcal{D})$ is a Q-factor decomposition of X, then f stabilizes to a near-homeomorphism.*

PROOF. Let $\epsilon > 0$ and let $\mathcal{D}$ be a Q-factor decomposition of Y with mesh less than $\epsilon/2$ as in the hypothesis. It is sufficient to show that there exists a homeomorphism $h: X \times Q \longrightarrow Y \times Q$ such that for each $D \in \mathcal{D}$, $h(f^{-1}(D) \times Q) = D \times Q$, since if we have such an h then we can define, for n large enough so that the diameter of I_{n+1}^∞ is less than $\epsilon/2$,

$$h_1 : X \times I_1^n \times I_{n+1}^\infty \longrightarrow Y \times I_1^n \times I_{n+1}^\infty$$

as follows. Assume h is defined on $X \times I_{n+1}^\infty$, where we are viewing Q as I_{n+1}^∞, let id be the identity function on I_1^n, and let $h_1 = h \times \mathrm{id}$. Then $d(f, h) < \epsilon$ and hence f is a near-homeomorphism.

We will now construct $h: X \times Q \longrightarrow Y \times Q$. For $\mathcal{D}_1 \subset \mathcal{D}$, let $\min(\mathcal{D}_1) = \{D \in \mathcal{D}_1 : \text{no proper subset of } D \text{ belongs to } \mathcal{D}_1\}$. Let $\mathcal{D}^{-1} = \varnothing$ and for $i \geqslant 0$, let $\mathcal{D}^i = \mathcal{D}^{i-1} \cup \min(\mathcal{D} \backslash \mathcal{D}^{i-1})$. By the minimality condition, the members of $\mathcal{D}^0$ are disjoint Q-factors and hence there exists a homeomorphism

$$h_0 : \bigcup \{f^{-1}(D) : D \in \mathcal{D}^0\} \times Q \longrightarrow \bigcup \{D : D \in \mathcal{D}^0\} \times Q$$

such that for each $D \in \mathcal{D}^0$, $h_0(f^{-1}(D) \times Q) = D \times Q$. Let $j \geqslant 0$ and assume there exists a homeomorphism

$$h_j : \bigcup \{f^{-1}(D) : D \in \mathcal{D}^j\} \times Q \longrightarrow \{D : D \in \mathcal{D}^j\} \times Q$$

such that for each $D \in \mathcal{D}^j$, $h_j(f^{-1}(D) \times Q) = D \times Q$. For each $D \in \mathcal{D}^{j+1} \backslash \mathcal{D}^j$, let $\dot{D} = \bigcup \{D_1 \in \mathcal{D} : D_1$ is a proper subset of $D\}$. Then $\dot{D}$ is either empty, or as a finite union of Z-sets, is a Z-set in D and $\dot{D}$ is a union of members of $\mathcal{D}^j$. Since $f^{-1}(\mathcal{D})$ is a Q-factor decomposition of X, we also have that $f^{-1}(\dot{D})$ is a Z-set in $f^{-1}(D)$ and by the inductive hypothesis $h_j(f^{-1}(\dot{D}) \times Q) = \dot{D} \times Q$. Thus, by the Homeomorphism Extension Theorem, h_j can be extended to a homeomorphism of $f^{-1}(D) \times Q$ onto $D \times Q$ and hence we have constructed the required homeomorphism h_{j+1}. Thus, by finite induction we have constructed the required homeomorphism.

7. **Q-factor decompositions of A_n and the proof that $f_n : A_{n+1} \longrightarrow A_n$ stabilizes to a near-homeomorphism.** In this section we set up the major machinery for the rest of the paper and prove that f_n stabilizes to a near-homeomorphism.

Recall that $A_n^1 = \{[0, u] \cup [v, 1] \in A_n : 1/n \leqslant u \leqslant v \leqslant 1 - 1/n\}$ and let $\Delta_n^1 = \Delta(A_n^1) = \{(u, v) : [0, u] \cup [v, 1] \in A_n^1\}$. Then A_n^1 can be identified as the quotient space of Δ_n^1 where the diagonal of Δ_n^1 is shrunk to a point. All of the points of the diagonal correspond to the single point I in A_n^1. The corresponding equivalence relation on Δ_n^1 is induced by the evaluation map $\delta : \Delta_n^1 \longrightarrow A_n^1$ defined by $\delta(u, v) = [0, u] \cup [v, 1]$. Let S be a Q-factor decomposition of Δ_n^1 as in Figure 2 where the stair-step element containing the diagonal is denoted by s_0 and where the other elements s of S are of the form $s = s^1 \times s^2$ where s^1 and s^2 are closed subintervals of length less than $1/2n$ (possibly degenerate) of I. The $1/2n$ condition is needed so that the soon to be defined collection $\mathcal{D}$ covers A_n. Let us take all of the nondegenerate intervals s^i to be of the same length so that all of the principal members of $S \backslash \{s_0\}$ are square. We specifically include in S all of the edges and vertices of each of the square elements. Thus, the collection $S' = S \backslash \{s_0\}$ will be closed under finite intersections and it is just those elements s of S' that touch s_0 whose intersection with s_0, $s \cap s_0$, is the union of (two) elements of S. For a given $\epsilon > 0$ we can construct S so that the induced Q-factor decomposition $\delta(S)$ of A_n^1 has mesh less than ϵ.

We will use the Q-factor decomposition S to induce a Q-factor decomposition $\mathcal{D}$ on A_n. A *gap* in an element A of A_n is a component of $I \backslash A$. If u, v are the endpoints of a gap, then $(u, v) \in \Delta_n^1$ and if the gap is sufficiently small, then $(u, v) \in s_0$. Such gaps will be called *small* and otherwise they will be called *large*. The Q-factors D in $\mathcal{D}$ will be determined by the larger gaps of the members A of D. All A in a given D will have the same number of larger gaps and these gaps will all be in approximately the same place in I. Specifically, the members $s \in S'$

will determine the larger gaps in the elements A of a given D.

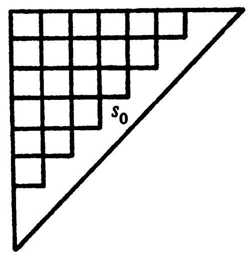

FIGURE 2

Call $s_1, \ldots, s_k \in S'$, $k \geqslant 0$, an *admissible sequence from S'* if for each i, $\sup s_i^2 < \inf s_{i+1}^1$, and for such an admissible sequence let

$$D(s_1, \ldots, s_k) = \{A \in A_n : \text{for each } i = 1, \ldots, k, \text{ there exists a pair of adjacent components } [a_i, b_i], [c_i, d_i] \text{ of } A \text{ such that } (b_i, c_i) \in s_i \text{ and for all other pairs of adjacent components } [a, b], [c, d] \text{ of } A, (b, c) \in s_0\}.$$

Thus, $s_1, \ldots, s_k$ determine the larger gaps of A and we allow as many smaller gaps as permitted for elements of A_n. $D(s_1, \ldots, s_k)$ may be empty, if for example, $s_1^1 = [0, a]$ where $a < 1/n$, recalling that for each element A of A_n; the components of A that contain 0 and 1 must be at least $1/n$ long and the other components must be at least $2/n$ long. For $k = 0$, denote the set $D(\ )$ by $D(s_0)$ and let $\mathcal{D}$ be the collection of all such $D(s_1, \ldots, s_k)$. In showing that $\mathcal{D}$ is a Q-factor decomposition of A_n, we find it convenient to analyze, for each $D \in \mathcal{D}$, the precise structure of those parts of the elements of D between pairs of adjacent larger gaps determined, for example by s_i and s_{i+1} of S'. In applications, J_1, J_2 will correspond to s_i^2, s_{i+1}^1 respectively. If $B_n = F_n(2^I)$, then an interval $[u, v] \in B_n$ if $v - u \geqslant 2/n$, or $[u, v]$ contains either 0 or 1 and $v - u \geqslant 1/n$. Let J_1, J_2 be subintervals of I, possibly degenerate, where $\sup J_1 < \inf J_2$ and let

$$E(J_1, J_2) = \{[u, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, v] : k \geqslant 0, u \in J_1, v \in J_2, \text{ each } (a_i, b_i) \in s_0, \text{ and each interval } [u, a_1], [b_1, a_2], \cdots, [b_k, v] \text{ is an element of } B_n\}.$$

Our next goal is to show that each $E(J_1, J_2)$ is a Q-factor. In doing this it is convenient to define, for $k \geqslant 0$, the set of all elements of $E(J_1, J_2)$ with exactly k gaps. We extend our definition of a gap to allow degenerate gaps, i.e., where $a_i = b_i$. For $k \geqslant 0$, let

$$E_k(J_1, J_2) = \{[u, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, v] : u \in J_1, v \in J_2, \text{ each } (a_i, b_i) \in s_0, \text{ and each interval is an element of } B_n\}.$$

Note, because of the specified minimum lengths of the intervals, that $E_k(J_1, J_2) = \emptyset$ for sufficiently large k. Thus, $E_{k-1}(J_1, J_2)$ need not be a subset of $E_k(J_1, J_2)$.

LEMMA 7.1. *The set*

$$\Delta_k = \{u, a_1, b_1, \ldots, a_k, b_k, v) \in E^{2k+2} : [u, a_1] \cup \cdots \cup [b_k, v] \in E_k(J_1, J_2)\}$$

is (a) *either empty or a Q-factor, and* (b) *if Δ_k is nondegenerate, then*

$$\Delta_k' = \{(u, a_1, b_1, \ldots, a_k, b_k, v) \in \Delta_k : \textit{for some } i,\ a_i = b_i\}$$

is a Q-factor and is a Z-set in Δ_k.

PROOF. (a) As mentioned above, for sufficiently large k, $\Delta_k = \emptyset$. Otherwise, Δ_k is a subset of a Euclidean space that fails to be convex because each $(a_i, b_i) \in s_0$ and s_0 is not convex. It is well known [16] that convex linear cells in E^m are characterized as being compact solutions to a finite set of linear inequalities. Since s_0 is a finite union of convex linear cells and since the side conditions on the coordinates of Δ_k are linear conditions (i.e., $a_i \leqslant b_i$ and $b_i + 2/n \leqslant a_{i+1}$), then Δ_k is a finite union of convex linear cells and hence is a polyhedron. In addition, Δ_k is contractible (in itself). To see this, first contract Δ_k to $C_k = \{(u, a_1, b_1, \ldots, a_k, b_k, v) \in \Delta_k$: for each i, $a_i = b_i\}$ as follows. Define a homotopy $h_t : s_0 \longrightarrow s_0$ by $h_0 = \mathrm{id}$; for each $(a, b) \in s_0$, let $h_1(a, b) = ((a + b)/2, (a + b)/2)$; and for $t \in (0, 1)$, let $h_t = (1 - t)h_0 + th_1$. This homotopy corresponds to sliding each $(a, b) \in s_0$ (see Figure 2) to the diagonal of s_0 along the line through (a, b) with slope -1. Then $H_t : \Delta_k \longrightarrow \Delta_k$ is defined by simultaneously applying h_t to each pair a_i, b_i. Now C_k is a convex cell and thus contractible and hence Δ_k is contractible. Thus Δ_k is a contractible polyhedron and, by [12, Corollary 5.6] is a Q-factor.

(b) The same argument as above shows that Δ_k' is a contractible polyhedron and is therefore a Q-factor. By the characterization of Z-sets, it is sufficient to construct, for each $\epsilon > 0$, a map $f : \Delta_k \longrightarrow \Delta_k$ such that $f(\Delta_k) \cap \Delta_k' = \emptyset$ and $d(f, \mathrm{id}) < \epsilon$. If $0 \notin J_1$ and $1 \notin J_2$ and Δ_k is nondegenerate, then $\sup J_2 - \inf J_1 > (k + 1)(2/n)$ and this provides room to simultaneously break apart the degenerate gaps, i.e., where $a_i = b_i$, in the members A of $E_k(J_1, J_2)$. This corresponds to a map of Δ_k that pushes Δ_k off Δ_k' and hence Δ_k' is a Z-set in Δ_k. If $0 \in J_1$ or $1 \in J_2$ the situation is similar. In fact, Δ_k is a topological cell and Δ_k' is a closed subset of its boundary and hence in a Z-set in Δ_k.

LEMMA 7.2. *Each $E(J_1, J_2)$ is either the empty set or a Q-factor.*

PROOF. If $\sup J_2 - \inf J_1$ is too small, then $E(J_1, J_2) = \emptyset$. In any case, $E(J_1, J_2) = \bigcup_{i \geqslant 0} E_i(J_1, J_2) \neq \emptyset$. In particular, $E_0 = E_0(J_1, J_2) = \{[u, v] : u \in J_1, v \in J_2\} \neq \emptyset$ and is a Q-factor since $\delta : \Delta_0 \longrightarrow E_0$ is a homeomorphism and Δ_0 is topologically either a 0, 1 or 2-cell depending upon whether both, one or neither of J_1, J_2 is degenerate. Let $k > 0$ and assume that $\bigcup_{i=0}^{k-1} E_i$ is a Q-factor. Then, as in the proof of 5.1, $\bigcup_{i=0}^{k} E_i$ is naturally homeomorphic to $\Delta_k \cup_{\delta'} \bigcup_{i=0}^{k-1} E_i$ where δ' is the evaluation map restricted to Δ'_k. Thus, by the Attaching Theorem, $\bigcup_{i=0}^{k} E_i$ is a Q-factor and hence by finite induction $E(J_1, J_2)$ is a Q-factor.

Let $J_1, \ldots, J_k$, for $k = 2, 4, \ldots$, be subintervals of I where $M_i = \sup J_i < \inf J_{i+1} = m_{i+1}$. In the following we will have occasions to use, for $D \in \mathcal{D}$, a function

$$h : D \longrightarrow E(J_1, J_2) \times \cdots \times E(J_{k-1}, J_k)$$

defined by $h(A) = (A \cap [m_1, M_2], \ldots, A \cap [m_{k-1}, M_k])$. This function will be called the *canonical map.*

PROPOSITION 7.3. *The collection $\mathcal{D}$ is a Q-factor decomposition of A_n and furthermore, for any $\epsilon > 0$, $\mathcal{D}$ can be constructed to have mesh less than ϵ.*

PROOF. If S is constructed so that mesh $\delta(S) < \epsilon$, then the mesh condition will be automatically satisfied. To show that $\mathcal{D}$ covers A_n, let $A = [0, u_1] \cup [v_1, u_2] \cup \cdots \cup [u_m, 1] \in A_n$ and for each i, let s_i be the smallest element of S containing (u_i, v_i). From the sequence $s_1, \ldots, s_m$ delete any s_i that is equal to s_0 and renumber the sequence $s_1, \ldots, s_k$ $(k \leqslant m)$ maintaining the inherited order. Then $A \in D(s_1, \ldots, s_k)$ and $s_1, \ldots, s_k$ is an admissible sequence since in the construction of S, the elements of S' have sides of length less than $1/2n$ and hence $\mathcal{D}$ covers A_n. We will now show that each member of $\mathcal{D}$ is a Q-factor. If $D(s_1, \ldots, s_k) \in \mathcal{D}$, then the canonical map

$$h : D(s_1, \ldots, s_k) \longrightarrow E(s_0^2, s_1^1) \times \cdots \times E(s_k^2, s_{k+1}^1),$$

where $s_0^2 = \{0\}$ and $s_{k+1}^1 = \{1\}$, is a homeomorphism, and hence, $D(s_1, \ldots, s_k)$ is a Q-factor since it is homeomorphic to a product of Q-factors.

We will now observe that $\mathcal{D}$ inherits the intersection property from S. Let $D_1 = D(s_1, \ldots, s_k)$ and $D_2 = D(t_1, \ldots, t_r)$ be members of $\mathcal{D}$ with a nonempty intersection. Then each s_i intersects either some t_j or s_0 and each t_j intersects either some s_i or s_0. Then $D_1 \cap D_2$ is the union of all $D(u_1, \ldots, u_k)$ where $u_1, \ldots, u_k$ is an admissible sequence from S' such that each u_α is contained in the intersection of either (a) some s_i and some t_j, or (b) some s_i and s_0, or (c) some t_j and s_0.

We also observe that $\mathcal{D}$ inherits the Z-set property from S. If $D_1 = D(s_1, \ldots, s_k) \subset D_2 = D(t_1, \ldots, t_r)$ then each s_i is either a subset of some t_j or is a subset of s_0. Furthermore, if $D_1 \neq D_2$, then some s_i is either a proper subset of some t_j or of s_0. In either case, since S has the Z-set property, we can see that maps on the appropriate elements of S within a given $\epsilon > 0$ of the identity, of the type discussed in the proof of 7.1 (b), induces a corresponding map on D_2 and hence D_1 is a Z-set in D_2.

We now restate and prove

PROPOSITION 3.4. *Each $f_n: A_{n+1} \longrightarrow A_n$ stabilizes to a near-homeomorphism.*

PROOF. Let $\epsilon > 0$ and let S be a Q-factor decomposition of Δ_n^1 as described in the first part of this section such that the induced Q-factor decomposition $\mathcal{D}$ on A_n has mesh less than ϵ. Let u, v be the endpoints of a gap in an element A of A_{n+1}, and let $\alpha = \min\{(v-u)/2, 1/n(n+1)\}$. Then the endpoints of the corresponding gap in $f_n(A)$ are $u+\alpha, v-\alpha$. This correspondence $(u, v) \longrightarrow (u+\alpha, v-\alpha)$ defines a map $g_1: \Delta_{n+1}^1 \longrightarrow \Delta_n^1 \cup \Delta' \cup \Delta''$ where $\Delta' = \{(u, u): 1/n+1 \leqslant u \leqslant 1/n\}$ and $\Delta'' = \{(u, u): 1-1/n \leqslant u \leqslant 1-1/n+1\}$. Let $g_2: \Delta_n^1 \cup \Delta' \cup \Delta'' \longrightarrow \Delta_n^1$ be the map that is the identity on Δ_n^1 and takes Δ' to $(1/n, 1/n)$ and Δ'' to $(1-1/n, 1-1/n)$. Then $g = g_2 \circ g_1$ is a map from Δ_{n+1}^1 onto Δ_n^1. Then $g^{-1}(S)$ is a Q-factor decomposition of Δ_{n+1}^1 of the same type as S. In fact, the square elements of $g^{-1}(S)$ and the stair-step boundary of $g^{-1}(s_0)$ are just translates away from the diagonal of the corresponding elements of S. Then $f_n^{-1}(\mathcal{D})$ is precisely the collection induced by $g^{-1}(S)$ which by 7.3 is a Q-factor decomposition of A_{n+1} and hence, by 5.1, f_n stabilizes to a near-homeomorphism.

8. **The proof that $h_n: A_n \longrightarrow A_{n,t}$ stabilizes to a near homeomorphism.** In this section we may assume that $1/n$ is small relative to t and $1-t$. Recall that $A_{n,t} = F_n(H(0, t, 1))$ and is the set of all members of A_n that contain $J = [t-1/n, t+1/n]$. We will construct, for a given $\epsilon > 0$, a Q-factor decomposition $\mathcal{D}_t$ of $A_{n,t}$ with mesh less than ϵ. This will directly follow the construction of the corresponding decomposition of A_n. Let $A_{n,t}^1$ be those members of A_n^1 that contain J, i.e., $A_{n,t}^1 = \{[0, u] \cup [v, 1]: 1/n \leqslant u \leqslant v \leqslant t-1/n$, or $t-1/n \leqslant u = v \leqslant t+1/n$, or $t+1/n \leqslant u \leqslant v \leqslant 1-1/n\}$, and let $\Delta_{n,t}^1 = \Delta(A_{n,t}^1)$. Let S_t be a Q-factor decomposition of $\Delta_{n,t}$, as in Figure 3, obtained by intersecting $\Delta_{n,t}$ with each element of a Q-factor decomposition S of Δ_n as constructed in §7. Let $\mathcal{D}$ be the Q-factor decomposition of A_n induced by S and let $\mathcal{D}_t = \{D \cap A_{n,t}: D \in \mathcal{D}\}$. Then $\mathcal{D}_t$ can be thought of as being induced by S_t in the same way thta $\mathcal{D}$ was induced by S. Again, denote the large element of S_t containing the diagonal by s_0.

PROPOSITION 8.1. *A collection $\mathcal{D}_t$ is a Q-factor decomposition of $A_{n,t}$ and furthermore, for any $\epsilon > 0$, $\mathcal{D}_t$ can be constructed to have mesh less than ϵ.*

PROOF. This proof is the same as the proof of 7.3 except that we modify the canonical map as follows. For $D(s_1, \ldots, s_k) \in \mathcal{D}_t$, let m be the integer such that J is contained between the intervals s_m^2 and s_{m+1}^1. Then the canonical map $h\colon D(s_1, \ldots, s_k) \longrightarrow E(s_0^2, s_1^1) \times \cdots \times E(s_{m-1}^2, s_m^1) \times E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1) \times E(s_{m+1}^2, s_{m+2}^1) \times \cdots \times E(s_k^2, s_{k+1}^1)$ is a homeomorphism.

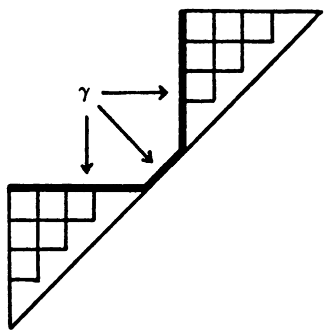

FIGURE 3

Our main goal now is to prove that $h_n^{-1}(\mathcal{D}_t)$ is a Q-factor decomposition of A_n. We first will review the definition of h_n. The retraction $h_t\colon H(0, 1) \longrightarrow H(0, t, 1)$ was defined by $h_t(A) = A \cup [u, u + \alpha] \cup [v - \alpha, v]$ where u is the maximal point of A less than or equal to t, v is the minimal point of A greater than or equal to t, and α is the minimum of $t - u$ and $v - t$. Then $h_n = F_n \circ h_t \circ s_n$ where s_n is the discontinuous section of F_n described in §3. Furthermore, if $A \in A_n$, then J is not a subset of A iff A has exactly one nondegenerate gap (u, v) where $u < t + 1/n$ and $v > t - 1/n$. It directly follows from the above that $h_n\colon A_n \longrightarrow A_{n,t}$ can be described as follows. For $A \in A_n$, if $J \subset A$, then $h_n(A) = A$ and if J is not contained in A, where (u, v) is the nondegenerate gap of A where $u < t + 1/n$ and $v > t - 1/n$, then $h_n(A) = A \cup [u, u + \alpha] \cup [v - \alpha, v]$ where α is the minimum of $(t + 1/n) - u$ and $v - (t - 1/n)$. This α also has the property that it is the minimum d such that $J \subset A \cup [u, u + d] \cup [v - d, v]$. Although this retraction h_n was defined on A_n, the definition would be equally valid for elements of 2^I having nonvoid intersections with both $[0, t - 1/n]$ and $[t + 1/n, 1]$. Henceforth, we will consider h_n to be extended to these elements of 2^I. We now analyze the precise structure of those components of the elements of $A_{n,t}$ that are affected by h_n^{-1}.

Let J_1, J_2 be closed subintervals of I such that $\sup J_1 \leqslant t - 1/n$ and $t + 1/n \leqslant \inf J_2$. We emphasize that in writing $[u, v]$ we mean $u \leqslant v$. Let $s \subset \Delta_{n,t}^1$ and let $C(J_1, s, J_2) = \{[u_1, u_2] \cup [u_3, u_4] \in 2^I\colon u_1 \in J_1, u_4 \in J_2, (u_2, u_3) \in s$, and each interval is an element of $B_{n,t}\}$ where $B_{n,t} = F_n(H(t))$. This latter condition can equivalently be stated as each interval is a component of some element of $A_{n,t}$. We now introduce small gaps as allowed by our side conditions. Let

$$E(J_1, s, J_2) = \{[u_1, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, u_2] \cup [u_3, c_1] \cup [d_1, c_2] \cup \cdots \cup [d_m, u_4] : k \geq 0, m \geq 0, (a_i, b_i) \in s_0, (c_i, d_i) \in s_0, [u_1, u_2] \cup [u_3, u_4] \in C(J_1, s, J_2), \text{ and each interval is an element of } B_{n,t}\}.$$

Since h_n is a retraction, our attention is on those elements A of $A_{n,t}$ such that $h_n^{-1}(A)$ contains elements other than A. Such elements A have a gap (u, v) such that the corresponding 2-tuple (u, v) is in

$$\gamma = \{(u, v) \in \Delta^1_{n,t} : u = t + 1/n, \text{ or } v = t - 1/n, \text{ or } t - 1/n \leq u = v \leq t + 1/n\}.$$

For $s \in S_t$, let $s' = s \cap \gamma$ (see Figure 3) and let

$$E_{0,0}(J_1, s', J_2) = \{[u_1, u_2 - d] \cup [u_3 + d, u_4] : d \in I, [u_1, u_2] \cup [u_3, u_4] \in C(J_1, s', J_2), \text{ and each interval is an element of } B_n\}.$$

We emphasize that J_1 and J_2 are on either side of J and that we will only be concerned here with those $s \in S_t$ such that $s' \neq \varnothing$. It follows from the extended definition of h_n that $h_n^{-1}C(J_1, s', J_2) = E_{0,0}(J_1, s', J_2)$ and that $h_n^{-1}C(J_1, s, J_2) = C(J_1, s, J_2) \cup E_{0,0}(J_1, s', J_2)$. We now specify the elements of $h_n^{-1}E(J_1, s, J_2)$ with different specific numbers of small gaps on either side of J. For $s \in S_t$ where $s' \neq \varnothing$, $k \geq 0$, and $m \geq 0$, let

$$E_{k,m}(J_1, s', J_2) = \{[u_1, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, u_2 - d] \cup [u_3 + d, c_1] \cup [d_1, c_2] \cup \cdots \cup [d_m, v_4] : d \in I, [u_1, u_2] \cup [u_3, u_4] \in C(J_1, s', J_2), \text{ each } (a_i, b_i) \text{ and } (c_i, d_i) \text{ belong to } s_0, \text{ and each interval is an element of } B_n\}.$$

The proof of Lemma 7.1 is also a proof for

LEMMA 8.2. (a) *Each set $\Delta_{km} = \Delta(E_{km}(J_1, s', J_2))$ is either empty or a Q-factor, and*

(b) *if Δ_{km} is nondegenerate, then*

$$\Delta'_{km} = \{(u_1, a_1, b_1, \ldots, a_k, b_k, u_2 - d, u_3 + d, c_1, d_1, \ldots, c_m, d_m, v_4) \in \Delta_{km} : d = 0 \text{ or for some } i, a_i = b_i \text{ or for some } j, c_j = d_j\}$$

is a Q-factor which is a Z-set in Δ_{km}.

LEMMA 8.3. *Each set $h_n^{-1}E(J_1, s, J_2)$ is either empty or a Q-factor.*

PROOF. If $q \geq n/2$ and $r \geq n/2$, then

$$h_n^{-1}E(J_1, s, J_2) = \bigcup\{E_{km}(J_1, s', J_2): k \leq q,\ m \leq r\} \cup E(J_1, s, J_2).$$

We first show that $E(J_1, s, J_2)$ is a Q-factor. If $s = s^1 \times s^2$, then the canonical map from $E(J_1, s, J_2)$ onto $E(J_1, s^1) \times E(s^2, J_2)$ is a homeomorphism and hence, by 7.2, $E(J_1, s, J_2)$ is homeomorphic to the product of Q-factors and hence is a Q-factor. Also, the canonical map from $E(J_1, s_0, J_2)$ onto $E(J_1, t + 1/n) \times E(t - 1/n, J_2)$ is a homeomorphism and hence $E(J_1, s_0, J_2)$ is a Q-factor.

We can assume that each Δ_{km} is nondegenerate, and then by 8.2 each Δ'_{km} is a Q-factor and a Z-set in the Q-factor Δ_{km}. Also, E_{km} is naturally identified with the quotient space of Δ_{km}, where the equivalence relation is induced by the evaluation map δ on Δ'_{km}. Thus, to attach a given E_{km} to the inductive step we formally take the attachment of Δ_{km} to the inductive step by the map δ on Δ'_{km} where $\delta(\Delta'_{km})$ must be contained in the inductive step. This containment condition is satisfied if we attach to $E(J_1, s, J_2)$ the sets E_{km} in the following prescribed order: $E_{00}, E_{01}, \ldots, E_{0r}; E_{10}, E_{11}, \ldots, E_{1r}; \ldots; E_{q0}, E_{q1}, \ldots, E_{qr}$. Thus, by the inductive use the Attaching Theorem, this lemma is proved.

LEMMA 8.4. *The collection $h_n^{-1}(\mathcal{D}_t)$ is a Q-factor decomposition of A_n.*

PROOF. For $D(s_1, \ldots, s_k) \in \mathcal{D}_t$, let m be the positive integer such that J is contained between s_m^2 and s_{m+1}^1. If $k = 0$, which means we are considering $D(s_0)$, then $h_n^{-1}D(s_0)$ is precisely the Q-factor $h_n^{-1}E(0, s_0, 1)$ of the previous lemma. If $k \geq 1$, then the canonical map from $D(s_1, \ldots, s_k)$ to

$$(*)\quad \begin{aligned} &E(0, s_1^1) \times \cdots \times E(s_{m-1}^2, s_m^1) \times E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1) \\ &\qquad \times E(s_{m+1}^2, s_{m+2}^1) \times \cdots \times E(s_k^2, 1) \end{aligned}$$

is a homeomorphism. If $\sup s_m^2 = t - 1/n$ and $\inf s_{m+1}^1 > t + 1/n$, see Figure 4(a)

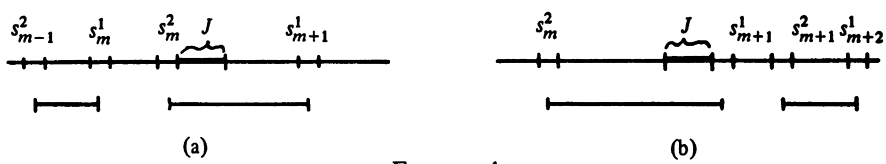

FIGURE 4

then $h_n^{-1}D(s_1, \ldots, s_k)$ is canonically homeomorphic to $(*)$ where $E(s_{m-1}^2, s_m^1) \times E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1)$ is replaced by $h_n^{-1}E(s_{m-1}^2, s_m, s_{m+1}^1)$ and hence, by 7.2 and 8.3, is a product of Q-factors and hence is a Q-factor. If $\sup s_m^2 < t - 1/n$ and $\inf s_{m+1}^1 = t - 1/n$, see Figure 4(b), then $h_n^{-1}D(s_1, \ldots, s_k)$ is canonically homeomorphic to $(*)$ where $E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1) \times E(s_{m+1}^2, s_{m+2}^1)$ is replaced by $h_n^{-1}E(s_m^2, s_{m+1}, s_{m+2}^1)$ and hence is a Q-factor. If $\sup s_m^2 = t - 1/n$ and $\inf s_{m+1}^1 = t + 1/n$, then $h_n^{-1}D(s_1, \ldots, s_k)$ is the union of the two Q-factors described in the above two cases and hence is a Q-factor since their intersection is a Q-factor that is a Z-set in each of them. Their intersection is precisely the Q-factor $(*)$ where $E(s_m^2, t + 1/n) \times E(t - 1/n, S_{m+1}^1)$ represents the single element J of 2^I. If $\sup s_m^2 < t - 1/n$ and $\inf s_{m+1}^1 > t + 1/n$, then $h_n^{-1} D(s_1, \ldots, s_k) = D(s_1, \ldots, s_k)$. This completes the verification that the elements of $h_n^{-1}(D_t)$ are Q-factors.

The verification that $h_n^{-1}(\mathcal{D}_t)$ has the intersection and Z-set properties is virtually the same as the corresponding argument for $\mathcal{D}$ in the proof of 7.3.

We now restate 3.11, the proof of which is a direct consequence of 5.1, 8.1 and 8.4.

PROPOSITION 3.11. *Each $h_n: A_n \longrightarrow A_{n,t}$ stabilizes to a near-homeomorphism.*

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