THE HYPERSPACE OF THE CLOSED UNIT INTERVAL
IS A HILBERT CUBE \(^{(1)}\)

BY
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ABSTRACT. Let \(X\) be a compact metric space and let \(2^X\) be the space of all nonvoid closed subsets of \(X\) topologized with the Hausdorff metric. For the closed unit interval \(I\) the authors prove that \(2^I\) is homeomorphic to the Hilbert cube \(I^\infty\), settling a conjecture of Wojdyslawski that was posed in 1938. The proof utilizes inverse limits and near-homeomorphisms, and uses (and develops) several techniques and theorems in infinite-dimensional topology.

1. Introduction. Let \(X\) be a compact metric space and let \(2^X\) be the space of all nonvoid closed subsets of \(X\) topologized with the Hausdorff metric. In the Bull. Amer. Math. Soc. [9], we announced that \(2^I\) is homeomorphic to \((\sim)\) the Hilbert cube \(Q\) and gave an outline of our proof. §3 of this paper is essentially an expanded version of [9]; The bulk of this paper, §§4–8 consists of the proofs of the alleged but unproved claims in [9]. In §5 we prove that a certain finite dimensional subspace \(A_n\) of \(2^I\) is a \(Q\)-factor, that is, \(A_n \times Q \approx Q\), and the rest of the paper is devoted to proving that two specific maps, \(f = f_n \times id\) and \(r_n\) (defined in §3), are near-homeomorphisms, i.e., uniform limits of homeomorphisms.

Our original and unpublished proof that \(f\) and \(r_n\) are near-homeomorphisms was based on a theory of reduced mapping cylinders. See [13] for a corresponding account of mapping cylinders. Since then we have obtained shorter and easier proofs that these maps are near-homeomorphisms using \(Q\)-factor decompositions, a notion introduced by D. W. Curtis in [3]. In this paper we present in §§6–8 these more recently obtained proofs.

Very recently T. A. Chapman announced a theorem that characterizes near-homeomorphisms between Hilbert cubes as being those continuous surjections with the property that the inverse image of each point has trivial shape. This result is a consequence of his paper [Cell-like mappings of Hilbert cube manifolds: Applications to simple-homotopy theory, Bull. Amer. Math. Soc. 79 (1973), 1286–

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1291, and relies on a great deal of apparatus including algebraic $K$-theory and computations of Whitehead groups. The use of this characterization of near-homeomorphisms of the Hilbert cube would shorten this paper. However, because of the extensive background needed for Chapman's proof, use of his result would significantly lessen the accessibility of the proof of the main result of this paper.

**Brief history of the problem.** Probably the first result in the direction of $2^I$ was by L. Vietoris when he proved in [10] that if $X$ is a Peano continuum, then so is $2^X$. In [11], Wazewski proved the converse. Mazurkiewicz [7] showed that if $X$ is compact and connected, then $2^X$ is a continuous image of the Cantor star and Wojiyslawski [14] showed that if $X$ is a Peano continuum, then $2^X$ is contractible and locally contractible and later [15] that $2^X$ is an absolute retract if and only if $X$ is a Peano continuum. In his earlier paper Wojdyslawski specifically asked if $2^I \approx Q$ and, more generally, he asked if $2^X \approx Q$ where $X$ is any nondegenerate Peano continuum. Professor Kuratowski has informed us that the conjecture that $2^I \approx Q$ was well known to the Polish topologists in the 1920's.

It was well known that $Q$ is contractible, locally contractible, an absolute retract, and that $2^I$ and $Q$ each contain homeomorphic copies of each other. These were some of the obvious reasons that Wojdyslawski's conjecture seemed reasonable. It was known that $Q$ is homogeneous [6], but it was not previously known that $2^I$ is. From the homogeneity of $Q$ it easily follows that each point of $Q$ is unstable and in [5] Neil Gray proved the corresponding property for $2^X$, where $X$ is any nondegenerate Peano space.

Other efforts in the direction of $2^X$ were focused on the study of the subspace of $2^X$ consisting of all nonvoid subsets of $X$ containing less than or equal to $n$ points ($n \geq 1$), denoted by $X(n)$ and called the $n$-fold symmetric product of $X$. See the introduction of [8] for an account of these results.

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2. Definitions and preliminaries. If $X$ is a compact metric space, then the Hausdorff metric $D$ on $2^X$ can be defined by

$$D(A, B) = \inf\{e > 0 : A \subset U(B, e) \text{ and } B \subset U(A, e)\}$$

where, for $C \subset X$, $U(C, e)$ is the open $e$-neighborhood of $C$ in $X$.

Let $I$ be the closed unit interval $[0, 1]$ and for $S \subset I$, let $H(S)$ denote the subspace of $2^I$ consisting of all closed subsets of $I$ that contain $S$. If $t_1, \ldots, t_n$ are points of $I$, denote $H(t_1, \ldots, t_n)$ by $H(t_1, \ldots, t_n)$. In related papers $H(0, 1)$ has often been denoted by $2^I_{01}$.

Let $Q$ denote the countable infinite product of copies of $I$ and define a
Hilbert cube as any space homeomorphic to $Q$. A space $X$ is a $Q$-factor if $X \times Q \approx Q$. This is equivalent to saying that there exists a space $Y$ such that $X \times Y \approx Q$ since if the later is true, then $Q \approx (X \times Y)^\omega \approx X \times (X \times Y)^\omega \approx X \times Q$.

A map is a continuous function and homeomorphisms are always onto. If $X$ and $Y$ are homeomorphic compact metric spaces, then a map $f : X \to Y$ is a near-homeomorphism if for each $\epsilon > 0$ there is a homeomorphism $h : X \to Y$ such that $d(h, f) < \epsilon$. It follows easily that the composition of two near-homeomorphisms is a near-homeomorphism and that the cartesian product of two near-homeomorphisms is a near-homeomorphism. We say that $f : X \to Y$ stabilizes to a near-homeomorphism if $f \times \text{id} : X \times Q \to Y \times Q$ is a near-homeomorphism.

An inverse sequence $(X_n, f_n)$ is a sequence of spaces $X_n$ and maps $f_n$ such that for each $n \geq 1$, $f_n : X_{n+1} \to X_n$. The inverse limit of $(X_n, f_n)$, denoted by $\lim (X_n, f_n)$, is the subspace of the product of the $X_n$ consisting of all points $(x_n) \in \prod_{n=1}^{\infty} X_n$ such that for each $n \geq 1$, $f_n(x_{n+1}) = x_n$.

We quote the following three results as they will be referred to in this paper.

**Theorem 2.1** (Morton Brown [2, Theorem 4, p. 482]). Let $S = \lim (X_n, f_n)$ where the $X_n$ are all homeomorphic to a compact metric space $X$ and each $f_n$ is a near-homeomorphism. Then $S$ is homeomorphic to $X$.

**Theorem 2.2** [9, Theorem 5.2, p. 405]. Let $S = \lim (X_n, f_n)$ and $T = \lim (Y_n, g_n)$ where all the spaces are compact and for each $n$, let $h_n : X_n \to Y_n$ be a map such that $g_n \circ h_{n+1} = h_n \circ f_n$. If for each $n$, both $f_n$ and $h_n$ are (stabilizes to) near-homeomorphisms, then the induced map $h : S \to T$ defined by $h(x_n) = (h_n(x_n))$ is a (stabilizes to a) near-homeomorphism.

The following lemma was first observed by Fort and Segal in [4]. It provides a useful method for recognizing certain inverse limits.

**Lemma 2.3** [4, Lemma 4, p. 132]. Let $X$ be a compact metric space, let $X_1, X_2, \ldots$ be closed subsets of $X$, and for each $n$ let $\varphi_n$ be a map of $X$ onto $X_n$ and let $f_n$ be a map of $X_{n+1}$ onto $X_n$ such that $\varphi_n = f_n \circ \varphi_{n+1}$ and $\varphi_1, \varphi_2, \ldots$ converges uniformly to the identity map on $X$. Then the function $\varphi$ on $X$ defined by $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots)$ is a homeomorphism of $X$ onto $\lim(X_n, f_n)$.

**3. The reduction of the proof.** This section is a more explicit and detailed version of §3 of [9]. The basic idea of the proof that $2^I \approx Q$ is to identify a nested sequence of Hilbert cubes, $Y_1 \subset Y_2 \subset \ldots$, contained in $H(0, 1)$, whose union is dense in $H(0, 1)$, and to define retractions $r_n : Y_{n+1} \to Y_n$ such that each $r_n$ is a near-homeomorphism and such that $H(0, 1)$ is homeomorphic to $\lim(Y_n, r_n)$. Then (Morton Brown's) Theorem 2.1 implies that $H(0, 1)$ is a Hilbert cube. Then Proposition 3.1, below, says that if $H(0, 1)$ is a Hilbert cube,
then so is \(2^I\). This inverse sequence \((Y_n, r_n)\) will be called the \textit{principal sequence}. We will reduce this program to considering certain finite-dimensional subspaces \(A_n\) and \(A_{n,t}\) of \(2^I\) and maps \(f_n : A_{n+1} \to A_n\) and \(h_n : A_n \to A_{n,t}\) and in this section we will do everything needed except for proving that each \(A_n\) is a \(Q\)-factor and that \(f_n\) and \(h_n\) stabilize to near-homeomorphisms.

**Proposition 3.1.** If \(H(0, 1)\) is a Hilbert cube, then so is \(2^I\).

**Proof.** In [8] it is shown that \(2^I\) is homeomorphic to \(CCH(0, 1)\), where \(CX\) denotes the cone over \(X\). (The formula \((A, s, t) \to ((1 - t)(1 - s)a + t : a \in A)\) defines a map from \(H(0, 1) \times I \times I\) to \(2^I\) producing the same identifications as the coning operations.) O. H. Keller proved in [6] that all infinite-dimensional, convex compacta of Hilbert space are Hilbert cubes, and since \(CQ\) has a geometric realization as such a subset of Hilbert space, then \(CQ\) and hence \(CQ\) is a Hilbert cube and the result follows.

Each \(Y_n\) is a Hilbert cube. Before defining the \(Y_n\) we will prove that \(H(0, 1)\) is a \(Q\)-factor. This result will be the main tool in proving that each \(Y_n\) is a Hilbert cube. We proceed as follows. For each \(n \geq 1\), let \(F_n : H(0, 1) \to H(0, 1)\) be the function assigning to an element \(A\) in \(H(0, 1)\) its closed \(1/n\)-neighborhood in \(I\), i.e., \(F_n(A) = \{s \in I : |s - a| \leq 1/n\text{ for some }a \in A\}\). These functions are continuous. Let \(A_n = F_n(H(0, 1))\) and let \(f_n : A_{n+1} \to A_n\) be defined by \(f_n = F_n(A_{n+1})\). Since \(1/n = 1/n + 1 + 1/n(n + 1)\), it follows that \(F_n = f_n \circ F_{n+1}\). Thus, \(f_n(A_{n+1}) = A_n\) and we call the inverse sequence \((A_n, f_n)\) the auxiliary sequence.

**Proposition 3.2.** The inverse limit of \((A_n, f_n)\) is homeomorphic to \(H(0, 1)\).

**Proof.** For each \(n\), \(F_n = f_n \circ F_{n+1} : H(0, 1) \to A_n\) and \(F_1, F_2, \ldots\) converges uniformly to the identity map on \(H(0, 1)\). Thus, by 2.3, the function \(F : H(0, 1) \to \lim (A_n, f_n)\) defined by \(F(A) = (F_1(A), F_2(A), \ldots)\) is a homeomorphism.

The proof of the next proposition is given in §5.

**Proposition 3.3.** Each \(A_n\) is a \(Q\)-factor.

A proof of the next proposition is given in §7.

**Proposition 3.4.** Each \(f_n : A_{n+1} \to A_n\) stabilizes to a near-homeomorphism.

**Theorem 3.5.** \(H(0, 1)\) is a \(Q\)-factor.

**Proof.** The proof is an immediate consequence of Theorem 2.1, the preceding three propositions, and the fact that \(\lim (A_n, f_n) \times Q\) is homeomorphic to \(\lim (A_n \times Q, f_n \times \text{id})\).
For each \( n \geq 1 \), let \( o(n) = \{0, 1, \frac{1}{n}, \frac{1}{n} + 1, \ldots \} \) and let \( Y_n = H(o(n)) \).

**Corollary 3.6.** Each \( Y_n \) is a Hilbert cube.

**Proof.** For a fixed \( n \geq 1 \), let \( J_m \) denote the \( m \)-th subinterval from the right determined by \( o(n) \), i.e., \( J_1 = [1/n, 1] \), \( J_2 = [\frac{1}{n} + 1, \frac{1}{n}] \), etc., and let \( H_m = \{ A \in 2^J_m : A \text{ contains the endpoints of } J_m \} \). Also let \( \alpha_m : J_m \to I \) be the order-preserving linear homeomorphism, let \( \beta_m : H_m \to H(0, 1) \) be the induced homeomorphism, and define \( \beta_m(A) = A \cap J_m \). Then \( \alpha : Y_n \to \Pi_{m=1}^\infty H(0, 1)_m \), where \( H(0, 1)_m = H(0, 1) \), defined by \( \alpha = (\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2, \ldots) \) is a homeomorphism. Since \( H(0, 1) \) is a Q-factor, it follows that \( Y_n \) is a Hilbert cube since it is known by [12, Theorem 6.2, p. 21] that the countable infinite product of nondegenerate Q-factors is a Hilbert cube.

**The principal inverse sequence.** We define the maps \( r_n : Y_{n+1} \to Y_n \) as follows. For \( A \in Y_{n+1} \), let \( u = \max\{x \in A : x < 1/n\} \) and \( v = \min\{x \in Y_n \} \), and let \( \alpha = \min\{d : A \cup [u, u + d] \cup [v - d, v] \in Y_n \} \). Then \( r_n(A) = A \cup [u, u + \alpha] \cup [v - \alpha, v] \). Note that \( \alpha = \min\{1/n - u, v - 1/n\} \). It is easy to see that each \( r_n \) is continuous and is a retraction. The inverse sequence \((Y_n, r_n)\) is called the principal sequence.

**Proposition 3.7.** The inverse limit of \((Y_n, r_n)\) is homeomorphic to \( H(0, 1) \).

**Proof.** For \( x \in I \) and \( B \subset I \), let \( d(x, B) = \inf\{|x - b| : b \in B\} \) and for \( U \subset I \) and \( n \geq 1 \), let \( \xi(U, n) = \max\{d(x, I \setminus U) : x \in o(n)\} \). Define \( R_n : H(0, 1) \to Y_n \) by letting \( R_n(A) \) be the union of \( A \) and

\[
\bigcup\{[u, u + \xi(U, n)] \cup [v - \xi(U, n), v] : U = (u, v) \text{ is a component of } I \setminus A\}.
\]

It easily follows that \( R_n = r_n \circ R_{n+1} \) by observing what happens, for \( A \in H(0, 1) \), on the component of \( I \setminus A \) that contains \( 1/n \), if it exists. Thus, we can define \( R : H(0, 1) \to \lim(Y_n, r_n) \) by \( R(A) = (R_1(A), R_2(A), \ldots) \) and this is a homeomorphism by 2.3 since \( R_1, R_2, \ldots \) converges uniformly to the identity map on \( H(0, 1) \).

The hardest part of our program is showing that each \( r_n \) is a near-homeomorphism. We start with the following reduction of \( r_n \). For each \( r \in (0, 1) \), let \( h_r : H(0, 1) \to H(0, r, 1) \) be the retraction analogous to \( r_n \), that is, \( h_r(A) = A \cup [u, u + r] \cup [v - r, v] \) where \( u \) is the maximal point of \( A \) less than or equal to \( r \).

(2) Our definition was motivated by the following. If \( s_n : Y_{n+1} \to Y_n \) is defined by \( s_n(A) = A \cup \{1/n\} \), then \( H(0, 1) \approx \lim(Y_n, s_n) \), but \( s_n \) patently cannot be a near-homeomorphism since if \( A \in Y_n \) where \( 1/n \) is an isolated point of \( A \), then \( s_n^{-1}(A) = \{A, A \setminus \{1/n\}\} \) is not connected and it is easy to prove that if \( r : Q \to Q \) is any near-homeomorphism, then the inverse images of points are connected. The authors are indebted to A. Verbeek for suggesting the \( s_n \) map which represented a step in the evolution of the argument.
t, v is the minimal point of A greater than or equal to t, and α is the minimum of t - u and v - t.

We say the map $f: X \to Y$ is topologically equivalent to $g: W \to Z$ if there exist homeomorphisms $\beta: X \to W$ and $\gamma: Y \to Z$ such that $\gamma \circ f = g \circ \beta$.

**Proposition 3.8.** Each map $r_n: Y_{n+1} \to Y_n$ is topologically equivalent to $h_t \times \text{id}: H(0, 1) \times Q \to H(0, t, 1) \times Q$, for some $t \in (0, 1)$.

**Proof.** Adopt all of the notation from the proof of 3.6, let $J_0 = J_1 \cup J_2 = [1/n + 1, 1]$, let $\alpha_0: J_0 \to I$ be the order-preserving homeomorphism and let $\beta_0$ be the intersection with $J_0$ map, and let $t = \alpha_0(1/n)$. Define $\gamma: Y_{n+1} \to H(0, 1)$ by $\gamma = \alpha_0^* \circ \beta_0$ and $\gamma_t: Y_n \to H(0, t, 1)$ by $\gamma_t = \alpha_0^* \circ \beta_0$, let $\psi: Y_{n+1} \to \prod_{m=3}^n H(0, 1)_m$ be defined by $(\alpha_0^* \circ \beta_m, \alpha_0^* \circ \beta_{m+1}, \ldots)$ and let $\varphi: \prod_{m=3}^n H(0, 1)_m \to Q$ be any homeomorphism. Then $(\gamma, \varphi \circ \psi): Y_{n+1} \to H(0, 1) \times Q$ and $(\gamma_t, \varphi \circ \psi): Y_n \to H(0, t, 1) \times Q$ are homeomorphisms and $(\gamma_t, \varphi \circ \psi) \circ r_n = (h_t \times \text{id}) \circ (\gamma, \varphi \circ \psi)$ and the proposition is proved.

The above proposition means that in order to prove that each $r_n: Y_{n+1} \to Y_n$ is a near-homeomorphism it is sufficient to prove that $h_t: H(0, 1) \to H(0, t, 1)$ stabilizes to a near-homeomorphism. We proceed as follows, again making use of the auxiliary system. For $n \geq 1$, let $A_{n,t} = F_n(H(0, t, 1))$ and let $f_{n,t}: A_{n+1,t} \to A_{n,t}$ be the restriction of $f_n$ to $A_{n+1,t}$.

**Proposition 3.9.** The inverse limit of $(A_{n,r}, f_{n,r})$ is homeomorphic to $H(0, t, 1)$.

**Proof.** As in the proof of 3.2, the map $F_t: H(0, t, 1) \to \lim(A_{n,t}, f_{n,t})$ defined by $F_t(A) = (F_1(A), F_2(A), \ldots)$ is a homeomorphism.

We now will represent $h_t$ as an inverse limit of maps. Let $F_{n,t}$ denote the restriction of $F_n$ to $H(0, t, 1)$. There is a natural map $h_n: A_n \to A_{n,t}$ such that $h_n \circ F_n = F_{n,t} \circ h_t$. We describe $h_n$ as follows. Let $s_n: A_n \to H(0, 1)$ be the natural (but discontinuous) section of $F_n$ (i.e., $F_n \circ s_n = \text{id}$) that cuts off an interval of length $1/n$ from each free end other than 0 or 1 of each component of $A$. That is, if $A \in A_n$ is of the form $A = [0, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, 1] (a_i < b_i)$, then $s_n(A) = [0, a_1 - 1/n] \cup [b_1 + 1/n, a_2 - 1/n] \cup \cdots \cup [b_k + 1/n, 1]$. Then $h_n: A_n \to A_{n,t}$ is defined by $h_n = F_{n,t} \circ h_t \circ s_n$. Thus we have

$$(*) \quad h_n \circ F_n = F_{n,t} \circ h_t.$$

In the following, let $A_\infty = \lim(A_{n,r}, f_{n,r})$ and $A_{\infty, t} = \lim(A_{n,r}, f_{n,r})$.

**Proposition 3.10.** The maps $(h_n)$ satisfy the condition $h_n \circ f_n = f_{n,t} \circ h_{n+1,t}$ for each $n \geq 1$, and thus induce a map $h_\infty: A_\infty \to A_{\infty, t}$ defined by $h_\infty(x_n) = (h_n(x_n))$. Moreover, $h_t: H(0, 1) \to H(0, t, 1)$ is topologically equivalent to $h_\infty$. 

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Proof. Since $F_n = f_n \circ F_{n+1}$ and $F_{n,t} = f_{n,t} \circ F_{n+1,t}$ we have from\[ (*) \]that $h_n \circ f_n \circ F_{n+1} = f_{n,t} \circ h_t \circ F_{n+1,t}$ and applying (*) for $n + 1$ we have $h_n \circ f_n \circ F_{n+1} = f_{n,t} \circ h_n+1 \circ F_{n+1}$. Since $F_{n+1}$ is onto it follows that $h_n \circ f_n = f_{n,t} \circ h_n+1$ and thus we have the induced map $h_\infty: A_\infty \to A_{\infty,t}$ defined by $h_\infty(x_n) = (h_n(x_n))$, for $(x_n) \in A_\infty$.

The homeomorphism $F: H(0, 1) \to A_\infty$ of the proof of 3.2 and defined by $F(A) = (F_1(A), F_2(A), \ldots)$ has the property that for each $n \geq 1$, $F_n = \pi_n \circ F$ where $\pi_n : A_\infty \to A_n$ is the projection map. We have the corresponding homeomorphism $F_t: H(0, t, 1) \to A_{\infty,t}$ and the corresponding property $F_{n,t} = \pi_{n,t} \circ F_t$. Since $h_n \circ F_n = F_{n,t} \circ h_t$, it follows that $h_n \circ \pi_n \circ F = \pi_{n,t} \circ F_t \circ h_t$, and since $h_\infty$ is induced by the $h_n$, we have that $h_n \circ \pi_n = \pi_{n,t} \circ h_\infty$. Thus, $\pi_{n,t} \circ h_\infty \circ F = \pi_{n,t} \circ F_t \circ h_t$, for all $n \geq 1$, and it immediately follows that $h_\infty \circ F = F_t \circ h_t$, which completes the proof.

A proof of the following proposition, which comprises the bulk of this paper, is found in §8.

**Proposition 3.11.** Each $h_n : A_n \to A_{n,t}$ stabilizes to a near-homeomorphism.

We finish this section with the main result.

**Theorem 3.12.** $2^I$ is a Hilbert cube.

**Proof.** If $f$ and $g$ are topologically equivalent maps and $f$ is a near-homeomorphism then so is $g$. This follows since the finite composition of near-homeomorphisms is a near-homeomorphism.

Since each $h_n$ stabilizes to a near-homeomorphism, it follows by 2.2 that so does $h_\infty$. Thus, since the maps $h_\infty$ and $h_t$ are topologically equivalent by 3.10 and the maps $h_t \times \text{id}$ and $r_n$ are topologically equivalent by 3.8, it follows that $r_n$
is a near-homeomorphism. Since $H(0, 1)$ is homeomorphic to $\lim(Y_n, r_n)$ by 3.7, and each $Y_n$ is a Hilbert cube by 3.6, it follows by 2.1 that $H(0, 1)$ is a Hilbert cube. Finally, 3.1 completes the proof.

4. The attaching theorem for $Q$-factors. In this section and in the rest of the paper, the notion of Z-set, introduced by R. D. Anderson in [1], is crucial. A closed subset $K$ of a topological space $X$ is a Z-set in $X$ if for every nonempty homotopically trivial ($n$-connected for all $n > 0$) open set $U$ in $X$, $U \setminus K$ is non-empty and homotopically trivial. A useful characterization of Z-sets in a special class of spaces, which includes polyhedra and the Hilbert cube, is proved in [3, Lemma 2.1] and is stated as follows. A closed subset $U$ of a metric ANR $X$ is a Z-set in $X$ if for each $\epsilon > 0$ there exists a map $f: X \rightarrow X \setminus K$ with $d(f, \text{id}) < \epsilon$.

For example, the Z-sets in an $n$-cell are precisely the closed subsets of the boundary. One of the fundamental theorems in infinite-dimensional topology and proved by Anderson in [1] is the

**Homeomorphism Extension Theorem.** If $K_1, K_2$ are Z-sets in Hilbert cubes $Q_1, Q_2$, respectively, and $f: K_1 \rightarrow K_2$ is a homeomorphism, then $f$ can be extended to a homeomorphism of $Q_1$ onto $Q_2$.

A corollary of this is the

**First Sum Theorem.** If each of $Q_1, Q_2$, and $Q_1 \cap Q_2$ is a Hilbert cube and $Q_1 \cap Q_2$ is a Z-set in each of $Q_1$ and $Q_2$, then $Q_1 \cup Q_2$ is a Hilbert cube.

If $X$ and $Y$ are disjoint compact metric spaces, $A$ a closed subset of $X$, and $f: A \rightarrow Y$ a map, then the adjunction space of $f$, denoted by $X \cup_f Y$, is $(X \cup Y)/\sim$, where $\sim$ is the equivalence relation on $X \cup Y$ generated by $a \sim f(a)$, for all $a \in A$. We say $X$ is attached to $Y$ by $f$. If $g: X \rightarrow Y$ is a map, then the mapping cylinder of $g$, denoted $M_g$, is the adjunction space $(X \times I) \cup_g Y$ where $g': X \times \{0\} \rightarrow Y$ is defined by $g'(x, 0) = g(x)$.

The following theorem is one of the basic theorems in the theory of $Q$-factors.

**Theorem 4.1** [13, Theorem 1, p. 114]. Let $X$ and $Y$ be $Q$-factors and let $g: X \rightarrow Y$ be a map of $X$ into $Y$, then the mapping cylinder of $g$ is also a $Q$-factor.

The following corollary is one of the basic tools of this paper.

**Corollary 4.2 (The Attaching Theorem).** Let $X$ and $Y$ be disjoint $Q$-factors and let $A$ be a closed subset of $X$ that is a $Q$-factor and a Z-set in $X$. If $f: A \rightarrow Y$ is any map, then the adjunction space $X \cup_f Y$ is a $Q$-factor.

**Proof.** Define $g: A \times Q \rightarrow A \times Q \times \{0\}$ by $g(x, y) = (x, y, 0)$. Since
A is a Z-set in X, it follows that A × Q is a Z-set in X × Q, and since they are both Hilbert cubes, we can extend g, by the Homeomorphism Extension Theorem, to a homeomorphism g_1 : X × Q —> A × Q × I. If f_1 : A × Q × {0} —> Y × Q is defined by f_1(x, y, 0) = (f(x, y)), then f_1 ∘ g = f × id and hence (X × Q) ⊔ f x id (Y × Q) is homeomorphic to (A × Q × I) ⊔ f_1 (Y × Q) which is the mapping cylinder of f × id : A × Q —> Y × Q and hence is a Q-factor by 4.1.

Since (X ⊔ f_1 Y) × Q ≅ (X × Q) ⊔ f x id (Y × Q), then (X ⊔ f_1 Y) × Q is a Q-factor and since Q × Q ≅ Q, then (X ⊔ Y) × Q ≅ Q and hence X ⊔ f_1 Y is a Q-factor.

5. Each A_n is a Q-factor. We now set up some notation for analysis of A_n which will be useful throughout the paper. Since A_n is the image of H(0, 1) under F_n, it is {[0, u_1] ∪ [v_1, u_2] ∪ ... ∪ [v_m, 1]} ∈ 2^I : 0 ≤ m < n/2, u_k < v_k for 1 ≤ k ≤ m, 1/n ≤ u_1, v_m ≤ 1 - 1/n, and v_k + 2/n ≤ u_{k+1} for 1 ≤ k ≤ m - 1}, where if m = 0 we mean [0, 1]. We let A_n^0 = {I} and for 1 ≤ m < n/2, A_n^m = {A ∈ A_n : A has less than or equal to (m + 1) components} and B_n^m = {A ∈ A_n : A has exactly m + 1 components}. Each B_n^m corresponds naturally to a subset Δ(B_n^m) of E^{2m} under the function φ which sends A = [0, u_1] ∪ [v_1, u_2] ∪ ... ∪ [v_m, 1] to the point φ(A) = (u_1, u_1, ..., u_m, v_m). It is easily checked that φ : B_n^m —> Δ(B_n^m) is a homeomorphism. The set Δ(B_n^m) is a convex set in E^{2m} and its closure, Δ_m = {(u_1, v_1, ..., u_m, v_m) ∈ E^{2m} : u_k < v_k for 1 ≤ k ≤ m, 1/n ≤ u_1, v_m ≤ 1 - 1/n, and v_k + 2/n ≤ u_{k+1}}. This, for n ≥ 3, is a 2m simplex in E^{2m} which can be obtained from the standard simplex σ = {(x_1, x_2, ..., x_{2m}) : 0 ≤ x_i ≤ x_{i+1} ≤ 1} by pushing every other face of σ towards its opposite vertex by either 1/n or 2/n. For example, the face σ_3 = {x ∈ σ : x_2 = x_3} is shifted over to become the face σ'_3 = {x ∈ σ : x_2 + 2/n = x_3} and call it the evaluation map. Note that the restriction of σ to Δ(B_n^m) is φ^{-1}.

Let Δ'_m = {(u_1, v_1, ..., u_m, v_m) ∈ Δ_m : for some i, u_i = v_i}. Then Δ'_m = Δ_m \ Δ(B_n^m) and is the union of every second face of Δ_m, the faces of σ that were not shifted over to obtain Δ_m, and is therefore, as easily seen by an inductive argument, topologically a (2m - 1)-cell in the boundary of Δ_m.

Proposition 5.1. Each A_n is a Q-factor.

Proof. We build up A_n inductively by attaching Q-factors as follows: A_n^0 is one point, namely {I}, and if m ≥ 1, and δ is the restriction of δ to Δ'_m, then A_n^m is naturally homeomorphic to Δ_m ∪ δ, A_n^{m-1} in the following sense. If i : A_n^{m-1} —> A_n^m is the injection map, then (δ ∪ i) : Δ_m ∪ A_n^{m-1} —> A_n^m induces the same equivalence relation on Δ_m ∪ A_n^{m-1} as δ' : Δ'_m —> A_n^m and hence if p : Δ_m ∪ A_n^{m-1} —> Δ_m ∪ δ, A_n^{m-1} is the quotient map, then h : Δ_m ∪ δ, A_n^{m-1} —> A_n^m defined by h(p(x)) = (δ ∪ i)(x) is a homeomorphism. It is in this sense that we say the spaces are naturally homeomorphic. Thus, by an inductive use

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of the Attaching Theorem, \( A_n \) is a \( Q \)-factor, since \( \Delta_n \) and \( \Delta'_m \) are \( Q \)-factors and since any closed subset of the boundary of \( \Delta_m \), namely \( \Delta'_m \), is a \( Z \)-set in \( \Delta_m \).

6. \( Q \)-factor decompositions. The concept of a \( Q \)-factor decomposition was introduced by D. W. Curtis in [3] and is a generalization of a simplicial complex where the \( Q \)-factors correspond to simplexes. Here we will generalize this notion to correspond to CW-complexes where the distinction is that, in the analogy with complexes, instead of insisting that the intersection of two simplexes be a face of each of them, we allow such an intersection to be a union of faces. Thus, we say that a collection of \( Q \)-factors has the intersection property if the nonempty intersection of two members of the collection is a union of members of the collection. The proper face relationship of simplexes corresponds to the following. A collection \( \mathcal{D} \) of \( Q \)-factors has the \( Z \)-set property provided that if \( D_1, D_2 \in \mathcal{D} \) where \( D_1 \) is a proper subset of \( D_2 \), then \( D_1 \) is a \( Z \)-set in \( D_2 \).

**Definition.** A \( Q \)-factor decomposition of a compact metric space \( X \) is a finite cover of \( X \) by \( Q \)-factors that has the intersection and \( Z \)-set properties.

The next theorem follows from a special case of [3, Theorem 2.4] by D. W. Curtis. We include the proof here for completeness. The mesh of a collection of subsets of a metric space is the maximum of the diameters of the members of the collection. For each \( i \), let \( l_i = i \) and if \( n < m \leq \infty \), let \( l^m_n = \Pi l_i=1 l^m_n l_i \).

**Theorem 6.1.** Let \( X \) and \( Y \) be compact metric spaces and let \( f : X \to Y \) be a map. If for each \( \epsilon > 0 \) there exists a \( Q \)-factor decomposition \( \mathcal{D} \) of \( Y \) with mesh less than \( \epsilon \) such that \( f^{-1}(D) \) is a \( Q \)-factor decomposition of \( X \), then \( f \) stabilizes to a near-homeomorphism.

**Proof.** Let \( \epsilon > 0 \) and let \( \mathcal{D} \) be a \( Q \)-factor decomposition of \( Y \) with mesh less than \( \epsilon/2 \) as in the hypothesis. It is sufficient to show that there exists a homeomorphism \( h : X \times Q \to Y \times Q \) such that for each \( D \in \mathcal{D} \), \( h(f^{-1}(D) \times Q) = D \times Q \), since if we have such an \( h \) then we can define, for \( n \) large enough so that the diameter of \( I^\infty_{n+1} \) is less than \( \epsilon/2 \),

\[
h_1 : X \times I^1_n \times I^\infty_{n+1} \to Y \times I^1_n \times I^\infty_{n+1}
\]

as follows. Assume \( h \) is defined on \( X \times I^\infty_{n+1} \), where we are viewing \( Q \) as \( I^\infty_{n+1} \), let \( \text{id} \) be the identity function on \( I^1_n \), and let \( h_1 = h \times \text{id} \). Then \( d(f, h) < \epsilon \) and hence \( f \) is a near-homeomorphism.

We will now construct \( h : X \times Q \to Y \times Q \). For \( \mathcal{D}_1 \subseteq \mathcal{D} \), let \( \min(\mathcal{D}_1) = \{ D \in \mathcal{D}_1 : \text{no proper subset of } D \text{ belongs to } \mathcal{D}_1 \} \). Let \( \mathcal{D}^{-1} = \emptyset \) and for \( i > 0 \), let \( \mathcal{D}^i = \mathcal{D}^{i-1} \cup \min(\mathcal{D} \setminus \mathcal{D}^{i-1}) \). By the minimality condition, the members of \( \mathcal{D}^0 \) are disjoint \( Q \)-factors and hence there exists a homeomorphism

\[
h_0 : \bigcup \{ f^{-1}(D) : D \in \mathcal{D}^0 \} \times Q \to \bigcup \{ D : D \in \mathcal{D}^0 \} \times Q
\]
such that for each $D \in \mathcal{D}^0$, $h_0(f^{-1}(D) \times Q) = D \times Q$. Let $j \geq 0$ and assume there exists a homeomorphism

$$h_j : \bigcup \{f^{-1}(D) : D \in \mathcal{D}^j\} \times Q \rightarrow \{D : D \in \mathcal{D}^j\} \times Q$$

such that for each $D \in \mathcal{D}^j$, $h_j(f^{-1}(D) \times Q) = D \times Q$. For each $D \in \mathcal{D}^{j+1}\setminus\mathcal{D}^j$, let $\hat{D} = \bigcup \{D_1 \in \mathcal{D} : D_1 \text{ is a proper subset of } D\}$. Then $\hat{D}$ is either empty, or as a finite union of Z-sets, is a Z-set in $D$ and $\hat{D}$ is a union of members of $\mathcal{D}^j$. Since $f^{-1}(D)$ is a $Q$-factor decomposition of $X$, we also have that $f^{-1}(\hat{D})$ is a Z-set in $f^{-1}(D)$ and by the inductive hypothesis $h_j(f^{-1}(\hat{D}) \times Q) = \hat{D} \times Q$. Thus, by the Homeomorphism Extension Theorem, $h_j$ can be extended to a homeomorphism of $f^{-1}(D) \times Q$ onto $D \times Q$ and hence we have constructed the required homeomorphism $h_{j+1}$. Thus, by finite induction we have constructed the required homeomorphism.

7. $Q$-factor decompositions of $A_n$ and the proof that $f_n : A_{n+1} \rightarrow A_n$ stabilizes to a near-homeomorphism. In this section we set up the major machinery for the rest of the paper and prove that $f_n$ stabilizes to a near-homeomorphism.

Recall that $A_n = \{[0, u] \cup [v, 1] \in A_n : 1/n < u < v < 1 - 1/n\}$ and let $A_n^1 = \{[0, u] \cup [v, 1] \in A_n : 1/n < u < v < 1 - 1/n\}$. Then $A_n^1$ can be identified as the quotient space of $A_n^1$ where the diagonal of $A_n^1$ is shrunk to a point. All of the points of the diagonal correspond to the single point $I$ in $A_n^1$. The corresponding equivalence relation on $A_n^1$ is induced by the evaluation map $\delta : A_n^1 \rightarrow A_n^1$ defined by $\delta(u, v) = [0, u] \cup [v, 1]$. Let $S$ be a $Q$-factor decomposition of $A_n^1$ as in Figure 2 where the stair-step element containing the diagonal is denoted by $s_0$ and where the other elements $s$ of $S$ are of the form $s = s^1 \times s^2$ where $s^1$ and $s^2$ are closed subintervals of length less than $1/2n$ (possibly degenerate) of $I$. The $1/2n$ condition is needed so that the soon to be defined collection $\mathcal{D}$ covers $A_n^1$. Let us take all of the nondegenerate intervals $s'$ to be of the same length so that all of the principal members of $S \backslash \{s_0\}$ are square. We specifically include in $S$ all of the edges and vertices of each of the square elements. Thus, the collection $S' = S \backslash \{s_0\}$ will be closed under finite intersections and it is just those elements $s$ of $S'$ that touch $s_0$ whose intersection with $s_0$, $s \cap s_0$, is the union of (two) elements of $S$. For a given $\epsilon > 0$ we can construct $S$ so that the induced $Q$-factor decomposition $\delta(S)$ of $A_n^1$ has mesh less than $\epsilon$.

We will use the $Q$-factor decomposition $S$ to induce a $Q$-factor decomposition $\mathcal{D}$ on $A_n$. A gap in an element $A$ of $A_n$ is a component of $\Gamma A$. If $u, v$ are the endpoints of a gap, then $(u, v) \in A_n^1$ and if the gap is sufficiently small, then $(u, v) \in s_0$. Such gaps will be called small and otherwise they will be called large. The $Q$-factors $D$ in $\mathcal{D}$ will be determined by the larger gaps of the members $A$ of $D$. All $A$ in a given $D$ will have the same number of larger gaps and these gaps will all be in approximately the same place in $I$. Specifically, the members $s \in S'$
will determine the larger gaps in the elements $A$ of a given $D$.

\[ \text{Figure 2} \]

Call $s_1, \ldots, s_k \in S', k \geq 0$, an admissible sequence from $S'$ if for each $i$, \( \text{sup} s_i^2 < \text{inf} s_{i+1} \), and for such an admissible sequence let

\[ D(s_1, \ldots, s_k) = \{ A \in A_n : \text{for each } i = 1, \ldots, k, \text{there exists a pair of adjacent components} \ (a_i, b_i) \text{ or } (c_i, d_i) \text{ of } A \text{ such that} \ (b_i, c_i) \in s_i \text{ and for all other pairs of adjacent components} (a, b), (c, d) \text{ of } A, \ (b, c) \in s_0 \}. \]

Thus, $s_1, \ldots, s_k$ determine the larger gaps of $A$ and we allow as many smaller gaps as permitted for elements of $A_n$. $D(s_1, \ldots, s_k)$ may be empty, if for example, $s_1^2 = [0, a]$ where $a < 1/n$, recalling that for each element $A$ of $A_n$; the components of $A$ that contain 0 and 1 must be at least $1/n$ long and the other components must be at least $2/n$ long. For $k = 0$, denote the set $D(\ )$ by $D(s_0)$ and let $\mathcal{D}$ be the collection of all such $D(s_1, \ldots, s_k)$. In showing that $\mathcal{D}$ is a $Q$-factor decomposition of $A_n$, we find it convenient to analyze, for each $D \in \mathcal{D}$, the precise structure of those parts of the elements of $D$ between pairs of adjacent larger gaps determined, for example by $s_i$ and $s_{i+1}$ of $S'$. In applications, $J_1, J_2$ will correspond to $s_i, s_{i+1}$ respectively. If $B_n = F_n(2^l)$, then an interval $[u, v] \in B_n$ if $v-u \geq 2/n$, or $[u, v]$ contains either 0 or 1 and $v-u \geq 1/n$. Let $J_1, J_2$ be subintervals of $I$, possibly degenerate, where $\text{sup}J_1 < \text{inf}J_2$ and let

\[ E(J_1, J_2) = \{ [u, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, v] : k \geq 0, u \in J_1, v \in J_2, \text{ each } (a_i, b_i) \in s_0, \text{ and each interval } [u, a_1], \ [b_1, a_2], \cdots, [b_k, v] \text{ is an element of } B_n \}. \]

Our next goal is to show that each $E(J_1, J_2)$ is a $Q$-factor. In doing this it is convenient to define, for $k \geq 0$, the set of all elements of $E(J_1, J_2)$ with exactly $k$ gaps. We extend our definition of a gap to allow degenerate gaps, i.e., where $a_i = b_i$. For $k \geq 0$, let

\[ E_k(J_1, J_2) = \{ [u, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, v] : u \in J_1, v \in J_2, \text{ each } (a_i, b_i) \in s_0, \text{ and each interval } [u, a_1], \ [b_1, a_2], \cdots, [b_k, v] \text{ is an element of } B_n \}. \]
Note, because of the specified minimum lengths of the intervals, that \( E_k(J_1, J_2) = \emptyset \) for sufficiently large \( k \). Thus, \( E_{k-1}(J_1, J_2) \) need not be a subset of \( E_k(J_1, J_2) \).

**Lemma 7.1.** The set

\[
\Delta_k = \{u, a_1, b_1, \ldots, a_k, b_k, v) \in E^{2k+2} : [u, a_1] \cup \cdots \cup [b_k, v] \in E_k(J_1, J_2)\}
\]

is (a) either empty or a Q-factor, and (b) if \( \Delta_k \) is nondegenerate, then

\[
\Delta_k' = \{(u, a_1, b_1, \ldots, a_k, b_k, v) \in \Delta_k : \text{for some } i, a_i = b_i\}
\]
is a Q-factor and is a Z-set in \( \Delta_k \).

**Proof.** (a) As mentioned above, for sufficiently large \( k \), \( \Delta_k = \emptyset \). Otherwise, \( \Delta_k \) is a subset of a Euclidean space that fails to be convex because each \((a_i, b_i) \in s_0\) and \( s_0 \) is not convex. It is well known [16] that convex linear cells in \( E^m \) are characterized as being compact solutions to a finite set of linear inequalities. Since \( s_0 \) is a finite union of convex linear cells and since the side conditions on the coordinates of \( \Delta_k \) are linear conditions (i.e., \( a_i \leq b_i \) and \( b_i + 2/n \leq a_{i+1} \)), then \( \Delta_k \) is a finite union of convex linear cells and hence is a polyhedron. In addition, \( \Delta_k \) is contractible (in itself). To see this, first contract \( \Delta_k \) to \( C_k = \{(u, a_1, b_1, \ldots, a_k, b_k, v) \in \Delta_k : \text{for each } i, a_i = b_i\} \) as follows. Define a homotopy \( h_t : s_0 \rightarrow s_0 \) by \( h_0 = \text{id}; \) for each \((a, b) \in s_0\), let \( h_1(a, b) = ((a + b)/2, (a + b)/2)\); and for \( t \in (0, 1) \), let \( h_t = (1 - t)h_0 + th_1 \). This homotopy corresponds to sliding each \((a, b) \in s_0\) (see Figure 2) to the diagonal of \( s_0 \) along the line through \((a, b)\) with slope \(-1\). Then \( H_t : \Delta_k \rightarrow \Delta_{k-1} \) is defined by simultaneously applying \( h_t \) to each pair \( a, b \). Now \( C_k \) is a convex cell and thus contractible and hence \( \Delta_k \) is contractible. Thus \( \Delta_k \) is a contractible polyhedron and, by [12, Corollary 5.6] is a Q-factor.

(b) The same argument as above shows that \( \Delta_k' \) is a contractible polyhedron and is therefore a Q-factor. By the characterization of Z-sets, it is sufficient to construct, for each \( \epsilon > 0 \), a map \( f : \Delta_k \rightarrow \Delta_{k-1} \) such that \( f(\Delta_k) \cap \Delta'_{k-1} = \emptyset \) and \( d(f, \text{id}) < \epsilon \). If \( 0 \notin J_1 \) and \( 1 \notin J_2 \) and \( \Delta_k \) is nondegenerate, then \( \sup J_1 - \inf J_1 > (k + 1)(2/n) \) and this provides room to simultaneously break apart the degenerate gaps, i.e., where \( a_i = b_i \) in the members \( A \) of \( E_k(J_1, J_2) \). This corresponds to a map of \( \Delta_k \) that pushes \( \Delta_k \) off \( \Delta_k' \) and hence \( \Delta_k' \) is a Z-set in \( \Delta_k \). If \( 0 \in J_1 \) or \( 1 \in J_2 \) the situation is similar. In fact, \( \Delta_k \) is a topological cell and \( \Delta_k' \) is a closed subset of its boundary and hence in a Z-set in \( \Delta_k \).

**Lemma 7.2.** Each \( E(J_1, J_2) \) is either the empty set or a Q-factor.

**Proof.** If \( \sup J_2 - \inf J_1 \) is too small, then \( E(J_1, J_2) = \emptyset \). In any case, \( E(J_1, J_2) = \bigcup_{i \geq 0} E_i(J_1, J_2) \neq \emptyset \). In particular, \( E_0 = E_0(J_1, J_2) = \{[u, v] : u \in J_1, v \in J_2\} \neq \emptyset \) and is a Q-factor since \( \delta : \Delta_0 \rightarrow E_0 \) is a homeomorphism.
and \( \Delta_0 \) is topologically either a 0, 1 or 2-cell depending upon whether both, one or neither of \( J_1, J_2 \) is degenerate. Let \( k > 0 \) and assume that \( \bigcup_{i=0}^{k-1} E_i \) is a \( Q \)-factor. Then, as in the proof of 5.1, \( \bigcup_{i=0}^{k-1} E_i \) is naturally homeomorphic to \( \Delta_k \cup \delta' \cup \bigcup_{i=0}^{k-1} E_i \) where \( \delta' \) is the evaluation map restricted to \( \Delta'_k \). Thus, by the Attaching Theorem, \( \bigcup_{i=0}^{k-1} E_i \) is a \( Q \)-factor.

Let \( J_1, \ldots, J_k \), for \( k = 2, 4, \ldots \), be subintervals of \( I \) where \( M_i = \sup J_i < \inf J_{i+1} = m_{i+1} \). In the following we will have occasions to use, for \( D \in \mathcal{D} \), a function

\[
h : D \to E(J_1, J_2) \times \cdots \times E(J_{k-1}, J_k)
\]

defined by \( h(A) = (A \cap [m_1, M_2], \ldots, A \cap [m_{k-1}, M_k]) \). This function will be called the canonical map.

**Proposition 7.3.** The collection \( \mathcal{V} \) is a \( Q \)-factor decomposition of \( A_n \) and furthermore, for any \( \epsilon > 0 \), \( \mathcal{V} \) can be constructed to have mesh less than \( \epsilon \).

**Proof.** If \( S \) is constructed so that mesh \( \delta(S) < \epsilon \), then the mesh condition will be automatically satisfied. To show that \( \mathcal{V} \) covers \( A_n \), let \( A = [0, u_1] \cup [u_1, u_2] \cup \cdots \cup [u_m, 1] \in A_n \) and for each \( i \), let \( s_i \) be the smallest element of \( S \) containing \( (u_i, v_i) \). From the sequence \( s_1, \ldots, s_m \), delete any \( s_i \) that is equal to \( s_0 \) and renumber the sequence \( s_1, \ldots, s_k \) \( (k < m) \) maintaining the inherited order. Then \( A \in D(s_1, \ldots, s_k) \) and \( s_1, \ldots, s_k \) is an admissible sequence since in the construction of \( S \), the elements of \( S' \) have sides of length less than \( 1/2n \) and hence \( \mathcal{V} \) covers \( A_n \). We will now show that each member of \( \mathcal{V} \) is a \( Q \)-factor. If \( D(s_1, \ldots, s_k) \in \mathcal{V} \), then the canonical map

\[
h : D(s_1, \ldots, s_k) \to E(s^2_0, s^1_1) \times \cdots \times E(s^2_k, s^1_{k+1}),
\]

where \( s^2_0 = \{0\} \) and \( s^1_{k+1} = \{1\} \), is a homeomorphism, and hence, \( D(s_1, \ldots, s_k) \) is a \( Q \)-factor since it is homeomorphic to a product of \( Q \)-factors.

We will now observe that \( \mathcal{V} \) inherits the intersection property from \( S \). Let \( D_1 = D(s_1, \ldots, s_k) \) and \( D_2 = D(t_1, \ldots, t_p) \) be members of \( \mathcal{V} \) with a nonempty intersection. Then each \( s_i \) intersects either some \( t_j \) or \( s_0 \) and each \( t_j \) intersects either some \( s_i \) or \( s_0 \). Then \( D_1 \cap D_2 \) is the union of all \( D(u_1, \ldots, u_k) \) where \( u_1, \ldots, u_k \) is an admissible sequence from \( S' \) such that each \( u_\alpha \) is contained in the intersection of either (a) some \( s_i \) and some \( t_j \), or (b) some \( s_i \) and \( s_0 \), or (c) some \( t_j \) and \( s_0 \).

We also observe that \( \mathcal{V} \) inherits the Z-set property from \( S \). If \( D_1 = D(s_1, \ldots, s_k) \subseteq D_2 = D(t_1, \ldots, t_p) \) then each \( s_i \) is either a subset of some \( t_j \) or is a subset of \( s_0 \). Furthermore, if \( D_1 \neq D_2 \), then some \( s_i \) is either a proper subset of some \( t_j \) or of \( s_0 \). In either case, since \( S \) has the Z-set property, we can see that maps on the appropriate elements of \( S \) within a given \( \epsilon > 0 \) of the iden-
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Proposition 3.4. Each \( f_n : A_{n+1} \to A_n \) stabilizes to a near-homeomorphism.

Proof. Let \( \varepsilon > 0 \) and let \( S \) be a \( \mathcal{Q} \)-factor decomposition of \( \Delta^1_n \) as described in the first part of this section such that the induced \( \mathcal{Q} \)-factor decomposition \( \mathcal{D} \) on \( A_n \) has mesh less than \( \varepsilon \). Let \( u, v \) be the endpoints of a gap in an element \( A \) of \( A_{n+1} \), and let \( \alpha = \min \{ (v - u)/2, 1/n(n + 1) \} \). Then the endpoints of the corresponding gap in \( f_n(A) \) are \( u + \alpha, v - \alpha \). This correspondence \( (u, v) \mapsto (u + \alpha, v - \alpha) \) defines a map \( g_1 : \Delta^1_{n+1} \to \Delta^1_n \cup \Delta' \cup \Delta'' \) where \( \Delta' = \{(u, u) : 1/n + 1 < u < 1/n\} \) and \( \Delta'' = \{(u, u) : 1 - 1/n < u < 1 - 1/n + 1\} \).

Let \( g_2 : \Delta^1_n \cup \Delta' \cup \Delta'' \to \Delta^1_n \) be the map that is the identity on \( \Delta^1_n \) and takes \( \Delta' \) to \((1/n, 1/n)\) and \( \Delta'' \) to \((1 - 1/n, 1 - 1/n)\). Then \( g = g_2 \circ g_1 \) is a map from \( \Delta^1_{n+1} \) onto \( \Delta^1_n \). Then \( \mathcal{D}^{-1}(S) \) is a \( \mathcal{Q} \)-factor decomposition of \( \Delta^1_{n+1} \) of the same type as \( S \). In fact, the square elements of \( g^{-1}(S') \) and the stair-step boundary of \( g^{-1}(s_0) \) are just translates away from the diagonal of the corresponding elements of \( S \). Then \( f_n^{-1}(\mathcal{D}) \) is precisely the collection induced by \( g^{-1}(S') \) which by 7.3 is a \( \mathcal{Q} \)-factor decomposition of \( A_{n+1} \) and hence, by 5.1, \( f_n \) stabilizes to a near-homeomorphism.

8. The proof that \( h_n : A_n \to A_{n,t} \) stabilizes to a near homeomorphism. In this section we may assume that \( 1/n \) is small relative to \( t \) and \( 1 - t \). Recall that \( A_{n,t} = F_n(H(0, t, 1)) \) and is the set of all members of \( A_n \) that contain \( J = [t - 1/n, t + 1/n] \). We will construct, for a given \( \varepsilon > 0 \), a \( \mathcal{Q} \)-factor decomposition \( \mathcal{D}_t \) of \( A_{n,t} \) with mesh less than \( \varepsilon \). This will directly follow the construction of the corresponding decomposition of \( A_n \). Let \( A_{n,t} \) be those members of \( A_{n,t} \) that contain \( J \), i.e., \( A_{n,t} = \{[0, u] \cup [v, 1] : 1/n < u < v < t - 1/n, \text{ or } t - 1/n < u = v < t + 1/n, \text{ or } t + 1/n < u < v < 1 - 1/n\} \), and let \( A_{n,t} \) be the \( \mathcal{Q} \)-factor decomposition of \( A_{n,t} \), as in Figure 3, obtained by intersecting \( A_{n,t} \) with each element of a \( \mathcal{Q} \)-factor decomposition \( \mathcal{S} \) of \( A_n \) as constructed in \( \S 7 \). Let \( \mathcal{D} \) be the \( \mathcal{Q} \)-factor decomposition of \( A_n \) induced by \( S \) and let \( \mathcal{D}_t = \{D \cap A_{n,t} : D \in \mathcal{D}\} \). Then \( \mathcal{D}_t \) can be thought of as being induced by \( S_t \) in the same way that \( \mathcal{D} \) was induced by \( S \). Again, denote the large element of \( S_t \) containing the diagonal by \( s_0 \).

Proposition 8.1. A collection \( \mathcal{D}_t \) is a \( \mathcal{Q} \)-factor decomposition of \( A_{n,t} \) and furthermore, for any \( \varepsilon > 0 \), \( \mathcal{D}_t \) can be constructed to have mesh less than \( \varepsilon \).

Proof. This proof is the same as the proof of 7.3 except that we modify the canonical map as follows. For \( D(s_1, \ldots, s_k) \in \mathcal{D}_t \), let \( m \) be the integer such
that \( J \) is contained between the intervals \( s_m^2 \) and \( s_{m+1}^1 \). Then the canonical map 
\[ h: D(s_1, \ldots, s_k) \to E(s_2^2, s_1^1) \times \cdots \times E(s_{m-1}^2, s_m^1) \times E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1) \times E(s_{m+1}^2, s_{m+2}^1) \times \cdots \times E(s_k^2, s_{k+1}^1) \]
is a homeomorphism.

**Figure 3**

Our main goal now is to prove that \( h_n^{-1}(D_t) \) is a Q-factor decomposition of \( A_n \). We first will review the definition of \( h_n \). The retraction \( h_t: H(0, 1) \to H(0, t, 1) \) was defined by \( h_t(A) = A \cup [u, u + \alpha] \cup [v - \alpha, v] \) where \( u \) is the maximal point of \( A \) less than or equal to \( t \), \( v \) is the minimal point of \( A \) greater than or equal to \( t \), and \( \alpha \) is the minimum of \( t - u \) and \( v - t \). Then \( h_n = F_n \circ h_t \circ s_n \) where \( s_n \) is the discontinuous section of \( F_n \) described in §3. Furthermore, if \( A \in A_n \), then \( J \) is not a subset of \( A \) iff \( A \) has exactly one nondegenerate gap \( (u, v) \) where \( u < t + 1/n \) and \( v > t - 1/n \). It directly follows from the above that \( h_n: A_n \to A_{n,t} \) can be described as follows. For \( A \in A_n \), if \( J \subset A \), then \( h_n(A) = A \) and if \( J \) is not contained in \( A \), where \( (u, v) \) is the nondegenerate gap of \( A \) where \( u < t + 1/n \) and \( v > t - 1/n \), then \( h_n(A) = A \cup [u, u + \alpha] \cup [v - \alpha, v] \) where \( \alpha \) is the minimum of \( (t + 1/n) - u \) and \( v - (t - 1/n) \). This \( \alpha \) also has the property that it is the minimum \( d \) such that \( J \subset A \cup [u, u + d] \cup [v - d, v] \). Although this retraction \( h_n \) was defined on \( A_n \), the definition would be equally valid for elements of \( 2^I \) having nonvoid intersections with both \([0, t - 1/n]\) and \([t + 1/n, 1]\). Henceforth, we will consider \( h_n \) to be extended to these elements of \( 2^I \). We now analyze the precise structure of those components of the elements of \( A_{n,t} \) that are affected by \( h_n^{-1} \).

Let \( J_1, J_2 \) be closed subintervals of \( I \) such that \( \sup J_1 \leq t - 1/n \) and \( t + 1/n \leq \inf J_2 \). We emphasize that in writing \([u, v]\) we mean \( u \leq v \). Let \( s \subset \Delta_{n,t}^1 \) and let \( C(J_1, s, J_2) = ([u_1, u_2] \cup [u_3, u_4] \in \Delta'_1: u_1 \in J_1, u_4 \in J_2, (u_2, u_3) \subset s, \text{ and each interval is an element of } B_{n,t} \}) \text{ where } B_{n,t} = F_n[H(t)]. \) This latter condition can equivalently be stated as each interval is a component of some element of \( A_{n,t} \). We now introduce small gaps as allowed by our side conditions. Let
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\[ E(J_1, s, J_2) = \{[u_1, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, u_2] \cup [u_3, c_1] \]
\[ \cup [d_1, c_2] \cup \cdots \cup [d_m, u_4]: k \geq 0, m \geq 0, (a_i, b_i) \in s_0, \]
\[ (c_i, d_i) \in s_0, [u_1, u_2] \cup [u_3, u_4] \in C(J_1, s, J_2), \]
and each interval is an element of \( B_{n,t} \).

Since \( h_n \) is a retraction, our attention is on those elements \( A \) of \( A_{n,t} \) such that \( h_n^{-1}(A) \) contains elements other than \( A \). Such elements \( A \) have a gap \((u, v)\) such that the corresponding 2-tuple \((u, v)\) is in

\[ \gamma = \{(u, v) \in \Delta_{n,t}^1: u = t + 1/n, \text{ or } v = t - 1/n, \text{ or } \]
\[ t - 1/n \leq u = v \leq t + 1/n \} . \]

For \( s \in S_t \), let \( s' = s \cap \gamma \) (see Figure 3) and let

\[ E_{0,0}(J_1, s', J_2) = \{[u_1, u_2 - d] \cup [u_3 + d, u_4]: d \in I, \]
\[ [u_1, u_2] \cup [u_3, u_4] \in C(J_1, s', J_2), \text{ and each interval is an element of } B_n \}. \]

We emphasize that \( J_1 \) and \( J_2 \) are on either side of \( J \) and that we will only be concerned here with those \( s \in S_t \) such that \( s' \neq \emptyset \). It follows from the extended definition of \( h_n \) that \( h_n^{-1}C(J_1, s', J_2) = E_{0,0}(J_1, s', J_2) \) and that \( h_n^{-1}C(J_1, s, J_2) = C(J_1, s, J_2) \cup E_{0,0}(J_1, s', J_2) \). We now specify the elements of \( h_n^{-1}E(J_1, s, J_2) \) with different specific numbers of small gaps on either side of \( J \). For \( s \in S_t \) where \( s' \neq \emptyset \), \( k > 0 \), and \( m > 0 \), let

\[ E_{k,m}(J_1, s', J_2) = \{[u_1, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_k, u_2 - d] \]
\[ \cup [u_3 + d, c_1] \cup [d_1, c_2] \cup \cdots \cup [d_m, u_4]: d \in I, \]
\[ [u_1, u_2] \cup [u_3, u_4] \in C(J_1, s', J_2), \text{ each } \]
\[ (a_i, b_i) \text{ and } (c_i, d_i) \text{ belong to } s_0, \text{ and each interval is an element of } B_n \}. \]

The proof of Lemma 7.1 is also a proof for

**Lemma 8.2.** (a) Each set \( \Delta_{km} = \Delta(E_{km}(J_1, s', J_2)) \) is either empty or a Q-factor, and

(b) if \( \Delta_{km} \) is nondegenerate, then

\[ \Delta_{km}' = \{(u_1, a_1, b_1, \ldots, a_k, b_k, u_2 - d, u_3 + d, c_1, d_1, \ldots, c_m, d_m, u_4) \in \Delta_{km}: d = 0 \text{ or for some } i, a_i = b_i \text{ or for some } j, c_j = d_j \} \]

is a Q-factor which is a Z-set in \( \Delta_{km} \).

**Lemma 8.3.** Each set \( h_n^{-1}E(J_1, s, J_2) \) is either empty or a Q-factor.

**Proof.** If \( q \geq n/2 \) and \( r \geq n/2 \), then
We first show that \( E(J_1, s, J_2) \) is a \( Q \)-factor. If \( s = s^1 \times s^2 \), then the canonical map from \( E(J_1, s, J_2) \) onto \( E(J_1, s^1) \times E(s^2, J_2) \) is a homeomorphism and hence, by 7.2, \( E(J_1, s, J_2) \) is homeomorphic to the product of \( Q \)-factors and hence is a \( Q \)-factor. Also, the canonical map from \( E(J_1, s_0, J_2) \) onto \( E(J_1, t + 1/n) \times E(t - 1/n, J_2) \) is a homeomorphism and hence \( E(J_1, s_0, J_2) \) is a \( Q \)-factor.

We can assume that each \( \Delta_{km} \) is nondegenerate, and then by 8.2 each \( \Delta'_{km} \) is a \( Q \)-factor and a \( Z \)-set in the \( Q \)-factor \( \Delta_{km} \). Also, \( E_{km} \) is naturally identified with the quotient space of \( \Delta_{km} \), where the equivalence relation is induced by the evaluation map \( \delta \) on \( \Delta_{km} \). Thus, to attach a given \( E_{km} \) to the inductive step we formally take the attachment of \( \Delta_{km} \) to the inductive step by the map \( \delta \) on \( \Delta_{km} \) where \( \delta(\Delta'_{km}) \) must be contained in the inductive step. This containment condition is satisfied if we attach to \( E(J_1, s, J_2) \) the sets \( E_{km} \) in the following prescribed order: \( E_{00}, E_{01}, \ldots, E_{or}, E_{10}, E_{11}, \ldots, E_{1r}, \ldots, E_{q0}, E_{q1}, \ldots, E_{qr} \). Thus, by the inductive use the Attaching Theorem, this lemma is proved.

**Lemma 8.4.** The collection \( h^{-1}(D_t) \) is a \( Q \)-factor decomposition of \( A_n \).

**Proof.** For \( D(s_1, \ldots, s_k) \in D_t \), let \( m \) be the positive integer such that \( J \) is contained between \( s_m^1 \) and \( s_{m+1}^1 \). If \( k = 0 \), which means we are considering \( D(s_0) \), then \( h^{-1}D(s_0) \) is precisely the \( Q \)-factor \( h^{-1}E(0, s_0, 1) \) of the previous lemma. If \( k > 1 \), then the canonical map from \( D(s_1, \ldots, s_k) \) to
\[
E(0, s_1^1) \times \cdots \times E(s_{m-1}^2, s_m^1) \times E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1)
\]
\[(\ast)\]
\[
\times E(s_{m+1}^2, s_{m+2}^1) \times \cdots \times E(s_k^2, 1)
\]
is a homeomorphism. If sup \( s_m^2 = t - 1/n \) and inf \( s_{m+1}^1 > t + 1/n \), see Figure 4(a)

![Figure 4](image)

then \( h^{-1}D(s_1, \ldots, s_k) \) is canonically homeomorphic to \( (\ast) \) where \( E(s_{m-1}^2, s_m^1) \times E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1) \) is replaced by \( h^{-1}E(s_{m-1}^2, s_m, s_{m+1}^1) \) and hence, by 7.2 and 8.3, is a product of \( Q \)-factors and hence is a \( Q \)-factor. If sup \( s_m^2 < t - 1/n \) and inf \( s_{m+1}^1 = t - 1/n \), see Figure 4(b), then
\[
h^{-1}D(s_1, \ldots, s_k) \text{ is canonically homeomorphic to } (\ast) \text{ where } E(s_m^2, s_{m+1}^1) \times E(t - 1/n, s_{m+1}^1) \times E(s_{m+1}^2, s_{m+2}^1) \text{ is replaced by } h^{-1}E(s_m^2, s_{m+1}, s_{m+2}^1)\]
hence is a $Q$-factor. If $\sup s_m^2 = t - 1/n$ and $\inf s_{m+1}^1 = t + 1/n$, then $h_n^{-1}(D(s_1, \ldots, s_k))$ is the union of the two $Q$-factors described in the above two cases and hence is a $Q$-factor since their intersection is a $Q$-factor that is a $Z$-set in each of them. Their intersection is precisely the $Q$-factor $(*)$ where $E(s_m^2, t + 1/n) \times E(t - 1/n, s_{m+1}^1)$ represents the single element $J$ of $2^J$. If $\sup s_m^2 < t - 1/n$ and $\inf s_{m+1}^1 > t + 1/n$, then $h_n^{-1}(D(s_1, \ldots, s_k)) = D(s_1, \ldots, s_k)$. This completes the verification that the elements of $h_n^{-1}(D_x)$ are $Q$-factors.

The verification that $h_n^{-1}(D_x)$ has the intersection and $Z$-set properties is virtually the same as the corresponding argument for $D$ in the proof of 7.3.

We now restate 3.11, the proof of which is a direct consequence of 5.1, 8.1 and 8.4.

**Proposition 3.11.** Each $h_n : A_n \rightarrow A_{n,t}$ stabilizes to a near-homeomorphism.

**REFERENCES**


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