TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 213, 1975

Z-SETS IN ANR'S

BY

DAVID W. HENDERSON([1])

ABSTRACT. (1) *Let A be a closed Z-set in an ANR X. Let $\mathcal{F}$ be an open cover of X. Then there is a homotopy inverse f: $X \rightarrow X - A$ to the inclusion $X - A \rightarrow X$ such that f and both homotopies are limited by $\mathcal{F}$.*

(2) *If, in addition, X is a manifold modeled on a metrizable locally convex TVS, F, such that F is homeomorphic to F^ω, then there is a homotopy j: $X \times I \rightarrow X$ limited by $\mathcal{F}$ such that the closure (in X) of $j(X \times \{1\})$ is contained in $X - A$.*

We say that a closed subset A of a space X is a *Z-set* (or has Property Z) in X, if, for each open set $U \subset X$, the inclusion $U - A \rightarrow U$ is a homotopy equivalence. This concept was first introduced by R. D. Anderson [1] for subsets of Hilbert space and has been defined in many different ways. Our definition is equivalent to the others in spaces to which they are applied. (See [8] and [2, Lemma 1].)

In [2] it is proved that, if X is a manifold modeled on a separable, infinite-dimensional Fréchet space, then A is a closed Z-set in X if and only if, for each cover $\mathcal{V}$ of X, there is a homeomorphism X onto $X - A$ limited by $\mathcal{V}$. The Theorem (I.2) of this paper is used in [4] and [18] to extend this result to manifolds modeled on nonseparable Fréchet spaces, F, which are homeomorphic to F^ω The method of proof in this paper has been applied in [16].

I. THEOREM. *Let A be a closed Z-set in a space X such that X and $X - A$ are paracompact (Hausdorff). Let $\mathcal{F}$ be an open cover of X.*

(I.1) *If X is a retract of an open subset, O, of a convex set lying in some locally convex topological vector space (LCTVS), F, then there is a map f: $X \rightarrow X - A$ such that f is a homotopy inverse to the inclusion i: $X - A \subset X$, with both homotopies limited by $\mathcal{F}$.*

(I.2) *If, in addition, X is a paracompact connected manifold modeled on a metrizable LCTVS, F, such that F is homeomorphic to its countable cartesian product F^ω, then there is a map f: $X \rightarrow X - A$ such that*

Received by the editors November 29, 1973.

AMS (MOS) subject classifications (1970). Primary 54C55, 57A20; Secondary 54E60, 57A17.

([1]) This work was done in part while the author was an Alfred P. Sloan Fellow and an exchange scientist at the Steklov Institute in Moscow, 1970; and was partially supported by NSF Grant GP 33960X.

$$\mathrm{cl}(f(X)) \subset X - A \quad (\text{'cl'} = \textit{closure in } X),$$

and such f is homotopic to the identity by a homotopy limited by $\mathcal{F}$.

REMARK I.3. The hypothesis of (1) is satisfied in case X is an ANR(Metric) (see [3, p. 86]) or in case X is a paracompact manifold modeled on $R^\infty = \varinjlim R^n$ or on the conjugate of a separable Banach space with the bounded weak-* topology (see Heisey [11, Corollary II–4]).

II. **Covers.** We assume that all covers consist of nonempty sets. If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of X, then we say $\mathcal{A}$ *refines* $\mathcal{B}$ or $\mathcal{A} < \mathcal{B}$, if, for each $A \in \mathcal{A}$, there is a $B \in \mathcal{B}$, such that $A \subset B$. We shall use the following notation:

$\mathrm{cl}\, A$ = closure of A,
$\mathrm{st}(Y, \mathcal{A}) = \bigcup\{A \in \mathcal{A} \mid A \cap Y \neq \varnothing\}$,
$\mathrm{st}(\mathcal{B}, \mathcal{A}) = \{\mathrm{st}(B, \mathcal{A}) \mid B \in \mathcal{B}\}$,
$\mathrm{st}\, \mathcal{A} = \mathrm{st}(\mathcal{A}, \mathcal{A})$,
$\mathrm{cl}\, \mathcal{A} = \{\mathrm{cl}\, A \mid A \in \mathcal{A}\}$,
$\mathrm{ch}\, \mathcal{A} = \{\mathrm{ch}\, A \mid A \in \mathcal{A}\}$,
$r\, \mathrm{ch}\, \mathcal{A} = \{r\, \mathrm{ch}\, A \mid A \in \mathcal{A}\}$,

where ch denotes convex hull in F and $r\colon 0 \to X$ is the retract (it is assumed when $r\, \mathrm{ch}\, A$ is written that $\mathrm{ch}\, A \subset 0$).

(II.1) COVER LEMMA. *For each open cover* $\mathcal{C}$ *of* X (*or* $X - A$), *there are locally finite open covers* $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, *of* X (*or* $X - A$) *such that* $\mathrm{st}\, \mathcal{B}_1 < \mathcal{C}$, $\mathrm{cl}\, \mathcal{B}_2 < \mathcal{C}$, $r\, \mathrm{ch}\, \mathcal{B}_3 < \mathcal{C}$.

PROOF. $\mathcal{B}_1$ exists because X (and $X - A$) is paracompact. (See Dugundji [7, 3.3–3.5, pp. 167 and 168].) $\mathcal{B}_2$ can be constructed using normality. In order to construct $\mathcal{B}_3$, find, for each $x \in X$, a convex open set V_x such that

$$x \in V_x \subset r^{-1}(C) \subset 0$$

for some $C \in \mathcal{C}$. Such V_x exists because 0 is locally convex. Let $\mathcal{B}_3$ be any locally finite refinement of $\{V_x \cap X \mid x \in X\}$.

III. PROOF OF THEOREM. Using the Cover Lemma find a locally finite open cover $\mathcal{B}$ of $X - A$ and locally finite open covers $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ of X such that

$$r\, \mathrm{ch}\, \mathrm{st}\, \mathcal{B} < \{X - A\}, \tag{III.1}$$

and

$$\mathcal{B} < \mathcal{C} < r\, \mathrm{ch}\, \mathrm{st}\, \mathcal{C} < \mathcal{D} < r\, \mathrm{ch}\, \mathrm{st}\, \mathcal{D} < \mathcal{E} < \mathrm{st}\, \mathcal{E} < \mathcal{F}. \tag{III.2}$$

Let N, M, K be simplicial complexes such that N is the nerve of $\mathcal{B}$, M is the nerve of $\mathcal{C}$ and K is the nerve of $\mathcal{B} \vee \mathcal{C}$. ('V' denotes 'disjoint union'); The *nerve* of $\mathcal{B}$ is an abstract simplicial complex N whose n-simplices are all subsets $\{B_1, \ldots, B_{n+1}\} \subset \mathcal{B}$ such that $B_1 \cap \ldots \cap B_{n+1} \neq \varnothing$. (See Dugundji [7, pp. 171–

173].) By the proof of 5.4 on p. 172 of [7], there are maps (barycentric maps) b: $X \to A \to |N|$ and c: $X \to |M|$ into the geometric realizations of the nerves such that, for each $x \in X$, $c(x)$ belongs to the closed simplex $|\{C \in \mathcal{C} | x \in C\}| \in |M|$. Also, for each $x \in X - A$, $b(x)$ belongs to the closed simplex $|\{B \in \mathcal{B} | x \in B\}| \in |N|$. Consider N and M in the natural way as subcomplexes of K.

(III.3) Note that for each $x \in X - A$, $b(x)$ and $c(x)$ are in the same closed simplex of K and thus there is a homotopy j: $(X - A) \times I \to |K|$ joining cl $X - A$ to b and limited by the closed simplices of $|K|$.

We define a map g: $|K| \to X$ as follows: For vertex $\{Y\} \in K$ define $g'(\{Y\})$ so that $g'(\{Y\}) \in Y - A$ (note that $Y \in \mathcal{B} \vee \mathcal{C}$). $Y - A$ is nonempty, because A is a Z-set. Let g': $|K| \to 0$ be obtained by extending the vertex map linearly on each simplex. Then, let $g = r \circ g'$. (III.4) Note that $g(|N|) \subset X - A$ by (III.1).

(III.5) Note that, for each closed simplex $|\{Y_1, \ldots, Y_n\}| \in |K|$,

$$g'(|\{Y_1, \ldots, Y_n\}|) \subset \text{ch st}(Y_1, \mathcal{B} \cup \mathcal{C}) = \text{ch st}(Y_1, \mathcal{C})$$

and

$$g(|\{Y_1, \ldots, Y_n\}|) \subset r \text{ ch st}(Y_1, \mathcal{C}) < \mathcal{D}.$$

(III.6) Thus, for $x \in X - A$, $g' \circ b(x)$ and x belong to ch st$(B, \mathcal{B}) \subset r^{-1}(X - A)$, where $x \in B$. Let k be the straight line homotopy in $r^{-1}(X - A)$ joining $g' \circ b$ to id_{X-A}, (see (III.1)), then $r \circ k$: $(X - A) \times I \to (X - A)$ is a homotopy joining $g \circ b$ to id_{X-A} and limited by $\mathcal{D}$. Similarly, for each $x \in X$, $g \circ c(x)$ and x belong to the same element of $\mathcal{D}$.

We shall prove below in §IV.2,

(III.7) PROPOSITION. *For each cover $\mathcal{W} < \mathcal{D}$, there is a map f_∞: $|K| \to X - A$ such that*

(a) *f_∞ and g are $\mathcal{W}$-close (i.e., for each $x \in |K|$, $\{f_\infty(x)\} \cup \{g(x)\} \in W$, for some $W \in \mathcal{W}$.*

(b) $f_\infty | |N| = g | |N|$.

The homotopy inverse promised by the (I.1) is $f = f_\infty \circ c$: $X \to X - A$. Note that $f(x) = f_\infty \circ c(x)$ and x belong to the same element of st$(\mathcal{D}, \mathcal{W})$. Thus the straight line homotopy in 0 joining f to id_X is limited by ch st$(\mathcal{D}, \mathcal{W}) <$ ch st $\mathcal{D}$.

Applying the retract r we obtain the desired homotopy h: $X \times I \to X$ joining f and id_X and limited by r ch st $\mathcal{D} < \mathcal{F}$ (III.2).

We have (III.3) a homotopy j: $(X - A) \times I \to |K|$ joining cl $X - A$ to b and limited by the closed simplices of $|K|$. Thus $f_\infty \circ j$: $(X - A) \times I \to X - A$ joins $f_\infty \circ c | X - A = f | X - A$ to $f_\infty \circ b$ limited by st$(\mathcal{D}, \mathcal{W}) < \mathcal{E}$ (III.5) and (III.2). But $f_\infty \circ b = (f_\infty | |N|) \circ b = (g | |N|) \circ b = g \circ b$ which is homotopic to

id_{X-A} limited by $\mathcal{D}$ (III.6). Thus $f|X - A$ is homotopic in $X - A$ to id_{X-A} limited by $\mathrm{st}(\mathcal{D}, \mathcal{E}) < \mathrm{st}\ \mathcal{E} < \mathcal{F}$ (III.2).

IV.

(IV.1) Z-SET LEMMA. *Let K be any simplicial complex and let B be a subpolyhedron of K (i.e., a subcomplex of some subdivision of K). Let A be a Z-set in X as in Theorem* (I.1). *Let f: $|K| \to X$ be a map such that $(f(B)) \subset X - A$. Then, for each open cover $\mathcal{V}$ of X and each n-dimensional* ($n = 0, 1, 2, \ldots$) *subpolyhedron C of K, there is a map $\tilde{f}$: $|K| \to X$ such that $\tilde{f}(|B| \cup |C|) \subset X - A$, $\tilde{f}||B| = f||B|$, and $\tilde{f}$ and f are $\mathcal{V}$-close.*

PROOF OF LEMMA. (By induction on n.) If $n = -1$, the lemma is clearly true. Assume true for $n - 1$. Let $\mathcal{T}$ be an open cover of X such that r ch st $\mathcal{T}$ refines $\mathcal{V}$. (Here we assume as in the theorem that $X \subset 0 \subset F$, where there is a retract r: $0 \to X$.) Thus each $T \in \mathcal{T}$ must be so small that ch $\mathrm{st}(T, \mathcal{T}) \subset 0$. Let $\tilde{K}$ be a subdivision of K such that, for each simplex $s \in \tilde{K}$, $f(\mathrm{st}(s, \tilde{K}))$ refines $\mathcal{T}$. ($\mathrm{st}(s, \tilde{K}) = \bigcup \{t \subset K | s$ is a face of $t\}$.) Now apply the lemma, with $\mathcal{T}$ and $\tilde{C}^{n-1} = n - 1$ skeleton of the subdivision $\tilde{C}$ that $\tilde{K}$ induces on C. Thus there is a map $\hat{f}$: $|K| \to X$ such that cl $\hat{f}(|B| \cup |\tilde{C}^{n-1}|) \subset X - A$, $\hat{f}||B|$ and $\hat{f}$ and f are $\mathcal{T}$-close. Let s be an n-simplex of $\tilde{C}$ not in $\tilde{B}$. Note that, for some $T(s) \in \mathcal{T}$, $\hat{f}(s) \subset \mathrm{st}(T(s), \mathcal{T}) \equiv S(s)$ and $\hat{f}(\mathrm{bd}\ s) \subset S(s) - A$. By the definition of Z-set, there is a map g: $S(s) \to S(s) - A$ which is a homotopy inverse to the inclusion. Thus $g \circ \hat{f}|\mathrm{bd}(s)$ is homotopic to $\hat{f}$ in $S(s) - A$. This homotopy together with $g \circ \hat{f}|s$ gives h_s: $S(s) \to S(s) - A$, an extension of $\hat{f}|\mathrm{bd}(s)$. For each n-simplex s of $\tilde{C}$ not in $\tilde{B}$, pick open sets $U(s)$ such that

$$s - \mathrm{bd}(s) \subset U(s) \subset |\mathrm{st}(s, \tilde{K})| \subset |K| - |B|,$$

$$U(s) \cap U(s') = \emptyset, \quad \text{for } s \neq s', \quad \text{and}$$

$$\hat{f}(\text{closure } U(s)) \subset S(s).$$

The map $\hat{f}|\mathrm{bd}\,U(s)$ together with h_s: $S \to S(s) - A$ can be extended to a map $\tilde{h}_s$: $\mathrm{cl}(U(s)) \to \mathrm{ch}(S(s))$ and thus to $r \circ \tilde{h}_s$: $\mathrm{cl}(U(s)) \to r(\mathrm{ch}(S(s)))$. Do this separately for each s to obtain $\tilde{f}$: $|K| \to X$ where $\tilde{f}|U(s) = r \circ \tilde{h}_s$ and $\tilde{f}||K| - \bigcup \{U(s)\} = \hat{f}$. It is easy to check that $\tilde{f}$ is the desired map.

(IV.2) PROOF OF PROPOSITION. Let $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \ldots$ be a sequence of locally finite open covers of X such that, for each n,

$$\mathcal{X}_n < \mathrm{cl}\ \mathcal{X}_n < \mathcal{W}, \quad \text{where } \mathcal{X}_0 = \mathcal{W}_0 \text{ and } \mathcal{X}_n = \mathrm{st}(\mathcal{X}_{n-1}, \mathcal{W}_n).$$

We may construct these covers inductively as follows: Let $\mathcal{W}_0 = \mathcal{X}_0$ be any locally finite open covers of Y such that cl $\mathcal{X}_0 < \mathcal{W}$. Assume $\mathcal{X}_0, \ldots, \mathcal{X}_{n-1}$, and $\mathcal{W}_0, \ldots, \mathcal{W}_{n-1}$, have been constructed as above, so that each $\mathcal{X}_i$ is locally finite.

Then, for each $x \in X$, let W_x be a neighborhood of x such that W_x intersects only finitely many members of $\mathfrak{X}_{n-1}$, say $\{A_1, A_2, \ldots, A_m\}$. We may assume that x belongs to the closure of each A_i. (If $x \notin \mathrm{cl}(A_i)$, then replace W_x by $W_x - \mathrm{cl}(A_i)$.) Let $W(A_i) \in \mathcal{W}$ be such that $\mathrm{cl}(A_i) \subset W(A_i)$. Then

$$X \in \bigcap\{W(A_i) | i = 1, \ldots, m\}.$$

Let V_x be an open set such that

$$x \in V_x \subset \mathrm{cl}(V_x) \subset \bigcap\{W(A_i) | i = 1, \ldots, m\}.$$

Let $\mathcal{W}_n$ be a locally finite refinement of $\{W_x \cap V_x | x \in X\}$ and define $\mathfrak{X}_n = \mathrm{st}(\mathfrak{X}_{n-1}, \mathcal{W}_n)$. Each $W \in \mathcal{W}_n$ intersects only finitely many members of $\mathfrak{X}_{n-1}$, thus $\mathfrak{X}_n$ is locally finite. Also, if $V \in \mathfrak{X}_n$, then $V \doteq \mathrm{st}(A, \mathcal{W}_n)$, for some $A \in \mathfrak{X}_{n-1}$. But, if $W \in \mathcal{W}_n$ and $W \cap A \neq \emptyset$, then $\mathrm{cl}(W) \subset W(A)$. Thus, since $\mathcal{W}_n$ is locally finite,

$$\mathrm{cl}(V) = \mathrm{cl}(\mathrm{st}(A, \mathcal{W}_n)) \subset W(A).$$

(IV.3) Let K^n denote the union of all simplices of K of dimension less than or equal to n.

We now construct inductively a sequence of maps $\{f_n: |K| \to X\}$ and $\{Q_n\}$, where Q_n is a closed neighborhood of $|K^n| \cup |N|$, such that

(i) $f_{n-1}(|K^{n-1}| \cup |N|) \subset X - A$,

(ii) $f_{n-2}|Q_{n-2} = f_{n-2}$,

(iii) $f_{n-2}(Q_{n-2}) \subset X - A$,

(iv) $Q_{n-2} \subset Q_{n-1}$,

(v) f_{n-1} and g are $\mathfrak{X}_{n-1}$-close, where g is the map constructed between (III.3) and (III.4).

Let $f_0 = g$ (III.4) and assume inductively that $\{f_0, f_1, \ldots, f_{n-1}\}$ and $\{Q_0, Q_1, \ldots, Q_{n-2}\}$ have been constructed. Let Q_{n-1} be a closed polyhedral neighborhood of $|K^{n-1}| \cup |N|$ such that

$$|N| \cup |K^{n-1}| \cup Q_{n-2} \subset Q_{n-1} \subset f_{n-1}^{-1}(X - A).$$

Now apply the Z-set Lemma with $K = K$, $B = Q_{n-1}$, $f = f_{n-1}$, $\mathcal{V} = \mathcal{W}_n$ and $C = K^n$ to obtain $\tilde{f} = f_n$. It is easy to check that (i)–(v) are satisfied. Thus we may assume the existence of the sequences $\{f_n\}$ and $\{Q_n\}$.

If $x \in |K^n|$, then Q_n is a neighborhood of x, thus the $\{f_n\}$ converge to a map

$$f_\infty: |K| \to X - A, \qquad f_\infty | Q_n = f_n | Q_n.$$

Since f_n and g are $\mathfrak{X}_n$-close, f_∞ and g are $\bigcup\{\mathfrak{X}_n | n = 1, 2, 3, \ldots\}$-close. But $\bigcup\{\mathfrak{X}_n | n = 1, 2, 3, \ldots\}$ refines $\mathcal{W}$, thus f_∞ and g are $\mathcal{W}$-close. Note that $f_\infty || N| = f_0 || N| = g || N|$. Thus the proposition is proved.

V. **Metric complexes.** In the proof of (I.2) we must deal with metric complexes. This section will introduce metric complexes and prove some important lemmas.

(V.1) Let P be a simplicial complex (not necessarily locally finite) and let $|P|$ be its geometric realization with the usual weak topology. The *barycentric coordinates* $\{b_v | v \text{ a vertex of } P\}$ for P are maps $b_v\colon P \to [0, 1]$ such that, for each $x \in |P|$, $b_v^{-1}(0, 1] =$ open star of v in $P = \operatorname{ost}(v, P)$ $\Sigma_v\, b_v(x) = 1$, $b_v^{-1}(1) = \{v\}$, and $x = \Sigma_v\, b_v(x) \cdot v$. The *barycentric metric* d on P is defined by $d(x, y) = \frac{1}{2}\, \Sigma_v |b_v(x) - b_v(y)|$.

(V.2) The *metric realization,* or *metric complex,* of P is denoted by $|P|_m$ and is the point set $|P|$ with the barycentric metric.

The topologies on $|P|_m$ and $|P|$ are equivalent if and only if P is locally finite. The next lemma will help us determine when maps defined on metric complexes are continuous.

(V.3) LEMMA. *Let Q be a subcomplex of P. Let $q\colon |Q|_m \to F \subset N$ be a map into a bounded convex subset (F) of a normal topological vector space (N, $\| \ \|$). Further, suppose either* (i) *$Q =$ (vertices of P) or* (ii) *P is a subcomplex of $Q_1 * Q_2$ ($*$ denotes join) where $Q = Q_1 \cup Q_2$, $Q_1 \cap Q_2 = \emptyset$. Then the linear extension $\tilde{q}$ of q to all of $|P|_m$ is continuous.*

PROOF. *Case* (i). If $Q =$ (vertices of P), then $\tilde{q}(x) = \tilde{q}(\Sigma\, b_v(x) \cdot v) = \Sigma\, b_v(x) \cdot q(v)$. Then

$$\|\tilde{q}(x) - \tilde{q}(y)\| = \left\|\sum b_v(x) \cdot q(v) - \sum b_v(y) \cdot q(v)\right\|$$
$$\leq \sum |b_v(x) - b_v(y)| \cdot \|q(v)\|.$$

Then $\tilde{q}$ is continuous, because

$$\|\tilde{q}(x) - \tilde{q}(y)\| \leq 2d(x, y) \cdot D,$$

where

$$D = \sup\{\|q(v)\| \,|\, v \text{ a vertex of } P\}.$$

Case (ii). Assume P is a subcomplex of $Q_1 * Q_2$, then, for $x \in |P|_m$, $x = t_x x_1 + (1 - t_x)x_2$, where $x_1 \in Q_1$, $x_2 \in Q_2$, and x_1, x_2, t_x are unique and vary continuously with respect to $x \in |P|_m - |Q_1 \cup Q_2|_m$. Define $\tilde{q}(x) = t_x q(x_1) + (1 - t_x)q(x_2)$. This is clearly continuous for $x \notin |Q_1 \cup Q_2|_m$. Let $x \in |Q_1|_m$ and let $\{y^i\}$ be a sequence in $|P|_m$ converging to x. Then $y^i = t_i y_1^i + (1 - t_i)y_2^i$ and $\{y_1^i\} \to x$, and $\{t_i\} \to 1$. Then

$$\|\tilde{q}(x) - \tilde{q}(y^i)\| \leq \|q(x) - t_i q(y_1^i)\| + |1 - t_i| D,$$

where $D = \sup\{\|q(y)|y \in Q_2\|\}$. Thus $\tilde{q}(y^i)$ converges to $\tilde{q}(x)$ and $\tilde{q}$ is continuous.

(V.4) DEFINITION. A subdivision Q of a simplicial complex P is called a *proper subdivision* if the topology on $|Q|_m$ is the same as $|P|_m$. ($|Q|$ and $|P|$ are the same point sets.)

Not all subdivisions are proper. For example, let

$$P = \{e_i \mid i = 0, 1, 2, \ldots\} \cup \{(e_0, e_i) \mid i = 1, 2, 3, \ldots\}$$

(the cone over countable many points) and let Q be the subdivision obtained by adding, for each i, a new vertex on (e_0, e_i) at $(1 - 1/i)e_0 + e_i/i$. Then $|Q|_m$ does not have the same topology as $|P|_m$. (However, note that $|Q|_m$ is homeomorphic to $|P|_m$.) (See also [6].)

(V.5) LEMMA. *A subdivision Q of P is a proper subdivision if and only if each open star of a vertex in Q is open in $|P|_m$.*

PROOF. If Q is a proper subdivision then each open star of a vertex in Q is open in $|Q|_m$ and therefore open in $|P|_m$. Conversely, assume that each open star of a vertex in Q is open in $|P|_m$. Let x be a point in $|Q| = |P|$. Let s (resp. r) be the simplex of Q (resp. P) which contains x in its interior, then

$$\operatorname{ost}(s, Q) = \bigcap \{\operatorname{ost}(v, Q) \mid v \text{ a vertex of } s\}$$

is open in $\operatorname{ost}(r, P)$. For $0 < t < 1$ and $Y \subset \operatorname{ost}(r, P)$, let $t \cdot Y = \{(1-t)x + ty \mid y \in Y\}$. The collections $\{t \cdot \operatorname{ost}(s, Q)\}$ and $\{t \cdot \operatorname{ost}(r, P)\}$, $0 < t < 1$, form neighborhood bases for x in $|Q|_m$ and $|P|_m$, respectively. Thus, for some t_0, $t_0 \cdot \operatorname{ost}(r, P) \subset \operatorname{ost}(s, Q)$, and therefore, for any t,

$$t_0 t \cdot \operatorname{ost}(s, Q) \subset t_0 t \cdot \operatorname{ost}(r, P) \subset t \cdot \operatorname{ost}(s, Q).$$

Thus $|Q|_m$ and $|P|_m$ have the same topology at x and $|Q|_m = |P|_m$.

(V.6) COROLLARY. *Barycentric subdivisions are proper.*

(V.7) LEMMA. *For each open cover $\mathcal{C}$ of a metric complex $|P|_m$, there is a proper subdivision Q of P such that the simplices of Q refine $\mathcal{C}$.*

PROOF. If s is a simplex of P, let $N_n(s)$ denote the collection of all simplices and their faces in the nth barycentric subdivision of P which have s as a face.

Let $n(s)$ be an integer ($\geqslant 2$) so large that $N(s) \equiv N_{n(s)}(s)$ refines $\mathcal{C}$. Such $n(s)$ exists because s is compact. Pick the $n(s)$ so that if s is a face of r then $n(s) < n(r)$. Let

$$N^i = \bigcup \{N(s) \mid \dim(s) = i\} - \operatorname{int}\left|\bigcup \{N(s) \mid \dim(s) < i\}\right|.$$

[For A, a complex, and X, a set, $A - X = \{s \in A \mid |s| \subset |A| - X\}$ and $A \cap X = \{s \in A \mid |s| \subset |A| \cap X\}$.] Let s^i denote the ith barycentric subdivision of s. Define the subdivision Q as follows: If s is a 1-simplex of P, let $|s| \cap Q$ be $s^{n(s)} \cap |N^1|$

together with $\bigcup\{N(v) \cap s | v \in S\}$. If s is a 2-simplex, let $Q \cap |s| \cap |N^2|$ be $N^2 \cap |s|$. Now each simplex of $N^1 \cap |s|$ is the join of a simplex in $N^1 \cap |N^2| \cap |s|$, and $N^2 \cap |N^1| \cap |s|$ subdivides $N^1 \cap |N^2| \cap |s|$. Thus we may extend $N^1 \cap |\text{bd } s|$ and $N^2 \cap |N^1| \cap |s|$ to $|N^1| \cap |s|$ by joining. Similarly extend by joining the above-defined subdivision of $N^0 \cap (|N^1| \cup |N^2|) \cap |s|$ to all of $|N^0| \cap |s|$. Continue in this manner subdividing each skeleton in order of dimension. If v is a vertex of Q, then v is a vertex of $s^{n(s)} \cap N^{\dim(s)}$, for some s. It can be checked routinely that $\text{ost}(v, Q) \supset \text{ost}(v, P^{n(s)})$ and that thus $\text{ost}(v, Q)$ is open.

VI. **Proof of Theorem** (I.2). The proof of (I.2) follows the same outline as the proof of (I.1).

(VI.1) *Let F be a metrizable locally convex topological vector space, then F can be embedded as a bounded convex subset of some normed TVS and thus F has a convex metric.* By [15, p. 46], F can be linearly embedded as a subspace of a countable product $\Pi\{N_i | i = 1, 2, \ldots\}$ of normed spaces. For each i, let $B_i = \{x \in N_i | \|x\|_i < 1/2^i\}$, where $\| \; \|_i$ is the norm on N_i. Let $f_i\colon N_i \to B_i$ be a radial homeomorphism. Then ΠN_i is homeomorphic to ΠB_i using the $\{f_i\}$. Let d be a metric on ΠB_i defined by $d(\{x_i\}, \{y_i\}) = \Sigma \|x_i - y_i\|_i$. With this metric ΠB_i is naturally isomorphic to a convex subset of $\Sigma_{l_1}(N_i) = \{\{x_i\} | x_i \in N_i, \Sigma \|x_i\|_i < \infty\}$, with the norm $\|\{x_i\}\| = \Sigma \|x_i\|_i$. But the radial homeomorphism between ΠN_i and ΠB_i takes F into the convex subset $F \cap \Pi B_i \subset \Pi B_i$ and thus F can be embedded as a convex subset of the normed space $\Sigma_{l_1}\{N_i\}$.

(VI.2) *Let $X = 0$ be an open subset in F, a bounded convex subset of a normed space* $(N, \| \; \|)$. This can be done by (VI.1) and the open embedding theorem [12, p. 323].

Using the Cover Lemma (II.1) find locally finite open covers $\mathcal{C}$ and $\mathcal{D}$ of X such that

(VI.3) $\mathcal{C} < \text{ch st}(\text{ch st}(\mathcal{C}, \mathcal{C})) < \mathcal{D} < \text{ch st } \mathcal{D} < \mathcal{F}$

and with the additional assumption that

(VI.4) *(Cardinality of $\mathcal{C}$) $\leqslant$ (weight of F) and the nerve, M, of $\mathcal{C}$ is locally finite dimensional.* Such a $\mathcal{C}$ can be found by using a lemma due to Dowker (Lemma 3.3 of [5]) which states, in part, that every locally finite cover of a normal space has a locally finite refinement whose nerve is locally finite-dimensional. Also every open cover of X has a *sub*cover whose cardinality is not more that the weight of X (see [9, Theorem 6, p. 32]) which, in turn, is not more than the weight of F.

(VI.5) As in §III, let M be the nerve of $\mathcal{C}$ and $c\colon X \to |M|_m$ be the barycentric map into the metric realization of the nerve.

(VI.6) Construct $g\colon |M|_m \to X$ as follows:

(i) *Let Y be paracompact with* $\text{weight}(Y) \leqslant \text{weight}(F)$. *If each point of Y has a closed neighborhood which can be closed embedded in F, then Y can be* *closed embedded in F.* This follows immediately from the proof of the closed embedding theorem (Theorem 1 of [13]).

(ii) There is a closed embedding $e\colon |M|_m \to F$. Note that weight$(|M|_m) \leqslant$ weight(F). A basis for $|M|_m$ is the collection of all $1/i$-neighborhoods of points with all rational barycentric coordinates. It is enough to show that the star of each vertex can be closed embedded. Each star is finite dimensional (since $|M|_m$ is locally finite dimensional) and is the cone over a metric subcomplex of one lower dimension. Clearly 0-dimensional stars (points) can be closed embedded. Assume $(n-1)$-dimensional metric stars can be closed embedded in F. Then $(n-1)$-dimensional metric complexes of the appropriate weight can be closed embedded in F by (i), and thus n-dimensional metric stars can be closed embedded in the open cone over F $(= c(F))$ which is homeomorphic to $F \times R$. (See [14, Lemma 1.1].) By [10, Theorem 4.2], $F \cong G \times R$, for some G. Thus

$$F \cong F^\omega \cong (G \times R)^\omega \cong (G \times R)^\omega \times R \cong F \times R.$$

(iii) *There is a homeomorphism* $h\colon X \times F \to X$ *such that* h *is* $\mathcal{C}$*-close to the projection* $p_1\colon X \times F \to X$. (I.e. for each $(x, y) \in X \times F$, there is a $C \in \mathcal{C}$ such that $\{x\} \cup \{h(x, y)\} \in \mathcal{C}$.) This is Corollary 2.3 of [17].

(iv) $g'\colon |M|_m \to X$ be the same function as above (III.4). In particular, g' is the linear extension of a vertex map which takes each vertex $\{C\} \in M$ into $C - A \subset X - A$. By (V.3.i) g' is continuous.

(v) *There is a closed embedding* $g\colon |M|_m \to X$ *such that* g *is* $\mathcal{C}$*-close to* g'. Let g be the composition of $(g', e)\colon |M|_m \to X \times F$ and $h_i X \times F \to X$. (See (ii) and (iii).) It is easy to check that g is a closed embedding. Note that g satisfies

(VI.7) (See (III.5).) For each simplex $|\{C_1, \ldots, C_n\}| \in |M|_m$, $g(|\{C_1, \ldots, C_n\}|) \subset \mathcal{D}$, and

(VI.8)(See (III.6.))For each $x \in X$, $g \circ c(x)$ and x belong to the same element of $\mathcal{D}$.

We prove below in §VII:

(VI.9) Proposition. *Let* $\mathcal{W}$ *be a cover of* x *such that* $\mathcal{W} < \mathcal{D}$ *and, for each* $W \in \mathcal{W}$, $g^{-1}(\mathrm{st}(W, \mathcal{W}))$ *is finite dimensional. (Such covers exist because* g *is a closed embedding and* $|M|_m$ *is locally finite dimensional.) There is a map* f_∞: $|M|_m \to X - A$ *such that*

(a) f_∞ *and* g *are* $\mathcal{W}$*-close, and*

(b) $f_\infty(|M|_m)$ *is closed in* X.

The map promised by (I.2) is $f = f_\infty \circ c\colon X \to X - A$. Note that $f(X) = f_\infty(c(X)) \subset f_\infty(|M|_m)$ which is closed and contained in $X - A$. Thus $\mathrm{cl}(f(X))$ misses A. Also, note that $f(X) = f_\infty \circ c(X)$ and x belong to the same element of st($\mathcal{D}$, $\mathcal{W}$). Thus the straight line homotopy in X joining f to id_X is limited by ch st($\mathcal{D}$, $\mathcal{W}$) $<$ ch st $\mathcal{D} < \mathcal{F}$.

VII.

(VII.1) LEMMA. *Let U be an open subset of F and let h_1 and h_2 be embeddings of a simplex s into F, such that $h_i(\mathrm{int}\ s) \subset U$, $i = 1, 2$ (int = interior), $h_1|\mathrm{bd}\ s = h_2|\mathrm{bd}\ s$ (bd = boundary), and h_1 is homotopic (in U) to h_2 (rel bd s). [A homotopy (in U) rel bd s is a map j: $s \times I \to$ closure U such that $j(\mathrm{int}\ s \times I) \subset U$ and $j(x, t) = h_1(x) = h_2(x)$, for $x \in \mathrm{bd}\ s$.] Then there is a homeomorphism h: $F \to F$ such that $h|F - U =$ identity and $h \circ h_1 = h_2$.*

PROOF. That such a result is true is generally known but it does not seem to appear in the literature. We give here an outline of a proof. The sets $h_1(\mathrm{int}\ s)$ and $h_2(\mathrm{int}\ s)$ are σ-compact closed subsets of U, and thus there is a homeomorphism g: $U \to U \times F$ such that $g(h_i(\mathrm{int}\ s)) \subset U \times \{0\}$ for $i = 1, 2$. This follows from the fact that F has an l_2-factor [12, proof of Lemma 2], and that a set is l_2-deficient if and only if it is F-deficient. (See [4, Theorem 3.1]. This result does not use the theorem of this paper.) The proof is completed by following the outline: (i) Embed the homotopy in $U \times F$ such that the paths of points from near bd s are "small". (ii) Find a "tubular" neighborhood of the embedded homotopy. (iii) Find an ambient homeomorphism of the tubular neighborhood that moves h_1 to h_2, is the identity on the boundary of the tubular neighborhood, and approaches the identity near g(bd U). For details of similar (but harder) procedures, see [13, proof of Theorem 3].

(VII.2) Z-SET LEMMA. *Let M be any simplicial complex and let B be a proper subpolyhedron of M. (I.e., B is a subcomplex of a proper subdivision of M (V.4).) Let A be a Z-set in X, as in (I.2). Let f: $|M|_m \to X$ be a closed embedding such that $f(|B|_m) \subset X - A$. Then, for each open cover $\mathcal{V}$ of X and each n-dimensional ($n = 0, 1, 2, \ldots$) proper subpolyhedron C of M, there is a closed embedding $\tilde{f}$: $|M|_m \to X$ such that $\tilde{f}(|B|_m \cup |C|_m) \subset X - A$, $\tilde{f}\,||B|_m$, and $\tilde{f}$ and f are $\mathcal{V}$-close.*

PROOF. (By induction on n.) The proof follows the outline of the proof of (IV.1). Assume that the lemma is true for $n - 1$. Let $\mathcal{T}$ be an open cover of X such that ch st $\mathcal{T}$ refines $\mathcal{V}$. As in (VI.2), assume that X is open in F, a bounded convex subset of a normed space (N, $\|\ \|$). In particular, each $T \in \mathcal{T}$ must be so small that ch st(T, $\mathcal{T}$) $\subset X$. Let $\tilde{M}$ be a proper subdivision of M such that, for each simplex $s \in \tilde{M}$, $f(\mathrm{st}(s, \tilde{M}))$ refines $\mathcal{T}$. [Use (II.1) and (V.7).] Now apply the lemma with $\mathcal{T}$ and $\tilde{C}^{n-1} = n - 1$ skeleton of the subdivision, $\tilde{C}$, and $\tilde{M}$ induces on C. Thus there is a closed embedding $\hat{f}$: $|M|_m \to X$ such that $\hat{f}(|B| \cup |\tilde{C}^{n-1}|_m) \subset X - A$, $\hat{f}\,||B|_m = f||B|_m$, and $\hat{f}$ and f are $\mathcal{T}$-close. Let s

be an n-simplex of $\widetilde{C}$ not in $\widetilde{B}$. Note that, for some $T(s) \in \mathcal{T}$, $\hat{f}(s) \subset \mathrm{st}(T(s), \mathcal{T})$ and $\hat{f}(\mathrm{bd}\ s) \subset X - A$. Let, for each s, $S(s)$ be an open set such that $\hat{f}(\mathrm{int}\ s) \subset S(s) \subset \mathrm{st}(T(s), \mathcal{T}) - \hat{f}(|B|_m)$, $S(s) \cap \hat{f}(|C|_m \cup |B|_m) = \hat{f}(\mathrm{int}\ s)$, and, for $s \neq s'$, $S(s) \cap S(s') = \emptyset$. (This is possible because $|\widetilde{M}|_m = |M|_m$. For example, use (V.7) and simplicial neighborhoods.) Now, for some polyhedral n-cell $c \subset \mathrm{int}\ s$, $\hat{f}(s - c) \subset X - A$. Now apply (IV.1) with $K = s$, $B = s - (\mathrm{int}\ c)$, $f = \hat{f}|s$, $C = c$, and $\mathcal{V}$ a cover so small that $\mathcal{V}$ refines $\{S(s), X - \hat{f}(c)\}$ and such that if $\hat{f}$ and h_s are $\mathcal{V}$-close then $\hat{f}$ is homotopic to h_s in $S(s)$. We obtain h_s: $s \to c|(s(s)) - A$, an extension of $\hat{f}|\mathrm{bd}\ s$. Note that $h_s|\mathrm{int}\ s$ is homotopic (rel bd s) in $S(s)$ to $\hat{f}|\mathrm{int}\ s$. Also $h_s|\mathrm{int}\ s$ is homotopic (rel bd s) in $S(s) - A$ to a closed embedding $\hat{h}_s$: $s \to \mathrm{cl}(S(s)) - A$. (Use (VI.6.v) with a cover whose elements get small near bd $S(s)$, and note that in a locally convex space any two functions sufficiently close are homotopic.) Now apply (VII.1) to $h_1 = \hat{f}|s$ and $h_2 = \hat{h}_s$ in order to obtain a homeomorphism j_s: cl $S(s) \to$ cl $S(s)$ such that $j_s|\mathrm{bd}\ S(s) =$ identity and $j_s \circ \hat{f}|s = \hat{h}_s$. Now define j: $X \to X$ by $j|S(s) = j_s$, for $s \in \widetilde{C} - \widetilde{B}$, and $j =$ identity, otherwise. Then set $\widetilde{f} = j \circ f$. That $\widetilde{f}$ satisfies the desired conclusions can be seen by using the fact that $S(s) \cap f(|B|_s \cup |C|_s) = \mathrm{int}\ s$.

(VII.3) Proof of Proposition (VI.9). The proof follows very closely the proof in (IV.2). The necessary changes are to replace $|K|$ by $|M|_m$, assume inductively that f_n is a closed embedding, and, instead of starting with $f_0 = g_1$ apply (VII.2) with $M = M$, $B = \emptyset$, $f = g$, $\mathcal{V} = \mathcal{W}_0$ and $C = M^0$ to obtain $\widetilde{f} = f_0$. In getting the polyhedral neighborhood Q_{n-1} use proper subdivision and (V.7). References to $|N|$ can be ignored. The conclusion ($f_\infty(|M|_m)$ is closed in X) follows because, if $W \in \mathcal{W}$, let dimension$(g^{-1}|\mathrm{st}(W, \mathcal{W})) = d$ see (VI.9), then

$$f_\infty(|M|_m) \cap W = f_\infty(g^{-1}(\mathrm{st}\ W)) \cap W \subset f_\infty(|M^d|_m) \cap W = f_d(|M^d|_m) \cap W$$

which is closed in W.

REFERENCES

1. R. D. Anderson, *On topological infinite deficiency*, Michigan Math. J. 14 (1967), 365–383. MR 35 #4893.

2. R. D. Anderson, D. W. Henderson and J. E. West, *Negligible subsets of infinite-dimensional manifolds*, Compositio Math. 21 (1969), 143–150. MR 39 #7630.

3. K. Borsuk, *Theory of retracts*, Monografie Mat., tom 44, PWN, Warsaw, 1967. MR 35 #7306.

4. T. A. Chapman, *Deficiency in infinite-dimensional manifolds*, General Topology and Appl. 1 (1971), 263–272. MR 48 #1259.

5. C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. 69 (1947), 200–242. MR 8, 594.

6. ———, *On affine and euclidean complexes*, Dokl. Akad. Nauk SSSR 128 (1959), 655–656. (Russian) MR 22 #8483.

7. J. Dugundji, *Topolcgy*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.

8. J. Eells, Jr. and N. H. Kuiper, *Homotopy negligible subsets,* Compositio Math. 21 (1969), 155–161. MR 40 #6546.

9. R. Engelking, *Outline of general topology,* PWN, Warsaw, 1965; English transl., North-Holland, Amsterdam; Interscience, New York, 1968. MR 36 #4508; 37 #5836.

10. A. M. Gleason, *Spaces with a compact Lie group of transformations,* Proc. Amer. Math. Soc. 1 (1950), 35–43. MR 11, 497.

11. R. Heisey, *Manifolds modelled on* R^{∞} *or bounded weak-* topologies,* Trans. Amer. Math. Soc. 206 (1975), 295–312.

12. D. W. Henderson, *Corrections and extensions of two papers about infinite-dimensional manifolds,* General Topology and Appl. 1 (1971), 321–327. MR 45 #2754.

13. ———, *Stable classification of infinite-dimensional manifolds by homotopy-type* Invent. Math. 12 (1971), 48–56. MR 44 #7594.

14. ———, *Micro-bundles with infinite-dimensional fibers are trivial,* Invent. Math. 11 (1970), 293–303. MR 43 #8092.

15. J. Kelley, I. Namioka et al., *Linear topological spaces,* University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR 29 #3851.

16. W. K. Mason, *Deficiency in spaces of homeomorphisms* (to appear).

17. R. M. Schori, *Topological stability for infinite-dimensional manifolds,* Compositio Math. 23 (1971), 87–100. MR 44 #4789.

18. H. Torunczyk, *(G, K)-skeleton and absorbing sets in complete metric spaces,* Fund. Math. (to appear).

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853