TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 213, 1975

MÜNTZ-SZÁSZ THEOREM WITH INTEGRAL COEFFICIENTS. II

BY

LE BARON O. FERGUSON[1] AND MANFRED VON GOLITSCHEK

ABSTRACT. The classical Müntz-Szász theorem concerns uniform approximation on [0, 1] by polynomials whose exponents are taken from a sequence of real numbers. Under mild restrictions on the exponents or the interval, the theorem remains valid when the coefficients of the polynomials are taken from the integers.

Let $C[a, b]$ be the continuous real valued functions defined on a closed bounded interval $[a, b]$ and $\|\cdot\|$ the supremum norm on $C[a, b]$ ($\|f\| = \sup\{|f(x)|: a \leq x \leq b\}$). Let $\Lambda = \{\lambda_i\}$ be a sequence of real numbers satisfying $0 < \lambda_1 < \lambda_2 < \ldots$. A Λ-polynomial is a function of the form

$$p(x) = a_0 + \sum_{i=1}^{n} a_i x^{\lambda_i} \tag{1}$$

where the a_i's are any real numbers. One version of the classical Müntz-Szász theorem reads as follows (cf. Müntz [7]).

THEOREM 1. *The Λ-polynomials are dense in $C[0, 1]$ if and only if $\sum_{i=1}^{\infty} \lambda_i^{-1} = \infty$.*

It is also well known that the ordinary polynomials with integer coefficients, i.e. integral polynomials, are dense in the subspace.

$$C_0[0, 1] = \{f \in C[0, 1]: f(0) \text{ and } f(1) \text{ are integers}\}$$

of $C[0, 1]$. This seems to be due originally to Kakeya [6]. For generalizations see Ferguson [2], [3], and Cantor [1].

Thus it is interesting to ask if Theorem 1 remains true for integral Λ-polynomials, i.e. functions of the form (1) where the a_i's are restricted to the ring of rational integers $\{0, \pm 1, \pm 2, \ldots\}$. The answer is yes under certain restrictions on the functions to be approximated, the interval [0, 1], or the sequence of exponents Λ.

For $\alpha > 0$ the map $x \longrightarrow \alpha x$ induces an isometry between $C[0, 1]$ and

Received by the editors July 17, 1974 and, in revised form, February 6, 1975.
AMS (MOS) subject classifications (1970). Primary 41A10, 30A18.
Key words and phrases. Müntz's theorem, Müntz-Szász theorem, polynomials with integral coefficients, approximation by polynomials with integral coefficients.
(1) Research of the first author sponsored in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under AFOSR Grant Number 71-2030.

$C[0, \alpha]$ under which Λ-polynomials correspond to Λ-polynomials. Thus for a given $\alpha > 0$ and sequence Λ the Λ-polynomials are dense in $C[0, 1]$ iff they are dense in $C[0, \alpha]$.

From Theorem 1 we have that $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$ is a necessary condition for the density of the Λ-polynomials, and since the integral Λ-polynomials are a subset of these, the condition is also necessary for the density of the integral Λ-polynomials. This leads to obvious converses for the following theorems.

Clearly, every integral Λ-polynomial takes on integral values at $x = 0$ and $x = 1$. Since the integers form a closed subset of the reals, it is not possible to approximate functions outside of the set $C_0[0, 1]$ by integral Λ-polynomials.

THEOREM 2. *Let $\Lambda = \{\lambda_i\}$ be a sequence of integers satisfying $0 < \lambda_1 < \lambda_2 < \ldots$. If $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$, then the integral Λ-polynomials are dense in $C_0[0, 1]$.*

The proof will follow from a series of lemmas.

LEMMA 1. *For any two positive integers q and s, $q < s$, there exists a polynomial Q_{qs} of the form*

$$Q_{qs}(x) = \sum_{i=q+1}^{s} c_{iqs} x^{\lambda_i}$$

such that

$$A_{qs} = \|x^{\lambda_q} - Q_{qs}(x)\| \leqslant 2 \exp\left(-2\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1}\right) \tag{2}$$

and $Q_{qs}(1) = 1$, where the first equality in (2) *serves to define A_{qs}.*

PROOF. From von Golitschek [5, Lemma 2] there exist real numbers c_i, $q + 1 \leqslant i \leqslant s$, such that

$$\left\| x^{\lambda_q} - \sum_{i=q+1}^{s} c_i x^{\lambda_i} \right\| \leqslant \prod_{i=q+1}^{s} \frac{\lambda_i - \lambda_q}{\lambda_i + \lambda_q} \leqslant \prod_{i=q+1}^{s} \exp\left(\frac{-2\lambda_q}{\lambda_i}\right)$$

where the latter inequality follows from the inequality (applied factorwise) $(1 - x)/(1 + x) \leqslant e^{-2x}$, $x \geqslant 0$, which is proved by elementary methods. Now set

$$Q_{qs}(x) = \sum_{i=q+1}^{s} c_i x^{\lambda_i} + \left(1 - \sum_{i=q+1}^{s} c_i\right) x^{\lambda_s}. \quad \square$$

LEMMA 2. *Let r and s be positive integers, $r < s$. Suppose that $|\Sigma_{i=j}^{s} d_i| < 1$, $r + 1 \leqslant j \leqslant s$, and $\Sigma_{i=r+1}^{s} d_i = 0$. Then setting $p_{rs}(x) = \Sigma_{j=r+1}^{s} d_j x^{\lambda_j}$ we have $\|p_{rs}\| \leqslant (\lambda_s - \lambda_r)/\lambda_r$.*

PROOF. Since $p_{rs}(1) = \Sigma_{j=r+1}^{s} d_j = 0$ by hypothesis, we have

$$p_{rs}(x) = \sum_{\kappa=r+1}^{s} (x^{\lambda_\kappa} - x^{\lambda_{\kappa-1}})\left(\sum_{i=\kappa}^{s} d_i\right)$$

and for all x, $0 \leqslant x \leqslant 1$,

$$|p_{rs}(x)| \leqslant \sum_{\kappa=r+1}^{s} (x^{\lambda_{\kappa-1}} - x^{\lambda_\kappa}) = x^{\lambda_r} - x^{\lambda_s} \leqslant \frac{\lambda_s - \lambda_r}{\lambda_r}. \tag{3}$$

The second inequality in (3) can be established by elementary means. □

Suppose that $\Sigma_{i=1}^{\infty} \lambda_i = \infty$. In the following we will use implicitly the fact that there are infinitely many q such that $\lambda_q < q^{5/4}$. Indeed, if not, then $\lambda_q^{-1} \leqslant q^{-5/4}$ for all but finitely many q which contradicts the assumption $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$.

LEMMA 3. *Let $\Lambda = \{\lambda_i\}$ satisfy the hypotheses of Theorem* 2, $0 < \epsilon < 1/25$, $K > 0$ *and let N be an integer with $N \geqslant 1 + 1/\epsilon^2$ and $\lambda_{N+1} \leqslant (N+1)^{5/4}$. There exist integers r and s such that $N < r < s$ and*

$$\lambda_s \leqslant s^{5/4}, \quad \lambda_s \leqslant (1 + 4\epsilon)\lambda_r, \quad \sum_{i=N}^{s} \lambda_i^{-1} \geqslant K \tag{4}$$

and

$$\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1} > \frac{\epsilon}{12}\sqrt{q} \quad \textit{whenever } N \leqslant q \leqslant r. \tag{5}$$

PROOF. Choose an integer M such that

$$M > N, \quad \lambda_M \leqslant M^{5/4}, \quad \text{and} \quad \sum_{i=N}^{M} \lambda_i^{-1} \geqslant K + 2. \tag{6}$$

Claim 1. There exists an integer s_0, $N < s_0 \leqslant M$, satisfying the following three conditions:

$$\lambda_{s_0} \leqslant s_0^{5/4}, \tag{7}$$

$$\sum_{i=N}^{s_0} \lambda_i^{-1} \geqslant K + 1, \tag{8}$$

$$\lambda_q \sum_{i=q+1}^{s_0} \lambda_i^{-1} + 5\lambda_q/s_0^{1/4} > \sqrt{q} \quad \text{whenever } N \leqslant q < s_0. \tag{9}$$

PROOF OF CLAIM 1. Set

$$Q = \left\{ q \,|\, N \leqslant q < M, \lambda_q \sum_{i=q+1}^{M} \lambda_i^{-1} \leqslant \sqrt{q} \right\}. \tag{10}$$

If Q is empty take $s_0 = M$. Suppose Q is not empty. Define $M^* = \min Q$. Then from (10)

$$\sum_{i=M^*+1}^{M} \lambda_i^{-1} \leqslant \frac{\sqrt{M^*}}{\lambda_{M^*}} \leqslant \frac{1}{\sqrt{M^*}}. \tag{11}$$

Define $s_0 = \max\{q \mid N \leqslant q \leqslant M^*, \lambda_q \leqslant q^{5/4}\}$. By hypothesis this set is not empty and $N < s_0 \leqslant M^* < M$. Since $\lambda_q > q^{5/4}$ whenever $s_0 + 1 \leqslant q \leqslant M^*$ we have

$$\sum_{i=s_0+1}^{M^*} \lambda_i^{-1} < \int_{s_0}^{\infty} \frac{dx}{x^{5/4}} = 4\, s_0^{-1/4}. \tag{12}$$

From (11) and (12) we have

$$\sum_{i=s_0+1}^{M} \lambda_i^{-1} < \frac{4}{s_0^{1/4}} + \frac{1}{\sqrt{M^*}} < \frac{5}{s_0^{1/4}} \leqslant 1. \tag{13}$$

This, together with (6), establishes (8). Inequality (7) follows from the definition of s_0. From the definition of M^* and $s_0 \leqslant M^*$ it follows that $\lambda_q \Sigma_{i=q+1}^{M} \lambda_i^{-1} > \sqrt{q}$ whenever $N \leqslant q < s_0$. Inequality (9) follows from this and (13) which completes the proof of Claim 1.

We next define, by induction, a finite sequence $s_1, s_2, \ldots, s_{\kappa+1}$ satisfying

$$s_{j+1} + [\epsilon s_{j+1}] = s_j \ \text{ or } \ s_j - 1, \quad 0 \leqslant j \leqslant \kappa, \tag{14}$$

and $s_{\kappa+1} \leqslant N < s_\kappa$.

Since $s_j > N > 1 + 1/\epsilon^2$ and $\epsilon < 1/25$ by hypothesis, the sequence $\{s_j\}_{j=0}^{\kappa+1}$ is strictly decreasing. It is also well defined since the left-hand side of (14), as a function of s_{j+1}, decreases by at most 2 when s_{j+1} is decreased by 1.

Claim 2. Let $1 \leqslant k \leqslant \kappa$. If

$$\lambda_{s_j} > (1 + 4\epsilon)\lambda_{s_{j+1}}, \qquad 0 \leqslant j \leqslant k - 1, \tag{15}$$

then

$$\lambda_{s_j} \leqslant s_j^{5/4}, \qquad 0 \leqslant j \leqslant k, \tag{16}$$

and

$$\sum_{i=s_k+1}^{s_0} \lambda_i^{-1} \leqslant \frac{1}{2} \frac{s_k}{\lambda_{s_k}}. \tag{17}$$

PROOF OF CLAIM 2. Inequality (16) holds for $j = 0$ by Claim 1. We proceed by induction. By the induction hypothesis and (15)

$$\lambda_{s_{j+1}} < \frac{\lambda_{s_j}}{1+4\epsilon} < \frac{s_j^{5/4}}{1+4\epsilon}.$$

Hence by (14)

$$\lambda_{s_{j+1}} \leqslant \frac{(1+\epsilon+1/s_{j+1})^{5/4}}{1+4\epsilon} s_{j+1}^{5/4} \leqslant s_{j+1}^{5/4}$$

where the second inequality can be verified by taking logarithms and noting that $(x-1) \geqslant \ln x \geqslant (x-1)/2$ for $1 \leqslant x \leqslant 2$.

Inequality (17) is established as follows. Using (14) we see that

$$\text{(18)} \qquad \sum_{i=s_j+1}^{s_{j-1}} \lambda_i^{-1} \leqslant (s_{j-1} - s_j)\lambda_{s_j}^{-1} \leqslant \left(\epsilon + \frac{1}{s_j}\right) \frac{s_j}{\lambda_{s_j}}, \qquad 1 \leqslant j \leqslant \kappa + 1.$$

Also from (14)

$$s_j \leqslant \left(1 + \epsilon + \frac{1}{s_{j+1}}\right) s_{j+1}, \qquad 0 \leqslant j \leqslant \kappa,$$

hence

$$\text{(19)} \qquad s_j \leqslant \left(1 + \epsilon + \frac{1}{s_k}\right)^{k-j} s_k, \qquad 0 \leqslant j \leqslant k.$$

Iterating on (15) gives

$$\lambda_{s_j} > (1+4\epsilon)^{k-j} \lambda_{s_k}, \qquad 0 \leqslant j \leqslant k.$$

This, together with (19) gives

$$\frac{s_j}{\lambda_{s_j}} \leqslant \left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{k-j} \frac{s_k}{\lambda_{s_k}}, \qquad 0 \leqslant j \leqslant k,$$

and by (18) we have

$$\sum_{i=s_j+1}^{s_{j-1}} \lambda_i^{-1} \leqslant \left(\epsilon + \frac{1}{s_j}\right) \left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{k-j} \frac{s_k}{\lambda_{s_k}}, \qquad 1 \leqslant j \leqslant k.$$

Hence

$$\begin{aligned} \sum_{i=s_k+1}^{s_0} \lambda_i^{-1} &= \sum_{j=k}^{1} \sum_{i=s_j+1}^{s_{j-1}} \lambda_i^{-1} \\ &\leqslant \sum_{j=k}^{1} \left(\epsilon + \frac{1}{s_k}\right) \left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{k-j} \frac{s_k}{\lambda_{s_k}} \\ &\leqslant \frac{s_k}{\lambda_{s_k}} \left(\epsilon + \frac{1}{s_k}\right) \sum_{j=0}^{k-1} \left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{j}. \end{aligned}$$

But $s_k > N > 1/\epsilon^2$ so

$$\sum_{i=s_k+1}^{s_0} \lambda_i^{-1} \leqslant \frac{s_k}{\lambda_{s_k}}(\epsilon+\epsilon^2)\Big/\left(1-\left(\frac{1+\epsilon+\epsilon^2}{1+4\epsilon}\right)\right)$$
$$= \frac{s_k}{\lambda_{s_k}}\epsilon(1+\epsilon)\frac{(1+4\epsilon)}{\epsilon(3-\epsilon)}$$
$$\leqslant \frac{s_k}{\lambda_{s_k}}(1+1/25)\frac{1+4/25}{3-1/25}$$
$$< \frac{1}{2}\frac{s_k}{\lambda_{s_k}}$$

which establishes (17), hence Claim 2.

We have, using (14),

$$\sum_{i=N}^{s_\kappa} \lambda_i^{-1} \leqslant (1+s_\kappa - s_{\kappa+1})\lambda_N^{-1} \leqslant \frac{2+\epsilon s_{\kappa+1}}{N}$$
$$\leqslant (2+\epsilon N)/N = 2/N + \epsilon < 2\epsilon.$$

This, together with (8), shows that (17) does not hold with $k=\kappa$. Thus, by Claim 2, (15) does not hold for $k=\kappa$ and we can define l to be the smallest integer satisfying $0 \leqslant l < \kappa$ and

$$\lambda_{s_l} \leqslant (1+4\epsilon)\lambda_{s_{l+1}} \tag{20}$$

Setting $s=s_l$ and $r=s_{l+1}$, we see that (4) and (5) are satisfied as follows.

If $l=0$ then $\lambda_s \leqslant s^{5/4}$ by (7), $\lambda_s \leqslant (1+4\epsilon)\lambda_r$ by (20), and by (8) we have (4). Otherwise $l \geqslant 1$ and (15) is valid for $0 \leqslant j \leqslant l-1$. From (16), it follows that $\lambda_{s_l} = \lambda_s \leqslant s_l^{5/4} = s^{5/4}$. Also, $\lambda_s \leqslant (1+4\epsilon)\lambda_r$ follows from (20). Finally, from (8) and (17) there follows

$$\sum_{i=N}^{s} \lambda_i^{-1} = \sum_{i=N}^{s_0} \lambda_i^{-1} - \sum_{i=s+1}^{s_0} \lambda_i^{-1} \geqslant K + 1 - \frac{1}{2}\frac{s_l}{\lambda_{s_l}} > K$$

which establishes (4).

To establish (5) we note first that from (14) and the fact that $r > N > \epsilon^{-2}$ we have $s - r \geqslant \epsilon s/2$. Hence, for $N \leqslant q \leqslant r$,

$$\sum_{i=q+1}^{s} \lambda_i^{-1} \geqslant \sum_{i=r+1}^{s} \lambda_i^{-1} \geqslant \frac{s-r}{\lambda_s} \geqslant \frac{\epsilon s}{2\lambda_s}.$$

From (9) and (17)

$$\begin{aligned}\sqrt{q} &< \lambda_q \sum_{i=q+1}^{s_0} \lambda_i^{-1} + 5\lambda_q s_0^{-1/4} \\ &\leqslant \lambda_q \left(\sum_{i=q+1}^{s} \lambda_i^{-1} + \sum_{i=s+1}^{s_0} \lambda_i^{-1} + 5s/\lambda_s \right) \\ &\leqslant \lambda_q \left(\sum_{i=q+1}^{s} \lambda_i^{-1} + \frac{1}{2}\frac{s}{\lambda_s} + 5\frac{s}{\lambda_s} \right) \\ &\leqslant \lambda_q \left(\sum_{i=q+1}^{s} \lambda_i^{-1} \right) \left(1 + \frac{11}{2} \cdot \frac{2}{\epsilon} \right).\end{aligned}$$

Thus, for $N \leqslant q \leqslant r$,

$$\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1} > \sqrt{q}\left(1 + \frac{11}{\epsilon}\right)^{-1} > \sqrt{q}\epsilon/12$$

which establishes (5). □

LEMMA 4. *Let r and s be positive integers, $r < s$, $f \in C_0[0, 1]$, and $f(0) = 0$. Define*

$$E_s(f) = \inf_{a_j \in \mathbf{R}} \left\| f(x) - \sum_{j=1}^{s} a_j x^{\lambda_j} \right\|.$$

Then there exist integers b_j, $1 \leqslant j \leqslant s$, such that

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leqslant 2E_s(f) + \sum_{q=1}^{r} A_{qs} + \frac{\lambda_s - \lambda_r}{\lambda_r} \tag{21}$$

where A_{qs} is defined in (2).

PROOF. By a standard compactness argument there exists a polynomial $\widetilde{P}_s$ of degree s or less such that $\|f - \widetilde{P}_s\| = E_s(f)$. Setting $P_s = \widetilde{P}_s - \widetilde{P}_s(1)x^{\lambda_1} = \Sigma_{j=1}^{s} a_{j0}x^{\lambda_j}$ it is easy to see that

$$\|f - P_s\| \leqslant 2E_s(f) \quad \text{and} \quad P_s(1) = \sum_{j=1}^{s} a_{j0} = 0. \tag{22}$$

We define coefficients b_j and a_{jq} by induction on q. By (21) we have (23) and (24) below when $q = 0$:

$$\left\| f(x) - \sum_{j=1}^{q} b_j x^{\lambda_j} - \sum_{j=q+1}^{s} a_{jq} x^{\lambda_j} \right\| \leqslant 2E_s(f) + \sum_{j=1}^{q} \|x^{\lambda_j} - Q_{js}(x)\| = A_q \tag{23}$$

and

$$\sum_{j=1}^{q} b_j + \sum_{j=q+1}^{s} a_{jq} = 0, \tag{24}$$

where the equality in (23) serves to define A_q.

To describe the induction step we assume (23) and (24) hold. Define $b_{q+1} = [a_{q+1,q}]$ and $a_{j,q+1} = a_{jq} + (a_{q+1,q} - b_{q+1})c_{j,q+1,s}$ $(q + 2 \leqslant j \leqslant s)$ where $c_{j,q+1,s}$ are the coefficients of the polynomial $Q_{q+1,s}$ in Lemma 1. Then

$$\left\| f(x) - \sum_{j=1}^{q+1} b_j x^{\lambda_j} - \sum_{j=q+2}^{s} a_{j,q+1} x^{\lambda_j} \right\|$$

$$= \left\| f(x) - \sum_{j=1}^{q} b_j x^{\lambda_j} - \sum_{j=q+1}^{s} a_{jq} x^{\lambda_j} - (a_{q+1,q} - b_{q+1})(Q_{q+1,s}(x) - x^{\lambda_{q+1}}) \right\|$$

$$\leqslant A_q + \| Q_{q+1,s}(x) - x^{\lambda_{q+1}} \|,$$

and

$$\sum_{j=1}^{q+1} b_j + \sum_{j=q+2}^{s} a_{j,q+2} = \sum_{j=1}^{q+1} b_j + \sum_{j=q+2}^{s} (a_{jq} + (a_{q+1,q} - b_{q+1})c_{j,q+1,s})$$

$$= b_{q+1} - a_{q+1,q} + (a_{q+1,q} - b_{q+1}) \sum_{j=q+2}^{s} c_{j,q+1,s} = 0$$

since $Q_{q+1,s}(1) = \Sigma_{j=q+2}^{s} c_{j,q+1,s} = 1$. Thus (23) and (24) hold for $q + 1$ in place of q for this definition of b_{q+1} and $a_{j,q+1}$ $(q + 2 \leqslant j \leqslant s)$.

We stop the above induction at $q = r$ and proceed differently to define $b_{r+1}, \ldots, b_s$. Thus we have

$$\left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=r+1}^{s} a_{jr} x^{\lambda_j} \right\| \leqslant 2E_s(f) + \sum_{j=1}^{r} \| x^{\lambda_j} - Q_{js}(x) \| = A_r \tag{25}$$

and

$$\sum_{j=1}^{r} b_j + \sum_{j=r+1}^{s} a_{jr} = 0. \tag{26}$$

Define, recursively, for $j = s, s - 1, \ldots, r + 1$, $d_s = a_{sr} - [a_{sr}]$ and

$$d_j = \begin{cases} a_{jr} - [a_{jr}] & \text{if } \sum_{i=j+1}^{s} d_j \leqslant 0 \\ a_{jr} - [a_{jr}] - 1 & \text{if } \sum_{i=j+1}^{s} d_i > 0 \end{cases} \qquad (s - 1 \geqslant j \geqslant r + 1). \tag{27}$$

Then the d_j's satisfy the inequality in the hypotheses of Lemma 2. Also, from (27), $d_i \equiv a_{ir}$ (mod 1) $(r + 1 \leqslant i \leqslant s)$ so $\Sigma_{i=r+1}^{s} d_i \equiv \Sigma_{i=r+1}^{s} a_{ir}$ (mod 1). But, by (26), $\Sigma_{i=r+1}^{s} a_{ir} \equiv 0 \pmod 1$ and since $|\Sigma_{i=r+1}^{s} d_i| < 1$ we have $\Sigma_{i=r+1}^{s} d_i = 0$. Define $b_j = a_{jr} - d_j$ $(r+1 \leq j \leq s)$ and $p_{rs}(x) = \Sigma_{j=r+1}^{s} d_j x^{\lambda_j}$. The polynomial p_{rs} satisfies the hypotheses of Lemma 2. Hence $\|p_{rs}\| \leq (\lambda_s - \lambda_r)/\lambda_r$. Thus

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| = \left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=r+1}^{s} a_{jr} x^{\lambda_j} + p_{rs}(x) \right\|$$

$$\leq A_r + \|p_{rs}\| \leq A_r + (\lambda_s - \lambda_r)/\lambda_r. \quad \square$$

Proof of Theorem 2. Let $f \in C_0[0, 1]$. Since it suffices to approximate $f - f(1)x^{\lambda_1} - f(0)(1 - x^{\lambda_1})$, we can assume that $f(0) = 0 = f(1)$. Let $0 < \epsilon < 1/25$. By the classical Müntz theorem $E_i(f) \longrightarrow 0$ as $i \longrightarrow \infty$. Also $\lambda_i < i^{5/4}$ for infinitely many i or else we would have $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$. Thus there exists an integer N such that $E_N(f) \leq \epsilon$, $N \geq 4!(6/\epsilon)^5$, and $\lambda_N < N^{5/4}$. Choose $K > 0$ such that $\exp(-2K) \leq \epsilon$. By Lemma 3 there exist integers r and s, $N < r < s$, such that (4) and (5) hold. Applying Lemma 4 to these integers r and s we see that there exist integers b_j $(1 \leq j \leq s)$ such that (21) holds. We estimate the right-hand side of (21) as follows:

$$2E_s(f) \leq 2E_N(f) \leq 2\epsilon,$$

$$(\lambda_s - \lambda_r)/\lambda_r \leq 4\epsilon \quad \text{(using (4))},$$

$$\sum_{q=1}^{N-1} A_{qs} \leq 2 \sum_{q=1}^{N-1} \exp\left(-2\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1}\right) \quad \text{(using Lemma 1)}$$

$$\leq 2 \sum_{q=1}^{N-1} \exp(-2\lambda_q K) \quad \text{(using (4))}$$

$$\leq 2 \sum_{q=1}^{N-1} \epsilon^{\lambda_q} < 3\epsilon,$$

$$\sum_{q=N}^{r} A_{qs} \leq 2 \sum_{q=N}^{r} \exp\left(-2\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1}\right) \quad \text{(using Lemma 1)}$$

$$\leq 2 \sum_{q=N}^{r} e^{-\epsilon\sqrt{q}/6} \quad \text{(using (5))}$$

$$< 2 \sum_{q=N}^{r} 4! \left(\frac{6}{\epsilon}\right)^4 \frac{1}{q^2}$$

$$< 4 \cdot 4! \left(\frac{6}{\epsilon}\right)^4 \frac{1}{N} < \epsilon.$$

Thus (21) gives

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leq 2\epsilon + (3\epsilon + \epsilon) + 4\epsilon = 10\epsilon. \quad \square$$

Results similar to the above can be established more simply under certain conditions as follows. A preliminary version of these results appeared in Ferguson [4].

Let Λ be any subset of the positive real numbers.

THEOREM 3. *If the set* Λ *has a limit point* x_0 *with* $0 < x_0 < \infty$ *then the integral* Λ*-polynomials are dense in* $C_0[0, 1]$.

PROOF. Let $f \in C_0[0, 1]$, $\epsilon > 0$, and $\lambda \in \Lambda$. Since $f(0)$ and $f(1)$ are integers, it suffices to approximate $f - f(0) - (f(1) - f(0))x^{\lambda}$, and we assume without loss of generality that $f(0) = f(1) = 0$. Since x_0 is a positive limit point of Λ, it is easy to see that we can extract from Λ a sequence $\{\lambda_i\}$ satisfying

$$\lambda_i \longrightarrow x_0, \tag{1}$$

$$\lambda_i \text{ is monotone,} \tag{2}$$

$$\lambda_i > 1, \quad \text{all } i, \tag{3}$$

or

$$\lambda_i < 1, \quad \text{all } i, \tag{4}$$

and

$$|\lambda_1 - \lambda_j|/\lambda_k < \epsilon, \quad \text{all } j, k. \tag{5}$$

Since $x_0 > 0$ we have $\Sigma_{j=1}^{\infty} \lambda_j/(1 + \lambda_j^2) = \infty$; hence (cf. Paley-Weiner [8, Theorem XV]) there is a Λ-polynomial p_0 where $p_0(x) = a + \Sigma_{j=1}^{n} b_j x^{\lambda_j}$ with

$$\|f - p_0\| < \epsilon \tag{6}$$

and a constant. By (1), since $f(0) = 0$, $|a| < \epsilon$, hence

$$\|p_0 - p_1\| < \epsilon \tag{7}$$

where $p_1 = p_0 - a$. It is easy to see that we can write p_1 in the form $p_1(x) = cx^{\lambda_1} + \Sigma_{j=2}^{n} a_j(x^{\lambda_j} - x^{\lambda_{j-1}})$. From (6), (7) and $f(1) = 0$, $|c| < 2\epsilon$, and we have

$$\|p_1 - p_2\| < 2\epsilon \tag{8}$$

where $p_2 = p_1 - cx^{\lambda_1}$. Define an integral Λ-polynomial $[p_2]$ by $[p_2](x) = \Sigma_{j=2}^{n} [a_j](x^{\lambda_j} - x^{\lambda_{j-1}})$ where $[a_j]$ denotes the greatest integer less than or equal to a_j. Then

$$
\begin{aligned}
|p_2(x) - [p_2](x)| &= \left| \sum_{j=2}^{n} (a_j)(x^{\lambda_j} - x^{\lambda_{j-1}}) \right| \\
&\leq \sum_{j=2}^{n} (a_j)|x^{\lambda_j} - x^{\lambda_{j-1}}| \\
&\leq \sum_{j=2}^{n} |x^{\lambda_j} - x^{\lambda_{j-1}}| = \left| \sum_{j=2}^{n} (x^{\lambda_j} - x^{\lambda_{j-1}}) \right| \\
&= |x^{\lambda_n} - x^{\lambda_1}|
\end{aligned} \tag{9}
$$

where the second equality follows from the monotonicity of the numbers x^{λ_i} as i increases. This monotonicity in turn follows from the properties (2), (3) and (4) of the sequence $\{\lambda_i\}$ and well-known results concerning exponentiation.

An elementary analysis shows that $|x^{\lambda_n} - x^{\lambda_1}| \leq |\lambda_1 - \lambda_n|/\min\{\lambda_1, \lambda_n\}$; hence by (9) and (5)

$$\|p_2 - [p_2]\| \leq \epsilon. \tag{10}$$

From (6), (7), (8) and (10), $\|f - [p_2]\| < 5\epsilon$. □

Another direction in which the above results can be extended is the following. Let $C_0[0, \alpha]$, $\alpha < 1$, denote the real valued continuous functions on the interval $[0, \alpha]$ which take on integer values at 0, and $\|\cdot\|$ the supremum norm on $C_0[0, \alpha]$.

THEOREM 4. *Let Λ be a subset of the positive real numbers with no finite limit point and $\Sigma_{\lambda\in\Lambda}\lambda^{-1} = \infty$. Then the integral Λ-polynomials are dense in $C_0[0, \alpha]$ for any $\alpha < 1$.*

PROOF. Let $f \in C_0[0, \alpha]$ and $\epsilon > 0$. Since Λ has no finite limit points, there are only finitely many λ's in any bounded interval and we can assume without loss of generality that $\alpha^{\lambda} < \epsilon$, all $\lambda \in \Lambda$. Next extract from Λ a sequence $\{\lambda_i\}$ which is monotone increasing and satisfies $\Sigma_i\lambda_i^{-1} = \infty$, hence $\Sigma_i\lambda_i/(1 + \lambda_i^2) = \infty$. Proceeding as in the proof of Theorem 3 above we construct a Λ-polynomial p_1 satisfying

$$\|f - p_1\| < 2\epsilon. \tag{11}$$

Then

$$
\begin{aligned}
\|p_1 - [p_1]\| &\leq \|x^{\lambda_1}\| + \|x^{\lambda_n} - x^{\lambda_1}\| \\
&\leq 2\alpha^{\lambda_1} + \alpha^{\lambda_n} \\
&< 3\epsilon.
\end{aligned}
$$

This and (11) gives $\|f - [p_1]\| < 5\epsilon$ by the triangle inequality. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92502

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT WÜRZBURG, FEDERAL REPUBLIC OF GERMANY

Current address: Stephan Banach International Mathematical Center, P.O.B. 137, 00–950 Warsaw, Poland