

THE GAUSSIAN LAW AND THE
LAW OF THE ITERATED LOGARITHM FOR
LACUNARY SETS OF CHARACTERS

BY

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ABSTRACT. Salem and Zygmund showed that the Gaussian law holds for Hadamard sequences of real numbers while Mary Weiss proved a similar result for the law of the iterated logarithm. In the present paper, the author obtains corresponding results for lacunary sets of characters of an arbitrary infinite compact abelian group. It is shown that the laws are best satisfied for a certain class of lacunary sets but that modified results apply to more general classes.

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1.1. Edwards, Hewitt and Ross [1] conjecture that general classes of lacunary sets of characters of a compact abelian group exhibit properties similar to those known to hold for Hadamard sets of integers. Such a property which has been shown to hold in the abstract setting is the Fatou-Zygmund property (see Edwards, Hewitt and Ross [2] and Drury [1]). Two theorems describing the behaviour of Hadamard sequences are analogous to well-known theorems in probability theory dealing with sequences of independent random variables. These are the Gaussian law or Liapounov's theorem (see Gnedenko [1, §42]) and the law of the iterated logarithm or Kolmogorov's theorem (see Kolmogorov [1]). In this paper we demonstrate that both theorems hold for certain subclasses of Stečkin sets of characters of an arbitrary infinite abelian group. R. C. Baker [1] has produced results in connection with the Gaussian law. We clarify and extend these results. We begin by stating the theorems we will be concerned with as they apply to sequences of independent random variables on a probability space.

1.2. **LIAPOUNOV'S THEOREM (THE GAUSSIAN LAW).** *Let P be a probability measure on a probability space Ω and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of bounded*

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independent random variables such that $\int_{\Omega} f_k dP = 0$ for each $k = 1, 2, \dots$. We set $S_n = \sum_{k=1}^n f_k$, $B_n = (\sum_{k=1}^n \int_{\Omega} f_k^2 dP)^{1/2}$ and $M_n = \max_{k=1,2,\dots,n} \|f_k\|_{\infty}$. Then if it is true that (i) $B_n \rightarrow \infty$ as $n \rightarrow \infty$, and (ii) $M_n B_n^{-1} = o(1)$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} P\{t: S_n(t) < yB_n\} = (2\pi)^{-1/2} \int_{-\infty}^y \exp(-t^2/2) dt$$

for all real numbers y .

The function on the right side of the above equality is called the normal distribution with variance 1 and will be denoted by the symbol $G_1(y)$. The theorem stated is less general than that appearing in Gnedenko but is the most appropriate formulation for our purposes.

1.3. KOLMOGOROV'S THEOREM (THE LAW OF THE ITERATED LOGARITHM).

Let $\Omega, P, \{f_k\}_{k=1}^{\infty}, S_n, B_n$ and M_n be as in 1.2. Then if it is true that (i) $B_n \rightarrow \infty$, and (ii) $M_n B_n^{-1} (\log \log B_n)^{1/2} = o(1)$ as $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} S_n (2B_n^2 \log \log B_n)^{-1/2} = 1 \quad a.e.$$

1.4. Salem and Zygmund [1], [2] showed that the conclusion of 1.2 holds for $\Omega = T$, the circle group, for P , normalized Haar measure on T and $\{f_k\}_{k=1}^{\infty}$, a sequence of trigonometric functions defined by each $t \in T$ by

$$f_k(t) = a_k \cos(n_k t) + b_k \sin(n_k t)$$

where a_k and b_k are real numbers and $\{n_k\}_{k=1}^{\infty}$ is a Hadamard sequence of positive integers (that is, $n_{k+1}/n_k \geq q$ for some real number $q > 1$ and all k). Note that the f_k 's are not independent. After preliminary results were obtained by Salem and Zygmund [3] and Erdős and Gál [1], the corresponding theorem 1.3 was proved by Mary Weiss [1].

1.5. We now give a formulation of the abstract problem. Throughout this paper, unless otherwise specified, the symbol G will be used to denote an arbitrary infinite compact abelian group. λ will denote normalized Haar measure on G while the character group of G , which will be taken to be a multiplicative group, will be denoted by \hat{G} . Let $\Upsilon = \{X_n\}_{n=1}^{\infty}$ be a sequence of nonempty, pairwise disjoint, finite subsets of \hat{G} which are symmetric (that is, $\chi \in X_n$ implies that $\chi^{-1} \in X_n$). Suppose that, for a given countably infinite, symmetric subset Q of $\bigcup_{n=1}^{\infty} X_n$, there is defined a complex-valued function u on Q which is symmetric (that is, $u(\chi^{-1}) = \overline{u(\chi)}$). We put $Q_n = Q \cap X_n$ and set

$$S_n = \sum_{k=1}^n \sum_{\chi \in Q_k} u(\chi)\chi, \quad B_n = \left(\sum_{k=1}^n \sum_{\chi \in Q_k} |u(\chi)|^2 \right)^{1/2},$$

and

$$M_n = \max_{k=1,2,\dots,n} \max_{\chi \in Q_k} |u(\chi)|.$$

Obviously S_n, B_n and M_n depend on γ, Q and u but in most instances there will be no need to indicate this explicitly.

We investigate the behaviour of quadruples (G, γ, Q, u) in relation to the Gaussian law and the law of the iterated logarithm. Specifically, we suppose that R is a lacunary property of subsets of \hat{G} and we set

$$H_{R,G} = \{(G, \gamma, Q, u): Q = P \cup P^{-1}, P \text{ has property } R\}$$

where $P^{-1} = \{\chi^{-1}: \chi \in P\}$. It is then natural to ask for what properties R do the conclusions of 1.2 and 1.3 hold in the case when $\Omega = G, P = \lambda, (G, \gamma, Q, u) \in H_{R,G}$ and $f_k = \sum_{\chi \in Q_k} u(\chi)\chi$. We will show that the conclusion of 1.2 holds for a certain subclass of Stečkin sets but fails for other such sets and a modified law of the iterated logarithm holds to varying degrees for different subclasses of Stečkin sets.

The sets examined are defined in §2 and certain properties are proved which are intimately connected with the quasi independence of the sets in question. In §3, we prove the Gaussian law for a certain lacunary property and show that it fails to hold for other properties. In §4, we introduce parts of the theory of dyadic representations of positive integers to obtain an estimate of the L_p norm of functions each of whose values at a given point is the supremum of the values of a finite number of trigonometric polynomials. Finally in §5, we utilize the results obtained in §§2 and 4 to prove a law of the iterated logarithm for lacunary sets of characters.

1.6. Throughout this paper, we denote the set of integers by Z , the positive integers by N , the real numbers by R and the complex numbers by K . We use the symbols $[a, b],]a, b[, [a, b[$ and $]a, b]$ for $a, b \in R$ and $a \leq b$ to denote the appropriate closed interval, open interval and half open intervals of real numbers. For certain subsections which will be indicated, we will use this notation to denote intervals of integers. For S any set, we write $\text{card } S$ for the cardinality of S . For f , a real-valued function, and $a \in R$, we write $\{f > a\}$ for the set $\{t: f(t) > a\}$. For f , a nonzero complex-valued function, we denote by f^{-1} the function $f^{-1}: t \rightarrow (f(t))^{-1}$. Note that for $\chi \in \hat{G}$, χ^{-1} is the group inverse of χ . For $P \subset G$, we set $P^{-1} = \{\chi^{-1}: \chi \in P\}$. The end of a proof will be denoted by the symbol \square .

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2.1. We denote the identity element of \hat{G} by the symbol 1. For each $P \subset \hat{G}$, we set

$$P(2) = \{\chi \in P: \chi^2 = 1\} \quad \text{and} \quad P(>2) = \{\chi \in P: \chi^2 \neq 1\}.$$

The definitions which follow stem in part from Stečkin's work [1] and are in part similar to those appearing in §10 of Edwards, Hewitt and Ross [2]. For $n \in \mathbb{N}$ and $P \subset \hat{G}$, we let $S_2(P, n)$ be the set of all functions g defined on P which take values in $\{-2, -1, 0, 1, 2\}$ and have the properties that $\sum_{\chi \in P} |g(\chi)| = n$ and $g(\chi) \in \{-1, 0, 1\}$ for all $\chi \in P(2)$. We let $S_1(P, n)$ consist of all functions in $S_2(P, n)$ which take values in $\{-1, 0, 1\}$. For $\phi \in \hat{G}$ and $j = 1$ or 2 , we set

$$S_j(P, n, \phi) = \left\{ g \in S_j(P, n) : \prod_{\chi \in P} \chi^{g(\chi)} = \phi \right\}.$$

DEFINITIONS. Let P be a nonempty subset of G such that $P(>2) \cap P(>2)^{-1} = \emptyset$ and let $j = 1$ or 2 . Then we say that P has *property* (R_j) if there exists a positive number ω such that $\text{card } S_j(P, n, 1) \leq \omega^n$ for all $n \in \mathbb{N}$. The set of all $\omega \geq 1$ which have this property clearly has a minimum element called the R_j -*bound* for P . We will say that P is a j -*dissociate set* if, for each $n \in \mathbb{N}$, the set $S_j(P, n, 1)$ is empty. If P is the finite union of sets, each of which has the property (R_j) , then we will say that P is a j -*Stečkin set*.

2.2. COMMENT. Since we have $S_1(P, n, 1) \subset S_2(P, n, 1)$ for all $n \in \mathbb{N}$, it is clear that a set having property (R_2) has property (R_1) , that a 2-dissociate set is a 1-dissociate set and that a 2-Stečkin set is a 1-Stečkin set. However a 1-dissociate set may fail to be a 2-Stečkin set. An example of such a set will be given in 3.3.

2.3. The next two lemmas yield the properties critical for the theorems that follow.

LEMMA. Suppose that $P \subset \hat{G}$ has the property (R_2) with R_2 -bound equal to ω and let ξ be any positive number less than ω^{-1} . Further let V be a non-empty finite subset of P and f a complex-valued function defined on $V \times \{-2, -1, 0, 1, 2\}$ such that:

- (i) if $\chi \in V(2)$, then $f(\chi, k)$ is nonzero only if $k \in \{-1, 0, 1\}$; and
- (ii) for all $\chi \in V$ and $k \in \{-2, -1, 0, 1, 2\}$, $|f(\chi, k)| \leq \xi^{|k|} |f(\chi, 0)|$. Then

we have

$$(iii) \quad \left| \int_G \prod_{\chi \in V} \left(\sum_{k=-2}^2 f(\chi, k) \chi^k \right) d\lambda - \prod_{\chi \in V} f(\chi, 0) \right| \leq \xi \omega (1 - \xi \omega)^{-1} \prod_{\chi \in V} |f(\chi, 0)|.$$

PROOF. For convenience, set $A = \{-2, -1, 0, 1, 2\}$. We let B be the set of all nonzero functions from V into A and we set

$$C = \left\{ g \in B : \prod_{\chi \in V} \chi^{g(\chi)} = 1 \right\}.$$

It is clear that

$$(2.1) \quad \prod_{\chi \in V} \left(\sum_{k \in A} f(\chi, k) \chi^k \right) = \prod_{\chi \in V} f(\chi, 0) + \sum_{g \in B} \prod_{\chi \in V} f(\chi, g(\chi)) \prod_{\chi \in V} \chi^{g(\chi)}.$$

Recalling that $\int_G \chi d\lambda = \delta_{\chi 1}$ and integrating both sides of (2.1), we obtain

$$\int_G \prod_{\chi \in V} \left(\sum_{k \in A} f(\chi, k) \chi^k \right) d\lambda = \prod_{\chi \in V} f(\chi, 0) + \sum_{g \in C} \prod_{\chi \in V} f(\chi, g(\chi)).$$

To prove (iii), it is sufficient to show that

$$(2.2) \quad \sum_{g \in C} \prod_{\chi \in V} |f(\chi, g(\chi))| \leq \xi \omega (1 - \xi \omega)^{-1} \prod_{\chi \in V} |f(\chi, 0)|.$$

By condition (i), a summand in the left side of (2.2) is nonzero only if $g(\chi) \in \{-1, 0, 1\}$ for $\chi \in V(2)$. A function $g \in C$ which has this property belongs to $S_2(V, \sum_{\chi \in V} g(\chi), 1)$. Noting condition (ii) and the fact that $\sum_{\chi \in V} |g(\chi)| = n$ for $g \in S_2(V, n, 1)$, we obtain

$$\begin{aligned} \sum_{g \in C} \prod_{\chi \in V} |f(\chi, g(\chi))| &\leq \sum_{n=1}^{\infty} \sum_{g \in S_2(V, n, 1)} \prod_{\chi \in V} |f(\chi, g(\chi))| \\ &\leq \prod_{\chi \in V} |f(\chi, 0)| \sum_{n=1}^{\infty} \sum_{g \in S_2(V, n, 1)} \xi^n \\ &= \prod_{\chi \in V} |f(\chi, 0)| \sum_{n=1}^{\infty} \text{card } S_2(V, n, 1) \xi^n. \end{aligned}$$

Since we have $S_2(V, n, 1) \leq \text{card } S_2(P, n, 1) \leq \omega^n$, an easy calculation yields (2.2). \square

2.4. A similar argument yields the following:

LEMMA. Suppose that $P \subset \hat{G}$ has the property (R_1) with R_1 -bound equal to ω and let ξ be a positive number less than ω^{-1} . Further let V be a finite subset of P and f a complex-valued function defined on $V \times \{-1, 0, 1\}$ such that for all $\chi \in V$ and $k \in \{-1, 0, 1\}$, we have

$$(2.3) \quad |f(\chi, k)| \leq \xi^{|k|} |f(\chi, 0)|.$$

Then we have the following estimate:

$$\begin{aligned} \left| \int_G \prod_{\chi \in V} \left(\sum_{k \in \{-1, 0, 1\}} f(\chi, k) \chi^k \right) d\lambda - \prod_{\chi \in V} f(\chi, 0) \right| \\ \leq \xi \omega (1 - \xi \omega)^{-1} \prod_{\chi \in V} |f(\chi, 0)|. \end{aligned}$$

2.5. For the remainder of this section, we will consider nonempty sets $P \subset \hat{G}$ for which $P(> 2) \cap P(> 2)^{-1} = \emptyset$. We let V be any nonempty finite subset of P and we put $W = V \cup V^{-1}$. Furthermore we let u be a symmetric function defined on W and we set

$$S = \sum_{\chi \in W} u(\chi)\chi, \quad B = \left(\sum_{\chi \in W} |u(\chi)|^2 \right)^{1/2} \quad \text{and} \quad M = \max_{\chi \in W} |u(\chi)|.$$

There will be no necessity to indicate explicitly the dependence of S , B and M on W and n . We show that sets having the properties (R_1) or (R_2) satisfy certain integral estimates critical to the proof that such sets have the iterated logarithm property and the Gaussian property.

2.6. THEOREM. (a) *Suppose that P has property (R_2) with R_2 -bound equal to ω . Let ϵ be any positive number less than 12 and suppose that θ is any complex number such that*

$$(2.4) \quad 48\omega|\theta|M < \epsilon.$$

Then the following is true:

$$(2.5) \quad \int_G \exp(\theta S) d\lambda = (1 + \eta_1) \exp(\frac{1}{2}\theta^2 B^2(1 + \eta_2)) + \eta_3 |\theta|^2 B^2 \exp(3|\theta|^2 B^2)$$

where η_1, η_2 and η_3 are complex numbers each of which has absolute value at most equal to ϵ .

(b) *If, in addition, θ is positive, then we have*

$$\int_G \exp(\theta S) d\lambda = (1 + \eta_1) \exp(\frac{1}{2}\theta^2 B^2(1 + \eta_2))$$

where η_1 and η_2 are real numbers, each of which has absolute value at most equal to ϵ .

2.7. THEOREM. *Suppose that P has property (R_1) with R_1 -bound equal to ω . Let ϵ be any positive number less than 1 and suppose that θ is a positive number such that $4\omega\theta M < \epsilon$. We have*

$$(1 - \epsilon) \leq \int_G \exp(\theta S) d\lambda \leq (1 + \epsilon) \exp((1 + \epsilon)\theta^2 B^2).$$

2.8. PROOF OF 2.6(a). For $|z| < \frac{1}{2}$, we have, from the power series of the log function,

$$(2.6) \quad \exp z = (1 + z + \frac{1}{2}z^2) \exp h(z) \quad \text{where} \quad |h(z)| \leq 3|z|^3.$$

Noting that $2|\theta| |u(\chi)| < 2|\theta|M < \epsilon(24\omega)^{-1} < \frac{1}{2}$, we write

$$(2.7) \quad \begin{aligned} \exp(\theta S) &= \prod_{\chi \in V(>2)} \exp(\theta u(\chi)\chi) \prod_{\chi \in V(>2)} \exp(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}) \\ &= \Delta_{(2)} \Delta_{(>2)} \Phi_{(2)} \Phi_{(>2)} \end{aligned}$$

where

$$\begin{aligned} \Delta_{(2)} &= \prod_{\chi \in V(2)} (1 + \theta u(\chi)\chi + \frac{1}{2}\theta^2(u(\chi))^2), \\ \Delta_{(>2)} &= \prod_{\chi \in V(>2)} (1 + \theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1} + \frac{1}{2}(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1})^2), \\ \Phi_{(2)} &= \prod_{\chi \in V(2)} \exp h(\theta u(\chi)\chi) \end{aligned}$$

and

$$(2.8) \quad \Phi_{(>2)} = \prod_{\chi \in V(>2)} \exp h(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}).$$

From (2.6), we conclude that

$$\begin{aligned} &\left| \sum_{\chi \in V(2)} h(\theta u(\chi)\chi) + \sum_{\chi \in V(>2)} h(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}) \right| \\ &\leq \sum_{\chi \in V(2)} 3|\theta u(\chi)\chi|^3 + \sum_{\chi \in V(>2)} 3|\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}|^3 \\ &\leq 3 \sum_{\chi \in V(2)} |\theta|^3 M |u(\chi)|^2 + 24 \sum_{\chi \in V(>2)} |\theta|^3 M |u(\chi)|^2 \\ &\leq 12|\theta|^3 M \left(\sum_{\chi \in V(2)} |u(\chi)|^2 + 2 \sum_{\chi \in V(>2)} |u(\chi)|^2 \right). \end{aligned}$$

Since we are given that $48|\theta|M < \epsilon$ and since it is readily observed that $B^2 = \sum_{\chi \in V(2)} |u(\chi)|^2 + 2 \sum_{\chi \in V(>2)} |u(\chi)|^2$, we conclude that

$$(2.9) \quad \left| \sum_{\chi \in V(2)} h(\theta u(\chi)\chi) + \sum_{\chi \in V(>2)} h(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}) \right| \leq (\epsilon/4)|\theta|^2 B^2.$$

Since $|\exp z - 1| \leq |z| \exp|z|$ for all z , it follows that

$$(2.10) \quad |\Phi_{(2)}(t)\Phi_{(>2)}(t) - 1| \leq (\epsilon/4)|\theta|^2 B^2 \exp((\epsilon/4)|\theta|^2 B^2) \quad \text{for all } t \in G.$$

We now estimate $\int_G \Delta_{(2)} \Delta_{(>2)} d\lambda$. From (2.8), we write

$$\Delta_{(2)} = \prod_{\chi \in V(2)} (1 + \theta u(\chi)\chi + \frac{1}{2}\theta^2 |u(\chi)|^2)$$

and

$$\begin{aligned} \Delta_{(>2)} &= \prod_{\chi \in V(>2)} (1 + \theta^2 |u(\chi)|^2 + \theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1} \\ &\quad + \frac{1}{2}\theta^2 u(\chi)^2 \chi^2 + \frac{1}{2}\theta^2 u(\chi^{-1})^2 \chi^{-2}) \end{aligned}$$

We estimate $\int_G \Delta_{(2)} \Delta_{(>2)} d\lambda$ by applying 2.3. We set $\xi = 2|\theta|M$ and define f in the appropriate manner. It is clear that condition (i) of the lemma holds for our function f while condition (ii) is seen to hold for our choice of f and ξ once we

observe that

$$|1 + \theta^2 |u(x)|^2| \geq 1 - \theta^2 M^2 > \frac{1}{2} \quad (\text{see (2.4)}).$$

We obtain

$$(2.11) \quad \int_G \Delta_{(2)} \Delta_{(>2)} d\lambda = (1 + \eta_1) \prod_{x \in V(2)} (1 + \frac{1}{2} \theta^2 |u(x)|^2) \\ \times \prod_{x \in V(>2)} (1 + \theta^2 |u(x)|^2)$$

where

$$|\eta_1| \leq \xi \omega (1 - \xi \omega)^{-1} \\ = 2|\theta| M \omega (1 - 2|\theta| M \omega)^{-1} \\ < \epsilon \quad (\text{since } 2|\theta| M \omega < \epsilon/24 < \frac{1}{2} \text{ by (2.4)}).$$

For $|u| < \frac{1}{2}$, we have the formula

$$1 + u = \exp(u + k(u)) \quad \text{where } |k(u)| \leq 2|u|^2.$$

We rewrite (2.11) as

$$(2.12) \quad \int_G \Delta_{(2)} \Delta_{(>2)} d\lambda = (1 + \eta_1) \prod_{x \in V(2)} \exp(\frac{1}{2} \theta^2 |u(x)|^2 + k(\frac{1}{2} \theta^2 |u(x)|^2)) \\ \times \prod_{x \in V(>2)} \exp(\theta^2 |u(x)|^2 + k(\theta^2 |u(x)|^2)) \\ = (1 + \eta_1) \exp(\frac{1}{2} \theta^2 B^2 (1 + \eta_2))$$

where

$$\eta_2 = 2(\theta^2 B^2)^{-1} \left(\sum_{x \in V(2)} k(\frac{1}{2} \theta^2 |u(x)|^2) + \sum_{x \in V(>2)} k(\theta^2 |u(x)|^2) \right).$$

We have

$$|\eta_2| \leq 4(|\theta|^2 B^2)^{-1} \left(\sum_{x \in V(2)} \frac{1}{4} |\theta|^4 |u(x)|^4 + \sum_{x \in V(>2)} |\theta|^4 |u(x)|^4 \right) \\ \leq |\theta|^2 B^{-2} \left(\sum_{x \in V(2)} |u(x)|^4 + 2 \sum_{x \in V(>2)} |u(x)|^4 \right) \\ \leq |\theta|^2 M^2 B^{-2} \left(\sum_{x \in V(2)} |u(x)|^2 + 2 \sum_{x \in V(>2)} |u(x)|^2 \right) \\ = |\theta|^2 M^2 < \epsilon/2.$$

From (2.7), we have

$$\int_G \exp(\theta S) d\lambda = \int_G \Delta_{(2)} \Delta_{(>2)} \Phi_{(2)} \Phi_{(>2)} d\lambda \\ = \int_G \Delta_{(2)} \Delta_{(>2)} d\lambda - \int_G (1 - \Phi_{(2)} \Phi_{(>2)}) \Delta_{(2)} \Delta_{(>2)} d\lambda.$$

In view of (2.10) and (2.12), it remains to show that

$$\int_G |\Delta_{(2)}\Delta_{(>2)}| d\lambda < 4 \exp(3|\theta|^2 B^2).$$

By Hölder's inequality, it is sufficient to show that

$$(2.13) \quad \int_G |\Delta_{(2)}\Delta_{(>2)}|^2 d\lambda < 16 \exp(6|\theta|^2 B^2).$$

Noting that $u(\chi)\chi + u(\chi^{-1})\chi^{-1}$ is real for $\chi \in V(>2)$ and u symmetric, we obtain for $\chi \in V(>2)$ after a routine calculation using the estimate $|\theta| |u(\chi)| \leq |\theta|M < \frac{1}{4}$,

$$\begin{aligned} |1 + \theta(u(\chi)\chi + u(\chi^{-1})\chi^{-1}) + \frac{1}{2}\theta^2(u(\chi)\chi + u(\chi^{-1})\chi^{-1})^2|^2 \\ \leq 1 + (\theta + \bar{\theta})(u(\chi)\chi + u(\chi^{-1})\chi^{-1}) + 12|\theta|^2 |u(\chi)|^2. \end{aligned}$$

A similar calculation yields for $\chi \in V(2)$:

$$|1 + \theta u(\chi)\chi + \frac{1}{2}\theta^2 |u(\chi)|^2|^2 \leq 1 + (\theta + \bar{\theta})u(\chi)\chi + 6|\theta|^2 |u(\chi)|^2.$$

Noting (2.8), we combine these estimates to obtain

$$\begin{aligned} |\Delta_{(2)}\Delta_{(>2)}|^2 &\leq \prod_{\chi \in V(2)} (1 + 6|\theta|^2 |u(\chi)|^2 + (\theta + \bar{\theta})u(\chi)\chi) \\ &\quad \times \prod_{\chi \in V(>2)} (1 + 12|\theta|^2 |u(\chi)|^2 + (\theta + \bar{\theta})(u(\chi)\chi + u(\chi^{-1})\chi^{-1})). \end{aligned}$$

We apply 2.3 again for $\xi = |\theta|M$ to obtain

$$\begin{aligned} \int_G |\Delta_{(2)}\Delta_{(>2)}|^2 d\lambda &\leq (1 + \eta^1) \prod_{\chi \in V(2)} (1 + 6\theta^2 |u(\chi)|^2) \\ &\quad \times \prod_{\chi \in V(>2)} (1 + 12\theta^2 |u(\chi)|^2) \end{aligned}$$

where η^1 is necessarily real and $|\eta^1| \leq \xi\omega(1 - \xi\omega)^{-1} \leq \omega|\theta|M(1 - \omega|\theta|M)^{-1} < 1$.

Now since $1 + x \leq \exp x$ for all $x > 0$, we have

$$\begin{aligned} \int_G |\Delta_{(2)}\Delta_{(>2)}|^2 d\lambda &\leq 2 \prod_{\chi \in V(2)} \exp(6|\theta|^2 |u(\chi)|^2) \prod_{\chi \in V(>2)} \exp(12|\theta|^2 |u(\chi)|^2) \\ &\leq 2 \exp(6|\theta|^2 B^2). \end{aligned}$$

Thus we have shown (2.13) and our proof is complete. \square

2.9. PROOF OF 2.6(b). We note that, if z is real and $|z| > \frac{1}{2}$, then, in (2.6), the function $h(z)$ is real valued and $1 + z + \frac{1}{2}z^2 > \frac{1}{2} > 0$. It follows from (2.8) and (2.9) that $\exp(-(\epsilon/4)\theta^2 B^2) \leq \Phi_{(2)}\Phi_{(>2)} \leq \exp((\epsilon/4)\theta^2 B^2)$. Since $\Delta_{(2)}$ and $\Delta_{(>2)}$ are greater than 0, we obtain

$$(2.14) \quad \begin{aligned} \exp(-(\epsilon/4)\theta^2 B^2) \int_G \Delta_{(>2)} \Delta_{(>2)} d\lambda &\leq \int_G \Phi_{(>2)} \Phi_{(>2)} \Delta_{(>2)} \Delta_{(>2)} d\lambda \\ &\leq \exp((\epsilon/4)\theta^2 B^2) \int_G \Delta_{(>2)} \Delta_{(>2)} d\lambda. \end{aligned}$$

The desired result is obtained by combining (2.7) with (2.12) and (2.14). \square

2.10. PROOF OF 2.7. For $|z| < \frac{1}{2}$, we have

$$(2.15) \quad \exp z = (1 + z) \exp(\frac{1}{2}(1 + j(z))z^2) \quad \text{where } |j(z)| < |z|.$$

Note that if z is real then so is $j(z)$. Noting that $2\theta|u(\chi)| \leq 2\theta M < \frac{1}{2}$, we write

$$(2.16) \quad \begin{aligned} \exp(\theta S) &= \prod_{\chi \in V(2)} \exp(\theta u(\chi)\chi) \prod_{\chi \in V(>2)} \exp(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}) \\ &= \prod_{\chi \in V(2)} (1 + \theta u(\chi)\chi) \prod_{\chi \in V(>2)} (1 + \theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}) \\ &\times \prod_{\chi \in V(2)} \exp(\frac{1}{2}(1 + j(\theta u(\chi)\chi))\theta^2 u(\chi)^2) \\ &\times \prod_{\chi \in V(>2)} \exp(\frac{1}{2}(1 + j(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1})) \\ &\quad \times \theta^2 (u(\chi)\chi + u(\chi^{-1})\chi^{-1})^2). \end{aligned}$$

Note that $\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}$ for $\chi \in V(>2)$ and $u(\chi)$ for $\chi \in V(2)$ are real valued and have absolute value less than or equal to $2\theta|u(\chi)| \leq 2\theta M$. We use the bound on $j(z)$ given in (2.15) and the fact that $B^2 = \sum_{\chi \in V(2)} |u(\chi)|^2 + 2 \sum_{\chi \in V(>2)} |u(\chi)|^2$ to obtain

$$(2.17) \quad \begin{aligned} 1 &\leq \prod_{\chi \in V(2)} \exp(\frac{1}{2}(1 + j(\theta u(\chi)\chi))\theta^2 u(\chi)^2) \\ &\quad \times \prod_{\chi \in V(>2)} \exp(\frac{1}{2}(1 + j(\theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1})) \\ &\quad \times \theta^2 (u(\chi)\chi + u(\chi^{-1})\chi^{-1})^2) \\ &\leq \prod_{\chi \in V(2)} \exp(\frac{1}{2}(1 + \epsilon)\theta^2 |u(\chi)|^2) \\ &\quad \times \prod_{\chi \in V(>2)} \exp(2(1 + \epsilon)\theta^2 u(\chi)\chi^2) \\ &\leq \exp((1 + \epsilon)\theta^2 B^2). \end{aligned}$$

Since $1 + \theta u(\chi)\chi \geq \frac{3}{4} > 0$ for $\chi \in V(2)$ and $1 + \theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1} \geq \frac{1}{2} > 0$ for $\chi \in V(>2)$, we infer from (2.16) and (2.17) that

$$\Delta \leq \exp(\theta S) \leq \exp((1 + \epsilon)\theta^2 B^2) \Delta$$

where

$$\Delta = \prod_{\chi \in V(2)} (1 + \theta u(\chi)\chi) \prod_{\chi \in V(>2)} (1 + \theta u(\chi)\chi + \theta u(\chi^{-1})\chi^{-1}).$$

It remains to observe that 2.4 can be used to evaluate $\int_G \Delta d\lambda$. We obtain $\int_G \Delta d\lambda = 1 + \eta$ where $|\eta| \leq \theta M\omega(1 - \theta M\omega)^{-1}$. \square

2.11. An upper bound for $\int_G \exp(\theta S) d\lambda$ is easily computed using Hölder's inequality for the case in which P is a 2-Stečkin set and $\theta > 0$.

COROLLARY. *Suppose that P is a 2-Stečkin set which is the union of m sets P_1, P_2, \dots, P_m , each P_k having the property (R_2) with R_2 -bound equal to ω_k . Let $\omega = \max_{k=1,2,\dots,m} \omega_k$. Let ϵ be any positive number less than 1 and suppose that θ is a positive number such that $48\omega M\theta M < \epsilon$. It follows that*

$$(2.18) \quad \int_G \exp(\theta S) d\lambda \leq (1 + \epsilon) \exp((m/2)\theta^2 B^2(1 + \epsilon)).$$

PROOF. We write $V = \bigcup_{k=1}^m V_k$ where $V_k \subset P_k$. Without loss of generality, we may suppose that this union is a disjoint union of nonempty sets. Setting $W_k = V_k \cup V_k^{-1}$, we define

$$S_k = \sum_{\chi \in W_k} u(\chi)\chi, \quad B_k = \left(\sum_{\chi \in W_k} |u(\chi)|^2 \right)^{1/2} \quad \text{and} \quad M_k = \max_{\chi \in W_k} |u(\chi)|.$$

It is clear that

$$(2.19) \quad S = \sum_{k=1}^m S_k, \quad B = \left(\sum_{k=1}^m B_k^2 \right)^{1/2} \quad \text{and} \quad M = \max_{k=1,\dots,m} M_k.$$

We apply 2.6(b) to obtain

$$\int_G \exp(m\theta B_k) d\lambda \leq (1 + \epsilon) \exp(1/2 m^2 \theta^2 B_k^2(1 + \epsilon)).$$

We rewrite this as

$$(2.20) \quad \|\exp(m\theta S_k)\|_1^{1/m} \leq (1 + \epsilon)^{1/m} \exp(1/2 m\theta^2 B_k^2(1 + \epsilon)).$$

Since we have $\exp(\theta S) = \prod_{k=1}^m (\exp(m\theta S_k))^{1/m}$, we can apply (12.5) of Hewitt and Ross, Vol. I, to obtain

$$\int_G \exp(\theta S) d\lambda \leq \prod_{k=1}^m \|\exp(m\theta S_k)\|_1^{1/m}.$$

Noting (2.19) and (2.20), we conclude that (2.18) holds. \square

2.12. A similar argument yields the corresponding corollary to 2.7.

COROLLARY. *Suppose that P is a 1-Stečkin set which is the union of m sets P_1, P_2, \dots, P_m , each P_k having property (R_1) with R_1 -bound equal to ω_k . Put $\omega = \max_{k=1,2,\dots,m} \omega_k$. Let ϵ be any positive number less than 1 and suppose*

that θ is a positive number such that $4\omega M\theta M < \epsilon$. We have

$$(2.21) \quad \int_G \exp(\theta S) d\lambda \leq (1 + \epsilon) \exp((1 + \epsilon)m\theta^2 B^2).$$

2.13. The inequalities (2.18) and (2.21) are those used to prove our version of the law of iterated logarithm. It will prove convenient to formulate the following definition.

DEFINITION. A nonempty set of characters P such that $P(> 2) \cap P(> 2)^{-1} = \emptyset$ is said to be Φ -bounded by a positive number ϕ if there exist positive numbers k_1, k_2 and k_3 , such that the following property holds. Given a finite nonempty subset V of P , a symmetric function u defined on $W = V \cup V^{-1}$ and a positive number ϵ less than k_1 , then if θ is a positive number such that $k_2\theta M < \epsilon$, we have

$$\int_G \exp(\theta S) d\lambda \leq k_3 \exp(\phi(1 + \epsilon)\theta^2 B^2).$$

2.14. We reformulate 2.11 and 2.12 as:

THEOREM. (a) A 2-Stečkin set which is the union of m sets having the property (R_2) is Φ -bounded by $\frac{1}{2}m$.

(b) A 1-Stečkin set which is the union of m sets having the property (R_1) is Φ -bounded by m .

2.15. We have the following easy consequences of the definition.

THEOREM. Suppose that P is Φ -bounded by ϕ . Let V be a finite nonempty subset of P and suppose that u is a symmetric function defined on $W = V \cup V^{-1}$. Let ϵ be a positive number such that $k_2 M < \epsilon$. Then for any real number y , we have

$$\lambda\{S \geq y\} \leq k_3 \exp(\phi(1 + \epsilon)\theta^2 B^2 - \theta y).$$

PROOF. The proof follows readily from the inequality:

$$\lambda\{S \geq y\} \exp(\theta y) \leq \int_G \exp(\theta S) d\lambda. \quad \square$$

3

3.1. We now prove the Gaussian law for sets having the property (R_2) and show that it can fail for the union of two such sets and for sets having the property (R_1) .

THEOREM. Let $\Upsilon = \{X_n\}_{n=1}^{\infty}$ be as described in 1.5. Suppose that $P \subset \bigcup_{n=1}^{\infty} X_n$ is a countably infinite set having the property (R_2) with R_2 -bound equal to ω . Let u be a symmetric function defined on $Q = P \cup P^{-1}$ and suppose

that (i) $B_n \rightarrow \infty$, and (ii) $M_n B_n^{-1} = o(1)$ as $n \rightarrow \infty$. Then we have

$$\lim_{n \rightarrow \infty} \lambda\{S_n < yB_n\} = (2\pi)^{-1/2} \int_{-\infty}^y \exp(-1/2t^2) dt = G_1(y)$$

for all $y \in R$.

PROOF. Setting $F_n: y \rightarrow \lambda\{S_n < yB_n\}$, we define the characteristic function f_n of F_n by the formula:

$$f_n(u) = \int_{-\infty}^{\infty} \exp(iuy) dF_n(y) = \int_G \exp(iuS_n B_n^{-1}) d\lambda.$$

Since $\exp(-1/2u^2)$ is the characteristic function of $G_1(y)$, in order to prove our theorem, it suffices to show that, for any $A > 0$, $\lim_{n \rightarrow \infty} f_n(u) = \exp(-1/2u^2)$ uniformly for $|u| < A$ (see Gnedenko [1, §38]). Let ϵ be any positive number less than 12. Choose $n_\epsilon \in N$ such that $48\omega AM_n B_n^{-1} < \epsilon$ for all $n > n_\epsilon$. For each $n > n_\epsilon$ and each u such that $|u| < A$, we apply 2.6(a) to the set $V = \bigcup_{k=1}^n P_k$ where $P_k = X_k \cap P$, for ϵ as given and for $\theta = iuB_n^{-1}$. We obtain

$$f_n(u) = (1 + \eta_1) \exp(-1/2u^2(1 + \eta_2)) + \eta_3 u^2 \exp(3^2)$$

where η_1, η_2 and η_3 depend on n and u but have absolute value less than ϵ . Clearly we have $\lim_{n \rightarrow \infty} f_n(u) = \exp(-1/2u^2)$ uniformly for $|u| < A$ as required. \square

3.2. We now produce an example to show that the Gaussian law fails for a set which is the union of two 2-dissociate sets. Let $G = Z(2)^N$, the set of all sequences $x = \{x_j\}_{j=1}^{\infty}$ of elements of $\{0, 1\}$ with componentwise addition modulo 2 as the group operation. For F a nonempty finite subset of N , we define the character

$$\chi(j: j \in F): x \rightarrow \prod_{j \in F} (-1)^{x_j}.$$

For each $j \in N$, we set $X_j = \{\chi(1, j + 1), \chi(j + 1)\}$. We let $P = \{\chi(1, j + 1), \chi(j + 1): j \in N\}$. Clearly we have $Q = P \cup P^{-1} = P$ and $Q_k = Q \cap X_k = X_k$. Further note that Q is the union of the 2-dissociate sets $\{\chi(j + 1): j \in N\}$ and $\{\chi(1, j + 1): j \in N\}$. Let u be the function which is identically equal to 1 on Q . We observe that, for $x \in Z(2)^N$,

$$\begin{aligned} S_n(x) &= \sum_{j=1}^n \chi(1, j + 1)(x) + \sum_{j=1}^n \chi(j + 1)(x) \\ &= (1 + (-1)^{x_1}) \sum_{j=1}^n \chi(j + 1)(x) \end{aligned}$$

and

$$B_n = (2n)^{1/2}.$$

Now, for $y > 0$, we have

$$\begin{aligned} \lambda\{S_n < yB_n\} &= \lambda\left(\{x: x_1 = 1\} \cup \left\{x: x_1 = 0 \text{ and } 2 \sum_{j=1}^n \chi(1+j)(x) < yB_n\right\}\right) \\ &= \frac{1}{2} + \frac{1}{2}\lambda\left\{2 \sum_{j=1}^n \chi(1+j) < yB_n\right\} \end{aligned}$$

while, for $y \leq 0$, we obtain

$$\lambda\{S_n < yB_n\} = \frac{1}{2}\lambda\left\{2 \sum_{j=1}^n \chi(1+j) < yB_n\right\}.$$

Since $\{\chi(1+j): j \in N\}$ is a 2-dissociate set, we apply 3.1 to conclude that

$$\lim_{n \rightarrow \infty} \lambda\{S_n < yB_n\} = \begin{cases} \frac{1}{2} + \frac{1}{2}G_1(2^{-1/2}y) & \text{if } y > 0, \\ \frac{1}{2}G_1(2^{-1/2}y) & \text{if } y \leq 0. \end{cases}$$

Note that the function on the right side of the above expression is discontinuous at zero.

3.3. A similar example shows that the Gaussian law fails to hold for 1-dissociate sets. Let $G = Z(4) \times Z(2)^N$. Each element x of G can be expressed in the form $x = (x_0, x^1)$ where $x_0 \in \{0, 1, 2, 3\}$ and $x^1 \in Z(2)^N$. Let χ_j be the character $\chi_j: (x_0, x^1) \rightarrow i^{x_0}\chi(j)(x^1)$. Thus we readily calculate that

$$\chi_j(x_0, x^1) + \chi_j^{-1}(x_0, x^1) = \begin{cases} 2\chi(j)(x^1) & \text{if } x_0 = 0, \\ 0 & \text{if } x_0 = 1 \text{ or } 3, \\ -2\chi(j)(x^1) & \text{if } x_0 = 2. \end{cases}$$

The set $P = \{\chi_j: j \in N\}$ is easily shown to be a 1-dissociate set but it fails to be a 2-Stečkin set. (Observe that $\chi_j^2\chi_k^{-2} = 1$ for all $j, k \in N$.) A similar argument to the one given in 3.2 shows that the Gaussian law fails for the set P .

4

4.1. In this section, we suppose that $\Upsilon = \{X_n\}_{n=1}^\infty$ is a partition of a countably infinite subset of \hat{G} by finite nonempty symmetric sets. Further we let Q be an infinite symmetric subset of $\bigcup_{n=1}^\infty X_n$ and we let u be a symmetric function defined on Q . We will obtain an estimate for

$$\left\| \max_{r:m \leq r \leq m+p} (S_r - S_m)^2 \right\|_S \quad \text{for } m, p \in N, s > 2,$$

in the cases where Q is a Λ_q set for all $q > 2$. This will enable us to show that

$$\max_{r:p_k \leq r \leq p_{k+1}} |S_r(2B_r^2 \log \log B_r)^{-1/2} - S_{p_k}(2B_{p_k}^2 \log \log B_{p_k})^{-1/2}|$$

can be made small almost everywhere for a suitable sequence $\{p_k\}_{k=1}^\infty$ and large k . In establishing our result, we use properties of the dyadic representations of positive integers. We refine a technique used by Révész [1] in his proof of the law of the iterated logarithm for equinormed strongly multiplicative systems.

We will agree that all intervals used in this section are intervals of integers. For $k \in N$, we set

$$E_k = \{ \{\delta_j\}_{j=0}^{k-1} : \delta_j = 0 \text{ or } 1 \text{ and } \delta_0 = 0 \}.$$

For $\{\delta_j\}_{j=0}^n \in E_{n+1}$, we use the symbol $(\delta_0\delta_1 \cdots \delta_n)_2$ to denote the nonnegative integer $\sum_{j=0}^n \delta_j 2^{n-j}$ and refer to $(\delta_0\delta_1 \cdots \delta_n)_2$ as the n th dyadic representation of $\sum_{j=0}^n \delta_j 2^{n-j}$. In what follows, we will agree to use the symbols δ_j and δ_j^1 for each $j \in N$ only to denote elements of $\{0, 1\}$. Further we will only have cause to refer to n th dyadic representations where n is an unspecified positive integer. For $\Delta = \{\delta_j\}_{j=0}^{k-1} \in E_k$ where $0 < k \leq n$, we put

$$\Delta(0) = (\delta_0\delta_1 \cdots \delta_{k-1}00 \cdots 0)_2 \quad \text{and} \quad \Delta(1) = (\delta_0\delta_1 \cdots \delta_{k-1}10 \cdots 0)_2.$$

It is readily seen that each nonnegative integer $p < 2^n$ has a unique n th dyadic representation and that $(\delta_0\delta_1 \cdots \delta_n)_2 < (\delta_0^1\delta_1^1 \cdots \delta_n^1)_2$ if and only if $\delta_r < \delta_r^1$ where $r = \min\{k : \delta_k \neq \delta_k^1\}$. It will prove convenient to define for $p, n \in N$ and $k \in]0, n]$, $D(p, n, k) = \{\Delta \in E_k : \Delta(1) \leq p\}$.

4.2. The following lemma furnishes us with the properties of dyadic representations we need.

LEMMA. Let $m \in Z$ and $p, n \in N$ such that $p < 2^n$. For $k = 1, 2, \dots, n$, define

$$\Omega_k = \{[m + \Delta(0), m + \Delta(1)[: \Delta \in D(p, n, k)\}$$

and set $\Omega = \bigcup_{k=1}^n \Omega_k$. Then the following are true:

- (i) $\bigcup_{A \in \Omega} A = [m, m + p[$;
- (ii) for each $x \in [m, m + p[$, the set $\{A \in \Omega : x \in A\}$ has at most n elements;
- (iii) if f is a real-valued nondecreasing function defined on $[m, m + p]$, then

$$\sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} f(m + \Delta(1)) - f(m + \Delta(0)) \leq n(f(m + p) - f(m)).$$

PROOF. Without loss of generality, we may suppose that $m = 0$. Let $p = (\delta_0\delta_1 \cdots \delta_n)_2$.

Let us prove (i). It is clear that $\bigcup_{A \in \Omega} A \subset [0, p[$ since each set which is an element of Ω_k is a subset of $[0, p[$. To prove the reverse inclusion, let $x = (\delta_0^1\delta_1^1 \cdots \delta_n^1)_2 \in [0, p[$. Then if $r = \min\{j : \delta_j \neq \delta_j^1\}$, we know that $\delta_r^1 < \delta_r$ and therefore we have

$$x \in [(\delta_0^1 \delta_1^1 \cdots \delta_{r-1}^1 0 \cdots 0)_2, (\delta_0^1 \delta_1^1 \cdots \delta_{r-1}^1 10 \cdots 0)_2].$$

Further since $\delta_k^1 = \delta_k$ for $k = 0, 1, \dots, r - 1$ and $\delta_r = 1$, it is clear that the right endpoint of the above interval is at most equal to p and the interval is consequently an element of Ω_r . Part (i) is thus established.

To conclude that (ii) is true, we observe that the sets contained in Ω_k are mutually disjoint sets (that is, if $\Delta(0) < \Delta'(0)$ for $\Delta, \Delta' \in E_k$, then $\Delta(1) < \Delta'(0)$).

Finally we take up (iii). If f is a real-valued nondecreasing function defined on $[0, p]$, then the function $g(x) = f(x + 1) - f(x)$ defined on $[0, p[$ is a non-negative function and we have

$$f(b) - f(a) = \sum_{k \in [a, b[} g(k) \quad \text{for } 0 \leq a < b \leq p.$$

Thus we have

$$\begin{aligned} \sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} f(\Delta(1)) - f(\Delta(0)) &= \sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} \sum_{x \in [\Delta(0), \Delta(1)[} g(x) \\ &= \sum_{k=1}^n \sum_{A \in \Omega_k} \sum_{x \in A} g(x) = \sum_{A \in \Omega} \sum_{x \in A} g(x) \\ &\quad \text{(note that } \Omega \text{ is the disjoint union of the } \Omega_k \text{'s)} \\ &= \sum_{x \in \bigcup_{A \in \Omega} A} \text{card}\{A \in \Omega: x \in A\} g(x) \\ &\leq n \sum_{x \in [0, p[} g(x) \quad \text{(by parts (i) and (ii))} \\ &\leq n(f(p) - f(0)). \quad \square \end{aligned}$$

4.3. For the remainder of this section, we will let m denote an arbitrary nonnegative integer and n an arbitrary positive integer. For $j \leq n$ and $\Delta \in E_j$, we define

$$(4.1) \quad T(m, n, j, \Delta) = S_{m+\Delta(1)} - S_{m+\Delta(0)}.$$

LEMMA. We have the following decomposition:

$$S_m + (\delta_0 \delta_1 \cdots \delta_n)_2 - S_m = \sum_{j=1}^n \delta_j T(m, n, j, \{\delta_k\}_{k=0}^{j-1}).$$

PROOF. For $j \geq 1$ and $\delta_j = 1$, we have

$$\delta_j T(m, n, j, \{\delta_k\}_{k=0}^{j-1}) = S_{m+(\delta_0 \delta_1 \cdots \delta_{j-1} \delta_j 0 \cdots 0)_2} - S_{m+(\delta_0 \delta_1 \cdots \delta_{j-1} 0 \cdots 0)_2}.$$

This inequality holds if $\delta_j = 0$ since both sides are zero. By summing both sides and noting that all the middle terms in the right side cancel, we obtain

$$\begin{aligned} \sum_{j=1}^n \delta_j T(m, n, j, \{\delta_k\}_{k=0}^{j-1}) &= S_{m+(\delta_0\delta_1\cdots\delta_n)_2} - S_{m+(\delta_000\cdots0)_2} \\ &= S_{m+(\delta_0\delta_1\cdots\delta_n)_2} - S_m. \quad \square \end{aligned}$$

4.4. LEMMA. For $j \leq n$ and $\Delta \in E_j$, we have

$$\int_G T^2(m, n, j, \Delta) d\lambda = B_{m+\Delta(1)}^2 - B_{m+\Delta(0)}^2.$$

PROOF. Noting (4.1), we obtain

$$\begin{aligned} \int_G T^2(m, n, j, \Delta) d\lambda &= \sum_{k=\Delta(0)+1}^{\Delta(1)} \sum_{\chi \in Q_{m+k}} u(\chi) \int_G \chi T(m, n, j, \Delta) d\lambda \\ &= \sum_{k=\Delta(0)+1}^{\Delta(1)} \sum_{\chi \in Q_{m+k}} u(\chi) \times \text{coefficient of } \chi^{-1} \text{ in } T(m, n, j, \Delta) \\ &= \sum_{k=\Delta(0)+1}^{\Delta(1)} \sum_{\chi \in Q_{m+k}} |u(\chi)|^2 = B_{m+\Delta(1)}^2 - B_{m+\Delta(0)}^2. \quad \square \end{aligned}$$

4.5. We now suppose that P is a Λ_s set for all $s > 2$ and let $Q = P \cup P^{-1}$. Note that Q is also a Λ_s set for all $s > 2$ by (37.21)(a) of Hewitt and Ross, Vol. II [2]. Since a 1-Stečkin set is a Sidon set (see Rider [1]), it will be a Λ_s set for all $s > 1$ by (37.10) of Hewitt and Ross, Vol. II. Hence our results will be valid whenever P is a 1-Stečkin set.

THEOREM. For each $s > 2$, there exists a constant $\phi_s > 0$ dependent only on s such that

$$\|T^2(m, n, j, \Delta)\|_s \leq \phi_s (B_{m+\Delta(1)}^2 - B_{m+\Delta(0)}^2).$$

PROOF. Since $T(m, n, j, \Delta)$ is a trigonometric polynomial of elements of Q , there exists, by (37.7) of Hewitt and Ross, Vol. II, a constant $\phi_s > 0$ such that

$$\|T(m, n, j, \Delta)\|_{2s}^2 \leq \phi_s \|T(m, n, j, \Delta)\|_2^2.$$

The result follows from 4.4. \square

4.6. LEMMA. Suppose that $p < 2^n$. Then we have

$$\max_{r:m \leq r < m+p} (S_r - S_m)^2 \leq n \left(\sum_{k=1}^n \sum_{\Delta \in D(p,n,k)} T^2(m, n, k, \Delta) \right).$$

PROOF. Let $r \in [m, n + p]$. Then we have $r = m + (\delta_0 \delta_1 \cdots \delta_n)_2$ where $(\delta_0 \delta_1 \cdots \delta_n)_2 < p$ and we obtain

$$\begin{aligned} (S_r - S_m)^2 &= (S_{m+(\delta_0 \delta_1 \cdots \delta_n)_2} - S_m)^2 \\ &= \left(\sum_{k=1}^n \delta_k T(m, n, k, \{\delta_j\}_{j=1}^{k-1}) \right)^2 \quad (\text{see 4.3}) \\ &\leq n \left(\sum_{k=1}^n \delta_k T^2(m, n, k, \{\delta_j\}_{j=1}^{k-1}) \right) \quad (\text{by Cauchy's inequality}). \end{aligned}$$

If $\delta_k T^2(m, n, k, \{\delta_j\}_{j=1}^{k-1})$ appears as a nonzero term in our summation, we must have $\delta_k = 1$ and hence $(\delta_0 \delta_1 \cdots \delta_{k-1} 10 \cdots 0)_2 \leq (\delta_0 \delta_1 \cdots \delta_n)_2 < p$ which implies that $\{\delta_j\}_{j=1}^{k-1} \in D(p, n, k)$. We conclude that

$$(S_r - S_m)^2 \leq n \left(\sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} T^2(m, n, k, \Delta) \right).$$

Our result follows since r was selected arbitrarily. \square

4.7. We now piece together 4.5 and 4.6 to obtain the key theorem in this section.

THEOREM. Let $p \in N$ and $s > 2$. Then setting $A_s = \phi_s (\log 2)^{-2}$ where ϕ_s is the constant appearing in 4.5, we have

$$\left\| \max_{r: m \leq r < m+p} (S_r - S_m)^2 \right\|_s \leq A_s (\log(2p))^2 (B_{m+p}^2 - B_m^2).$$

PROOF. Choose $n \in N$ such that $2^{n-1} \leq p < 2^n$. It is easily seen that

$$(4.2) \quad n \log 2 \leq \log(2p).$$

Using the inequality established in 4.6, we obtain

$$\begin{aligned} \left\| \max_{r: m \leq r < m+p} (S_r - S_m)^2 \right\|_s &\leq n \left\| \sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} T^2(m, n, k, \Delta) \right\|_s \\ &\leq \sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} \|T^2(m, n, k, \Delta)\|_s \quad (\text{by Minkowski's inequality}) \\ &\leq n \phi_s \sum_{k=1}^n \sum_{\Delta \in D(p, n, k)} (B_{m+\Delta(1)}^2 - B_{m+\Delta(0)}^2) \quad (\text{by 4.5}) \\ &\leq n^2 \phi_s (B_{m+p}^2 - B_m^2) \quad \text{by (4.2)(iii)} \\ &\leq A_s (\log 2p)^2 (B_{m+p}^2 - B_m^2) \quad (\text{by 4.2}). \quad \square \end{aligned}$$

5

5.1. We now proceed to give the statement and proof of a law of the iterated logarithm for lacunary sets of characters. We recall the description of the problem given in 1.5. Before stating our key theorem, we need the following definition.

DEFINITION. Let \mathfrak{T}, Q and u be given. Suppose that $\{n_k\}_{k=1}^\infty$ is an increasing sequence of positive integers such that

$$\log n_k (\log B_{n_k})^{-\gamma} = O(1) \quad \text{as } k \rightarrow \infty \text{ for some } \gamma > 0.$$

Then if $\beta \geq 0$ is the greatest lower bound for all such γ , the sequence $\{n_k\}_{k=1}^\infty$ is called a β -sequence with respect to \mathfrak{T}, Q and u .

5.2. THEOREM. Let P be a countably infinite subset of G which is a Λ_q set for all $q \geq 2$ and is Φ -bounded by some positive number ϕ (see 2.13). Set $Q = P \cup P^{-1}$ and let $\mathfrak{T} = \{X_n\}_{n=1}^\infty$ and u be given such that $Q \subset \bigcup_{n=1}^\infty X_n$ and the following conditions are satisfied:

- I. $B_n \rightarrow \infty$ as $n \rightarrow \infty$, and
- II. $M_n B_n^{-1} (\log \log B_n)^{1/2} = o(1)$ as $n \rightarrow \infty$.

Then if $\{n_k\}_{k=1}^\infty$ is a β -sequence, we have

$$(5.1) \quad \limsup_{k \rightarrow \infty} |S_{n_k}| (2B_{n_k}^2 \log \log B_{n_k})^{-1/2} \leq (2\phi)^{1/2} (1 + 2\beta)^{1/2} \quad \text{a.e.}$$

Noting 2.14, we have

$\phi = 1/2 m$ in (5.1) when P is a 2-Stečkin set which is the union of m sets having the property (R_2) , and

$\phi = m$ in (5.1) when P is a 1-Stečkin set which is the union of m sets having the property (R_1) .

5.3. We accomplish the proof of 5.2 by a succession of lemmas. In each of these lemmas, we suppose that \mathfrak{T}, Q and u are given and that $Q = P \cup P^{-1}$. In addition we suppose that u satisfies conditions I and II as stated in 5.2.

For notational convenience, we set

$$T_n = (2B_n^2 \log \log B_n)^{1/2} \quad \text{for } n \in N.$$

LEMMA. Let $\gamma \in R; \alpha, \beta > 0$ and $r \in N$. Suppose that P is Φ -bounded by ϕ and let $\{p_k\}_{k=1}^\infty$ be a sequence of positive integers with the property that for each $j \in N$,

$$(5.2) \quad B_{p_k} \geq \exp(\beta j^\alpha + \gamma) \quad \text{if } k \geq r_j.$$

It follows that

$$\limsup_{k \rightarrow \infty} |S_{p_k}| T_{p_k}^{-1} \leq \alpha^{-1/2} (2\phi)^{1/2} \quad \text{a.e.}$$

PROOF. Choose $j_0 \in N$ such that

$$(5.3) \quad \frac{1}{2}\beta j_0^\alpha + \gamma > e.$$

For $k \geq rj_0$, we have $B_{p_k} \geq \exp(\beta j_0^\alpha + \gamma) > \exp(e)$ and hence $\log \log B_{p_k} > 1$. Suppose that we are given a positive number η such that

$$(5.4) \quad \eta < k_1(2\phi)^{1/2}\alpha^{-1/2}$$

(the constants k_1 , k_2 and k_3 are those appearing in 2.13). For $k \geq rj_0$, we put

$$(5.5) \quad \Gamma_k = \lambda\{S_{p_k} > ((2\phi)^{1/2}\alpha^{-1/2} + \eta)T_{p_k}\} = \lambda\{S_{p_k} T_{p_k}^{-1} > (2\phi)^{1/2}\alpha^{-1/2} + \eta\}.$$

For convenience, we define, for $k < rj_0$, $\Gamma_k = \lambda\{S_{p_k} > 1\}$. It is well known that if $\sum_{k=1}^\infty \Gamma_k < \infty$ holds, then it will follow that

$$\limsup_{k \rightarrow \infty} S_{p_k} T_{p_k}^{-1} \leq (2\phi)^{1/2}\alpha^{-1/2} + \eta \quad \text{a.e.}$$

For all $n \geq p_{rj_0}$, let

$$(5.6) \quad \theta_n = (2\phi\alpha)^{-1/2} B_n^{-1} (2 \log \log B_n)^{1/2}.$$

Then by condition II on u , there exists a positive integer $n' \geq p_{rj_0}$ such that

$$k_2 \theta_n M_n < (2\phi)^{-1/2} \alpha^{1/2} \eta \quad \text{for } n \geq n'.$$

If $j_1 \geq j_0$ is chosen such that $B_{n'} \leq \exp(\beta j_1^\alpha + \gamma)$, then by (5.2) we know that if $k \geq rj_1$ holds, then $B_{p_k} \geq B_{n'}$. Hence for $k \geq rj_1$, we have $k_2 \theta_{p_k} M_{p_k} < (2\phi)^{-1/2} \alpha^{1/2} \eta$. We can now apply 2.15 for each $k \geq rj_1$ in the case where $W = \bigcup_{j=1}^p Q_j$, $\epsilon = (2\phi)^{-1/2} \alpha^{1/2} \eta$ and $\theta = \theta_{p_k}$. Noting (5.4), we have $\epsilon < k_1$ and hence for all $y \in R$, we conclude that

$$(5.7) \quad \lambda\{S_{p_k} > y\} \leq k_3 \exp(\phi(1 + \eta(2\phi)^{-1/2}\alpha^{1/2})\theta_{p_k}^2 B_{p_k}^2 - \theta_{p_k} y).$$

We set $y = ((2\phi)^{1/2}\alpha^{-1/2} + \eta)T_{p_k}$. Noting (5.5) and substituting for θ_{p_k} (see (5.6)) and y in (5.7), we obtain after a straightforward calculation

$$\Gamma_k \leq k_3 (\log B_{p_k})^{-(\alpha^{-1} + \eta(2\phi)^{-1/2}\alpha^{-1/2})}.$$

It follows by (4.2) that, for $j \geq j_1$ and $0 \leq l < r$,

$$\begin{aligned} \Gamma_{rj+e} &\leq k_3 (\log B_{p_{rj+e}})^{-(\alpha^{-1} + \eta(2\phi)^{-1/2}\alpha^{-1/2})} \\ &\leq k_3 (\beta j^\alpha + \gamma)^{-(\alpha^{-1} + \eta(2\phi)^{-1/2}\alpha^{-1/2})}. \end{aligned}$$

However, noting (5.3), we have for $j \geq j_1 \geq j_0$

$$\beta j^\alpha + \gamma \geq \frac{1}{2}\beta j^\alpha + \frac{1}{2}\beta j_0^\alpha + \gamma > \frac{1}{2}\beta j^\alpha.$$

We conclude that for $j \geq j_1$ and $0 \leq l < r$

$$\Gamma_{rj+e} \leq k_3(\frac{1}{2}\beta)^{-(\alpha^{-1} + \eta(2\phi)^{-\frac{1}{2}}\alpha^{-\frac{1}{2}})}j^{-(1 + \eta(2\phi)^{-\frac{1}{2}}\alpha^{\frac{1}{2}})}.$$

It follows that $\sum_{k=1}^{\infty} \Gamma_k < \infty$ and hence we have

$$\limsup_{k \rightarrow \infty} S_{p_k} T_{p_k}^{-1} \leq (2\phi)^{\frac{1}{2}}\alpha^{-\frac{1}{2}} + \eta \text{ a.e.}$$

A similar argument using the function $-u$ instead of u yields:

$$\limsup_{k \rightarrow \infty} -S_{p_k} T_{p_k}^{-1} \leq (2\phi)^{\frac{1}{2}}\alpha^{-\frac{1}{2}} + \eta \text{ a.e.}$$

Thus we conclude that

$$\limsup_{k \rightarrow \infty} |S_{p_k} T_{p_k}^{-1}| \leq (2\phi)^{\frac{1}{2}}\alpha^{-\frac{1}{2}} + \eta \text{ a.e.}$$

Our lemma follows since η can be made arbitrarily small. \square

5.4. COMMENT. If for each $n \in N$, we have $B_n \geq \exp(n^n)$, then we can apply 5.3 to conclude that $\limsup_{n \rightarrow \infty} |S_n|T_n^{-1} = 0$ a.e. It is not obvious that this may occur if the rate of increase of the B_n 's is restricted by condition II placed on u . However, if u is identically equal to 1 on Q , we have $M_n = 1$ and $B_n^2 = \sum_{k=1}^n \text{card } Q_k$. Condition II is clearly satisfied and by choosing the sequence $\Upsilon = \{X_n\}_{n=1}^{\infty}$ in such a way that $\text{card } Q_n \geq \exp(2n^n)$ we find that $\limsup_{n \rightarrow \infty} |S_n|T_n^{-1} = 0$ a.e.

If we specify that $\text{card } Q_n = O(1)$ as $n \rightarrow \infty$, then an argument which we outline below shows that $B_n \leq \exp \beta^n$ for some $\beta > 0$ and hence 5.3 will not permit us to conclude that $\limsup_{n \rightarrow \infty} |S_n|T_n^{-1} \leq 1$ a.e. If $\text{card } Q_n \leq B$ for all $n \in N$, then it follows readily from the definition that $M_n \geq B^{-1}(B_n^2 - B_{n-1}^2)^{\frac{1}{2}}$. If $B_n B_{n-1}^{-1}$ is not bounded in n , it will follow that $M_n B_n^{-1}$ does not approach 0 thus contravening condition II. However, for $B_n B_{n-1}^{-1} = O(1)$ as $n \rightarrow \infty$, we readily conclude that $B_n \leq \exp(\beta^n)$ for some $\beta > 0$.

5.5. We now choose a sequence $\{p_k\}_{k=1}^{\infty}$ satisfying (5.2) where $\alpha < 1$ can be chosen in such a way that 4.7 can be used to show that

$$(5.8) \quad \lim_{k \rightarrow \infty} \max_{n: p_k < n < p_{k+1}} |S_n T_n^{-1} - S_{p_k} T_{p_k}^{-1}| = 0 \text{ a.e.}$$

The choice of α will depend on the rate of growth of B_n with respect to n . The straightforward procedure would be to define p_k by $p_k = \min\{n \in N: B_n^2 > \exp(k^\alpha)\}$. This is inadequate since in order to show (5.8) we require that $(B_{p_{k+1}}^2 - B_{p_k}^2)(\exp(k+1)^\alpha - \exp(k)^\alpha)^{-1}$ be bounded for all k . In general this is not possible but for an appropriate choice of $\{p_k\}_{k=1}^{\infty}$ we can assure boundedness for all k such that $p_{k+1} > p_k + 1$ and this is sufficient.

LEMMA. Let $\alpha \in [0, 1]$ and P be as in 5.3. Then there exists a strictly increasing sequence $\{p_j\}_{j=1}^{\infty}$ of positive integers with the following properties. We define the sequence $\{m_j\}_{j=1}^{\infty}$ of positive integers by setting

$$(5.9) \quad m_j = \max\{k \in N: B_{p_j}^2 > \exp k\}.$$

Then we have:

- (1) $m_{2j} \geq j$ for each $j \in N$;
- (2) $\limsup_{k \rightarrow \infty} |S_{p_k}| T_{p_k}^{-1} \leq (2\phi)^{1/2} \alpha^{-1/2}$ a.e.; and
- (3) if $p_{k+1} > p_k + 1$, then $B_{p_{k+1}}^2 - B_{p_k}^2 < 2\alpha m_k^{-1} \exp(m_k^\alpha)$.

PROOF. We note that $\{B_n^2\}_{n=1}^\infty$ is an unbounded nondecreasing sequence of nonnegative numbers. For each $k \in N$, we set

$$(5.10) \quad A_k = \{j \in N: \exp k^\alpha \leq B_j^2 \leq \exp(k + 1)^\alpha\}.$$

It is possible that A_k is empty for some $k \in N$. We observe that if $j_1 \in A_{k_1}$, $j_2 \in A_{k_2}$ and $k_1 < k_2$ then $j_1 < j_2$. For each $k \in N$, we define

$$q_{2k-1} = \min \bigcup_{j=k}^\infty A_j, \quad \text{and} \quad q_{2k} = \begin{cases} \max A_k & \text{if } A_k \neq \emptyset, \\ q_{2k-1} & \text{if } A_k = \emptyset. \end{cases}$$

It is clear that $\{q_k\}_{k=1}^\infty$ is a nondecreasing sequence with the property that

$$(5.11) \quad q_{2k+1} \leq q_{2k} + 1 \quad \text{for all } k \in N.$$

Choose $k_0 \in N$ such that $k_0 > 2 \cdot 2^{(1-\alpha)^{-1}}$. We let $\{p_k\}_{k=1}^\infty$ be the increasing sequence of distinct elements of $\{q_k\}_{k=k_0}^\infty$.

To show that (1) holds, we first observe that $p_{2j} \geq q_{2j}$ for all $j \in N$. Since $q_{2j} \in A_l$ for some $l \geq j$, we conclude from (5.10) that $B_{p_{2j}}^2 \geq B_{q_{2j}}^2 \geq \exp(j)$. It follows that $m_{2j} \geq j$ (see (5.9)).

In order to prove (2), it suffices to show that our sequence $\{p_k\}_{k=1}^\infty$ satisfies (5.2) for $\beta = 1/2$, $\gamma = \theta$ and $r = 2j$. For $k \geq 2j$, we have $p_k \geq p_{2j}$ and hence

$$\begin{aligned} B_{p_k}^2 &\geq B_{2j}^2 > \exp(m_{2j}^\alpha) \quad (\text{note (5.9)}) \\ &\geq \exp(j^\alpha) \quad (\text{by part (1)}). \end{aligned}$$

The inequality $B_{p_k} > \exp(1/2 j^\alpha)$ holds for $k \geq 2j$ and part (2) is thereby proved.

Finally we turn to (3). For all $k \in N$ we have $p_k = q_h$ and $p_{k+1} = q_{h+1}$ where $h = \max\{l \in N: p_k = q_l\}$. Noting (5.11), we infer that if $p_{k+1} > p_k + 1$, then $q_{h+1} > q_h + 1$ and hence h is odd. Thus we conclude that for each $k \in N$ such that $p_{k+1} > p_k + 1$, there exists $j \in N$ such that $p_k = q_{2j-1}$ and $p_{k+1} = q_{2j}$. It follows that $A_j \neq \emptyset$ since $q_{2j-1} \neq q_{2j}$ and therefore

$$(5.12) \quad \exp(j^\alpha) \leq B_{p_k}^2 \leq B_{p_{k+1}}^2 < \exp((j + 1)^\alpha).$$

We have $B_{p_{k+1}}^2 - B_{p_k}^2 < \exp(j + 1)^\alpha - \exp(j^\alpha)$. We have $j > 1/2(2j - 1) > 2^{(1-\alpha)^{-1}}$ by construction of $\{p_k\}_{k=1}^\infty$ and a simple argument which we omit yields:

$$\exp(j + 1)^\alpha - \exp(j^\alpha) \leq 2j^{\alpha-1} \exp(j^\alpha) \quad \text{if } j > 2^{(1-\alpha)^{-1}}.$$

The proof of part (3) is complete since it is clear from (5.9) and (5.12) that $m_k = j$. \square

5.6. Before proving 5.2, we require a technical lemma.

LEMMA. Suppose that for a given sequence $\Upsilon = \{X_n\}_{n=1}^\infty$ of finite non-empty pairwise disjoint symmetric subsets of \hat{G} and a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers, we define a new sequence $\Upsilon^\dagger = \{X_k^\dagger\}_{k=1}^\infty$ where

$$X_1^\dagger = \bigcup_{j=1}^{n_1} X_j \quad \text{and} \quad X_k^\dagger = \bigcup_{j=n_{k-1}+1}^{n_k} X_j \quad \text{for } k > 1.$$

If corresponding to the terms S_n, B_n and M_n defined with respect to Υ , we define S_k^\dagger, B_k^\dagger and M_k^\dagger with respect to Υ^\dagger , then the following properties hold:

- (1) $S_k^\dagger = S_{n_k}, B_k^\dagger = B_{n_k}$ and $M_k^\dagger = M_{n_k}$;
- (2) if u satisfies conditions I and II with respect to Υ (see 5.2) then u satisfies conditions I and II with respect to Υ^\dagger ; and
- (3) $\{n_k\}_{k=1}^\infty$ is a β -sequence with respect to Υ (see 5.1) if and only if $\{k\}_{k=1}^\infty$ is a β -sequence with respect to Υ^\dagger .

PROOF. The proof is easy and is omitted.

5.7. PROOF OF 5.2. In view of 5.6, we may suppose that $\{n_k\}_{k=1}^\infty$ is the sequence $\{n\}_{n=1}^\infty$. For $\epsilon > 0$ and $\gamma > \beta$ given, we set $\alpha = (1 + 2\gamma + \epsilon)^{-1}$. Then if $\{p_k\}_{k=1}^\infty$ is a sequence of positive integers having the properties (1), (2) and (3) of 5.5 for this choice of α , we readily deduce from the second of these properties that if it is true that

$$(5.13) \quad \lim_{k \rightarrow \infty} \max_{n: p_k \leq n < p_{k+1}} |S_n T_n^{-1} - S_{p_k} T_{p_k}^{-1}| = 0 \quad \text{a.e.}$$

then it will follow that

$$\limsup_{n \rightarrow \infty} |S_n| T_n^{-1} \leq (2\phi)^{1/2} (1 + 2\gamma + \epsilon)^{1/2} \quad \text{a.e.}$$

Our proof will be complete since $\epsilon > 0$ and $\gamma > \beta$ are arbitrarily chosen.

We begin by showing that

$$(5.14) \quad \lim_{k \rightarrow \infty} \max_{n: p_k \leq n < p_{k+1}} |S_n - S_{p_k}| T_{p_k}^{-1} = 0 \quad \text{a.e.}$$

We set

$$(5.15) \quad s = \max\{2, 2(\epsilon\alpha)^{-1}\}.$$

Noting 5.5(1), we choose $k_0 \in N$ such that for all $k \geq k_0$

$$(5.16) \quad 2\alpha m_k^{\alpha-1} < 1.$$

Now if $p_{k+1} = p_k + 1$ for all $k \geq k_0$, then (5.13) is trivially true. Hence we suppose that $p_{k+1} > p_k + 1$ holds for some $k \geq k_0$ and for any such k , we have by 5.5(3),

$$(5.17) \quad B_{p_{k+1}}^2 - B_{p_k}^2 \leq 2\alpha m_k^{\alpha-1} \exp(m_k^\alpha).$$

By 4.7, we infer that

$$\left\| \max_{n: p_k \leq n < p_{k+1}} (S_n - S_{p_k})^2 \right\|_s \leq A_s (\log(2(p_{k+1} - p_k)))^2 (B_{p_{k+1}}^2 - B_{p_k}^2)$$

where A_s is the constant appearing in 4.7. We rewrite this inequality as

$$(5.18) \quad \int_G \max_{n: p_k \leq n < p_{k+1}} (S_n - S_{p_k})^{2s} d\lambda \\ \leq A_s^s (\log(2(p_{k+1} - p_k)))^{2s} (B_{p_{k+1}}^2 - B_{p_k}^2)^s.$$

We estimate for $k \geq k_0$:

$$(5.19) \quad \log(2(p_{k+1} - p_k)) < \log(2p_{k+1}) \leq 2 \log p_{k+1} \\ < 2B(\log B_{p_{k+1}})^\alpha \quad \text{for some } B > 0 \text{ (see 5.1)} \\ = 2 \cdot 2^{-\alpha} B (\log B_{p_{k+1}}^2)^\alpha.$$

Noting (5.17) and (5.9), we have for $k \geq k_0$ such that $p_{k+1} > p_k + 1$.

$$(5.20) \quad \log(B_{p_{k+1}}^2) \leq \log(B_{p_k}^2 + 2\alpha m_k^{\alpha-1} \exp(m_k^\alpha)) \\ \leq \log(\exp((m_k + 1)^\alpha) + 2m_k^{\alpha-1} \exp(m_k^\alpha)) \\ < \log(2 \exp((m_k + 1)^\alpha)) \quad \text{(see (5.16))} \\ < 2(m_k + 1)^\alpha \leq 2 \cdot 2^\alpha m_k^\alpha \\ \leq A_0 m_k^{2\alpha\gamma s} m_k^{s(\alpha-1)} (\exp(m_k^\alpha))^s.$$

Substituting (5.19), (5.20) and (5.17) in (5.18), we have for $k \geq k_0$,

$$\int_G \max_{n: p_k \leq n < p_{k+1}} (S_n - S_{p_k})^{2s} d\lambda \leq A_0 m_k^{2\alpha\gamma s} m_k^{s(\alpha-1)} (\exp(m_k^\alpha))^s$$

where A_0 is some constant independent of k . This inequality remains valid if $p_{k+1} = p_k + 1$ since the left side is equal to 0. We note that

$$2\alpha\gamma s + s(\alpha - 1) = \alpha s(2\gamma + 1) - s = -\alpha s\epsilon \quad (\text{since } 2\gamma + 1 = \alpha^{-1} - \epsilon)$$

and that

$$T_{p_k}^{-2s} = 2^{-s} B_{p_k}^{-2s} (\log \log B_{p_k})^{-s} \\ \leq 2^{-s} (\exp(m_k^\alpha))^{-s} (\log \log \exp(\frac{1}{2} m_k^\alpha))^{-s} \leq (\exp(m_k^\alpha))^{-s}.$$

Thus we obtain

$$\int_G \max_{n: p_k \leq n < p_{k+1}} (S_n - S_{p_k})^{2s} T_{p_k}^{-2s} d\lambda \leq A_0 m_k^{-\alpha s} \leq A_0 m_k^{-2} \quad (\text{see (5.15)}).$$

Summing for $k \geq k_0$ and noting 5.5(2), we obtain

$$\sum_{k=k_0}^{\infty} \int_G \max_{n:p_k \leq n < p_{k+1}} (S_n - S_{p_k})^{2s} T_{p_k}^{-2s} d\lambda < \infty.$$

We apply the monotone convergence theorem to obtain (5.14). Having established (5.14), we conclude the proof by showing (5.13) to be true. We let

$$H = \left\{ t \in G: \limsup_{k \rightarrow \infty} |S_{p_k}(t)| T_{p_k}^{-1} \leq (2\phi)^{1/2} \alpha^{1/2} \text{ and} \right. \\ \left. \lim_{k \rightarrow \infty} \max_{n:p_k \leq n < p_{k+1}} |S_n(t) - S_{p_k}(t)| T_{p_k}^{-1} = 0 \right\}.$$

By 5.5(2) and (5.14), we conclude that $\lambda(H) = 1$. Let $\eta > 0$ and $t \in H$. Choose $k(t) \in N$ such that

- (i) $B_{p_k(t)} > \exp(e)$,
- (ii) $m_k^{\alpha-1} < (1/8)(2\phi)^{-1/2} \alpha^{-1/2}$ for $k \geq k(t)$ (note that $m_k \rightarrow \infty$ by 5.5(1)),
- (iii) $\max_{k \geq k(t)} |S_{p_k}(t)| T_{p_k}^{-1} \leq 2(2\phi)^{1/2} \alpha^{-1/2}$, and
- (iv) $\max_{n:p_k \leq n < p_{k+1}} |S_n(t) - S_{p_k}(t)| T_{p_k}^{-1} \leq \eta/2$ for $k \geq k(t)$.

Our proof will be complete once we show for $k \geq k(t)$ that

$$\max_{n:p_k \leq n < p_{k+1}} |S_{p_k}(t) T_{p_k}^{-1} - S_n(t) T_n^{-1}| < \eta/2.$$

Clearly we may suppose that $p_{k+1} > p_k + 1$. Let n be any number such that $p_k \leq n < p_{k+1}$. Then since $T_{p_k} \leq T_n$, we have

$$(5.21) \quad |S_{p_k}(t) T_{p_k}^{-1} - S_n(t) T_n^{-1}| \\ \leq |S_{p_k} T_{p_k}^{-1} - S_{p_k}(t) T_n^{-1}| + T_n^{-1} |S_{p_k}(t) - S_n(t)| \\ \leq |S_{p_k}(t)| (T_{p_k}^{-1} - T_n^{-1}) + T_{p_k}^{-1} |S_{p_k}(t) - S_n(t)|.$$

A simple calculation yields for $B_{p_k} \geq \exp e$,

$$T_{p_k}^{-1} - T_n^{-1} \leq T_{p_k}^{-1} B_{p_k}^{-2} (B_n^2 - B_{p_k}^2) \leq T_{p_k}^{-1} B_{p_k}^{-2} (B_{p_{k+1}}^2 - B_{p_k}^2).$$

Substituting this in (5.21), we obtain

$$|S_{p_k}(t) T_{p_k}^{-1} - S_n(t) T_n^{-1}| \\ \leq |S_{p_k}(t)| T_{p_k}^{-1} B_{p_k}^{-2} (B_{p_{k+1}}^2 - B_{p_k}^2) + T_{p_k}^{-1} |S_{p_k}(t) - S_n(t)| \\ \leq 2(2\phi)^{1/2} \alpha^{-1/2} B_{p_k}^{-2} (B_{p_{k+1}}^2 - B_{p_k}^2) + \eta/2 \quad (\text{by (iii) and (iv)}) \\ < 2(2\phi)^{1/2} \alpha^{-1/2} 2\alpha m_k^{\alpha-1} + \eta/2 \quad (\text{by (5.9) and 5.5(3)}) \\ < \eta \quad (\text{by (ii)}). \quad \square$$

5.8. REMARKS. (1) Our result, 5.2, in the case where $k = n_k$ differs from that of the classical law of the iterated logarithm insofar as we impose an added condition on the B_k 's. We note that

$$\max_{\chi \in Q_k} |u(\chi)|^2 \leq \sum_{\chi \in Q_k} |u(\chi)|^2 = B_k^2 - B_{k-1}^2.$$

Thus we see that our function u satisfies conditions I and II if

(i) $B_n \rightarrow \infty$, and

(ii) $\max_{k=1, \dots, n} B_n^{-2} (B_k^2 - B_{k-1}^2) \log \log B_n = o(1)$ as $n \rightarrow \infty$.

This observation enables us to deduce that in the case where u satisfies conditions I and II, $\{n\}_{n=1}^\infty$ may be a β -sequence for a given $\beta \geq 0$ or may fail to be a β -sequence for any $\beta \geq 0$. Specifically we have:

(a) if $B_n = \exp(\log n)^{1/\beta}$ for some $\beta > 0$, then u satisfies conditions I and II and $\{n\}_{n=1}^\infty$ is a β -sequence;

(b) if $B_n = \exp(\exp(\log \log n)^2)$ then u satisfies conditions I and II and $\{n\}_{n=1}^\infty$ is a 0-sequence;

(c) if $B_n = \log n$, then u satisfies conditions I and II and $\{n\}_{n=1}^\infty$ fails to be a β -sequence for any $\beta > 0$.

The verification of these claims is straightforward and we omit it.

(2) In the cases where $\beta = 0$, the upper bounds given for the lim sup in 5.2 are the best possible for P , a 2-Stečkin set and P , a 1-Stečkin set. To see this, we examine the example given in 3.2. However, instead of setting u identically equal to 1, we define u to be a function $u: \chi(1, j+1) \rightarrow u_j$ which satisfies conditions I and II and for which $\{n\}_{n=1}^\infty$ is a 0-sequence. (For instance, set $u_1 = B_1$ and $u_n = (B_n^2 - B_{n-1}^2)^{1/2}$ where $B_n = \exp(\exp(\log \log n)^2)$ as in (b) above.) We note that if $x \in Z(2)^N$ is such that $x_1 = 0$, then

$$S_n(x) = 2 \sum_{j=1}^n u_j \chi(j+1)(x) \quad \text{and} \quad B_n = 2^{1/2} \left(\sum_{j=1}^n u_j^2 \right)^{1/2}.$$

It is easily verified that $\{\chi(j+1): j \in N\}$ is an independent set of functions on $Z(2)^N$ and we can apply 1.3 to conclude that for $x_1 = 0$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} S_n(x) (2B_n^2 \log \log B_n)^{-1/2} \\ &= 2^{1/2} \limsup_{n \rightarrow \infty} \sum_{j=1}^n \chi(j+1)(x) \\ &\quad \times \left(2 \left(\sum_{j=1}^n u_j^2 \right) \log \log \left(2^{1/2} \left(\sum_{j=1}^n u_j^2 \right)^{1/2} \right) \right)^{1/2} \\ &= 2^{1/2} \quad \text{for almost all } x \text{ such that } x_1 = 0. \end{aligned}$$

A similar result is obtained for an arbitrary finite union of 2-dissociate sets viz.

there exists a set P which is the union of m 2-dissociate sets such that for some Υ and u , we have

$$\limsup_{n \rightarrow \infty} S_n(2B_n^2 \log \log B_n)^{-1/2} = m^{1/2}$$

on a set of positive measure. In this case, we examine the set $P = \bigcup_{j=1}^m \{\chi(F_j, k+n) : k \in N\}$ where F_1, \dots, F_m are distinct subsets of $\{1, 2, \dots, m\}$ and compute the lim sup on the set $\{x \in Z(2)^N : x_1 = x_2 = \dots = x_m = 0\}$.

A similar analysis using the example given in 3.3 yields that the upper bound given in 5.2 is attained for a set having the property (R_1) . Obviously we can extend this result to the finite union of sets having property (R_1) in the manner suggested above.

(3) As remarked in 5.4, no positive lower bound for the lim sup exists unless we impose the restriction that $\text{card } X_n$ be bounded for all n . In any case, no such bound exists for the examples given in (2) since for these we have $S_n = 0$ on a set of positive measure. In the case where $\text{card } X_n$ is bounded for all n and P is a set having the property (R_2) (in particular a 2-dissociate set), we might expect a positive lower bound to exist in view of the fact that these sets are precisely the sets that satisfy the Gaussian law.

(4) For the special case where $G = T$, the circle group, our result 5.2 is only partially comparable with Mary Weiss's theorem (see 1.4). It is well known that the dual group \hat{G} , in this case, is isomorphic to the additive group Z . We observe that a 2-dissociate set $\{n_k : k \in N\}$ is one for which $\sum_{k \in F} \epsilon_k n_k \neq 0$ if F is a nonempty finite subset of N and $\epsilon_k = \pm 2, \pm 1$. It is readily shown that a Hadamard set $\{n_k : k \in N\}$ of positive integers for which $n_{k+1}/n_k > 3$ for all $k \in N$ is a 2-dissociate set and any Hadamard set is the finite union of sets of this kind. However, a 2-dissociate set may fail to be the finite union of Hadamard sets (see Hewitt and Zuckerman [1]). Clearly our results apply to a larger class of lacunary sets than does Mary Weiss's theorem though our conclusions are weaker. It should also be noted that our results work for any partition Υ of Z into finite symmetric sets whereas Mary Weiss's results are shown for the natural ordering of N .

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