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BV-FUNCTIONS, POSITIVE-DEFINITE FUNCTIONS AND MOMENT PROBLEMS

BY

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ABSTRACT. Let S be a commutative semigroup with identity 1 and involution. A complex valued function f on S is defined to be positive definite if $\Pi_j \Delta_j f(1) \geqslant 0$ where the Δ_j's belong to a certain class of linear sums of shift operators. For discrete groups the positive definite functions defined herein are shown to be the classically defined positive definite functions. An integral representation theorem is proved and necessary and sufficient conditions for a function to be the difference of two positive-definite functions, i.e. a BV-function, are given. Moreover the BV-function defined herein agrees with those previously defined for semilattices, with respect to the identity involution. Connections between the positive-definite functions and completely monotonic functions are discussed along with applications to moment problems.

1. **Introduction.** Let S be an involution semigroup, Γ a subset of the complex-valued semicharacters on S equipped with a locally compact topology and M a subcollection of real regular Borel measures on Γ. Solvability of the moment problem

$$f(x) = \int_\Gamma \chi(x)\, d\mu(\chi) \qquad (\mu \in M) \tag{1.0.1}$$

is considered. It should be noted that all of the classically studied moment problems mentioned in [12] admit appropriate abstractions when put in this setting. A well-known instance of this is the abstraction of the trigonometric moment problem and subsequent solution by Raikov, cf. [9, p. 410]. More recent theorems of this type appear in [1], [5], [6], [7], [8] and [10].

A new notion of finite difference is introduced herein and those functions which admit an integral representation of the form (1.0.1) are characterized in terms of this difference operator, much in the flavor of Hausdorff's solution to the "little moment problem", cf. [8]. The main results of the work are Theorems 2.3 and 3.7. The first considers integral equations of the type (1.0.1) with respect to nonnegative μ and the second considers the same equation for signed μ. The

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first mentioned result implies a new characterization of both positive-definite functions on groups and $*$-definite functions, cf. [7]. Applications of the theory to inverse and Hermitian semigroups are given in §4 and §5.

2. **Positive-definite functions.** Let S be a commutative semigroup with identity 1 and involution $*$, cf. [7, p. 771], and C be the class of all complex valued functions f on S. The displacement operator $E_x|C \longrightarrow C$ is defined by $(E_x f)(y) = f(xy)$ $(x, y \in S)$ and frequently f_x will be written instead of $E_x f$, and I instead of E_1. Let $\{\eta_j\}$, $\{\alpha_j\}$ and $\{x_j\}$ $(j = 1, \dots, k)$ be finite sequences of nonnegative integers, fourth roots of unity and members of S respectively. A *difference operator* ∇ on C is defined by

$$\nabla f(\cdot;\ \{\alpha_j n_j x_j\}) = \prod_{j=1}^{k} \left(I + \frac{\alpha_j}{2} E_{x_j} + \frac{\alpha_j^*}{2} E_{x_j^*}\right)^{n_j} f. \tag{2.0.1}$$

It follows that

$$\begin{aligned} &\nabla_{k+1} f(x; \alpha_1 x_1, \dots, \alpha_{k+1} x_{k+1}) \\ &\quad = \nabla_k f(x; \alpha_1 x_1, \dots, \alpha_k x_k) \\ &\qquad + (\sigma_{k+1}/2) \nabla_k f(x x_{k+1}; \alpha_1 x_1, \dots, \alpha_k x_k) \\ &\qquad + (\sigma_{k+1}^*/2) \nabla_k f(x x_{k+1}; \alpha_1 x_1, \dots, \alpha_k x_k). \end{aligned} \tag{2.0.2}$$

Here and throughout this paper $*$ will denote both the semigroup involution and complex conjugation. Since $E_x E_y = E_{xy}$, the commutivity of S implies that ∇ is independent of the order in which the increments x_j are chosen so that the operator ∇ is well defined. A subset X of S such that every $y \in S$ except possibly the identity(1) admits a finite factorization of the form $\Pi_j x_j$ where $x_j \in X \cup X^*$ $(X^* = \{x \in S | x^* \in X\})$ will be called a *generator set* for S. Henceforth X will denote an arbitrary but fixed generator set for S.

If $f \in C$ and both $f(1) \geqslant 0$ and $\nabla f(1\colon\ \{\alpha_j x_j\}) \geqslant 0$ for all choices of $x_j \in X$ and for all choices of fourth roots of unity α_j, then f will be called *positive X-definite.* Later it will be shown that this notion of positive definite is equivalent to the classical concept when S is a discrete group and agrees with that introduced in [7] when $X = S$.

The difference operator Δ used in [8] can be extended by defining

$$\begin{aligned} &\Delta_{n+1} f(\cdot; \alpha_1 x_1, \dots, \alpha_{n+1} x_{n+1}) \\ &\quad = \Delta_n f(\cdot; \alpha_1 x_1, \dots, \alpha_n x_n) - \alpha_{n+1} \Delta_n f_{x_{n+1}}(\cdot; \alpha_1 x_1, \dots, \alpha_n x_n). \end{aligned} \tag{2.0.3}$$

(1) Although not explicity stated in [8] the generator set there need not contain the identity.

If S is Hermitian [7, p. 772] i.e. $x = x^*$ then

$$\nabla f(\cdot\,; \{\alpha_j x_j\}) = \prod_j \left[I + \left(\frac{\alpha_j + \alpha_j^*}{2} \right) E_{x_j} \right] f = \Delta f\left(\cdot\,; \left\{ -\left(\frac{\alpha_j + \alpha_j^*}{2} \right) x_j \right\}\right). \tag{2.0.4}$$

Thus $f \in K_X$ if and only if $\Delta f(1; \{\sigma_j x_j\}) \geq 0$ for all choices of $x_j \in X$ and square roots of unity σ_j.

If S is a group, $\{x_j\} \subset S$ and $\{\alpha_j\}$ is a set of fourth roots of unity for $j = 1, 2, \ldots, k$ then $\Pi_j(2I + \alpha_j E_{x_j^*} + \alpha_j^* E_{x_j^*}) = \Pi_j(I + \alpha_j E_{x_j^*} + \alpha_j^* E_{x_j^*} + \alpha_j \alpha_j^* E_{x_j x_j^*})$ so that the difference operator ∇ defined herein is related to the classical difference operator Δ used in [8] by

$$2^k \nabla f(\cdot\,; \{\alpha_j x_j\}) = \Delta f(\cdot\,; \{-\alpha_j x_j, -\alpha_j^* x_j^*\}). \tag{2.0.5}$$

Thus in terms of differences f is positive definite if and only if all differences of the form $\Delta f(1; \{\alpha_j x_j, \alpha_j^* x_j^*\})$ are nonnegative where $\{x_j\}$ is an arbitrary finite subset of a generator set X.

Let K_X denote the cone of positive X-definite functions. If X is a $*$-semicharacter i.e. if χ is a not necessarily bounded(2) complex-valued function on S such that $\chi \neq 0$, $\chi(xy) = \chi(x)\chi(y)$ and $\chi^*(x) = \chi(x^*)$ then

$$\nabla \chi(x; \{\sigma_j x_j\}) = \chi(x) \prod_j \left(1 + \frac{\alpha_j}{2} \chi(x_j) + \frac{\alpha_j^*}{2} \chi^*(x_j) \right) \tag{2.0.6}$$

thus χ is positive X-definite if and only if both

$$|\mathrm{Re}\, \chi(x)| \leq 1 \quad \text{and} \quad |\mathrm{Im}\, \chi(x)| \leq 1 \tag{2.0.7}$$

for all $x \in X$. The class of all such $*$-semicharacters is denoted by Γ_X. The cone K_X has a natural description in terms of Γ_X. The following proposition is essential to the development.

PROPOSITION 2.1. *Let $f \in K_X$, $\{x_j\}_{j=1,2,\ldots,k} \subset X \cup X^*$ and $x \in S$. Then*

$$\left| f\left(\prod_j x_j \right) \right| \leq 2^k f(1), \tag{2.1.1}$$

$$f(x^*) = f^*(x). \tag{2.1.2}$$

PROOF. Let A_k denote the set of all functions $\sigma \equiv \sigma_{(\cdot)}$ on the first k natural numbers whose range is contained in the three element set $\{0, 1, *\}$. If $\beta^4 = 1$ then

(2) In [7], all $*$-semicharacters were, by definition assumed bounded. Here they are not.

$$\sum_{\Pi\alpha_j^*=\beta} \nabla f(\cdot\,;\{\alpha_j x_j\}) = \sum_{\Pi_j\alpha_j^*=\beta} \Pi_j\left(I+\frac{\alpha_j}{2}E_{x_j}+\frac{\alpha_j^*}{2}E_{x_j^*}\right)f$$

$$= \sum_{(\Pi_j\alpha_j^*=\beta,\sigma\in A_k)} \Pi_j\left(\frac{\alpha_j}{2}\right)^{\sigma_j} f_{(x_1^{\sigma_1}\cdots x_k^{\sigma_k})}$$

$$= 4^{k-1}f + 2^{k-2}\beta^* f_{x_1\cdots x_k} + 2^{k-2}\beta^* f_{x_1^*\cdots x_k^*}$$

$$+ \sum_{(\sigma\neq 1,*,0)} \sum_{\Pi_j\alpha_j^*=\beta} \Pi_j\left(\frac{\alpha_j}{2}\right)^{\sigma_j} f_{(x_1^{\sigma_1}\cdots x_k^{\sigma_k})}.$$

But $\Sigma_{\Pi_j\alpha_j^*=\beta}\Pi_j(\alpha/2)^{\sigma_j} = 0$ so that if $x = \Pi_j x_j$ then

(2.1.3) $$\sum_{\Pi\alpha_j^*=\beta} \nabla f(\cdot\,,\{\alpha_j x_j\}) = [2^{k-1}f + \tfrac{1}{2}\beta f_x + \tfrac{1}{2}\beta^* f_{x^*}]$$

for all functions f.

For $f\in K$ and $x_j\in X$, evaluation of (2.1.3) at 1 shows

$$2^{k-1}f(1) + \tfrac{1}{2}\beta f(x) + \tfrac{1}{2}\beta^* f(x^*) \geq 0.$$

Setting $\beta = 1$ and i shows

$$\operatorname{Re} f(x) = \tfrac{1}{2}[f(x)+f(x^*)], \qquad \operatorname{Im} f(x) = (i/2)[f(x)-f(x^*)]$$

from which (2.1.1) and (2.1.2) follow.

PROPOSITION 2.2. *Every extreme point of B_X is a $*$-semicharacter in Γ_X.*

PROOF. First observe that $\nabla f(\cdot\,;\alpha x)$ $(\alpha^4 = 1)$ is in K_X whenever $f\in K_X$ and $x\in X$. Indeed if $x = x_{k+1}$ then

$$\nabla_k[\nabla f(\cdot\,;\alpha x)]\,(1;\alpha_1 x_1,\ldots,\alpha_k x_k) = \left[\Pi_j(I+\alpha_j E_{x_j}+\alpha_j^* E_{x_j^*})f\right](1)$$

$$= \nabla_{k+1} f(1;\alpha_1 x_1,\ldots,\alpha_{k+1}x_{k+1}).$$

Now suppose f is an extreme point of B_X. Direct computations shows $f = \tfrac{1}{4}\Sigma_{\alpha^4=1}\nabla f(\cdot\,;\alpha x)$ so that $f - \tfrac{1}{4}\nabla f(\cdot\,;\alpha x)\in K_X$. Since f is on an extreme ray of K_X there exists $\lambda_\alpha > 0$ such that $\lambda_\alpha f = \nabla f(\cdot\,;\alpha x)$. Evaluation at $y = 1$ shows $\lambda_1 = 1 + \tfrac{1}{2}f(x) + \tfrac{1}{2}f(x^*)$ and $\lambda_i = 1 + (i/2)\bar{f}(x) - (i/2)f(x^*)$. Evaluation at arbitrary y gives

$$f(x)f(y) + f(x^*)f(y) = f(xy) + f(x^*y),$$

$$f(x)f(y) - f(x^*)f(y) = f(xy) - f(x^*y).$$

Thus $f(x)f(y) = f(xy)$ for all $x \in X$ and $y \in S$ and since X is a generator set this same multiplicative property holds for arbitrary $x \in S$. The assertion follows from (2.0.7) and (2.1.2).

For each $x \in S$ let $\hat{x}$ denote the evaluation function on Γ_X, i.e. $\hat{x}(\chi) = \chi(x)$ for all $\chi \in \Gamma_X$.

THEOREM 2.3. *A real-valued function f is positive X-definite if and only if there exists a necessarily unique nonnegative regular Borel measure μ_f on Γ_X such that $f(x) = \int_{\Gamma_X} \hat{x}\, d\mu_f$.*

PROOF. The existence of the representing measure follows from the Kreĭn-Milman Theorem and Proposition 2.2. Since x is a continuous linear functional on E_X, uniqueness is a consequence of the Stone-Weierstrass theorem and follows along the lines of [5, Theorem 1.2]. If f admits such an integral representation then we must have

$$\nabla f(1; \{\sigma_j x_j\}_j) = \int_{\Gamma_X} \prod_j \left(1 + \frac{\sigma_j}{2}\hat{x}_j + \frac{\sigma_j^*}{2}\hat{x}_j^*\right) d\mu_f$$

(2.3.1)

$$\int \prod_{\sigma_j\text{-real}} (1 \pm \operatorname{Re} \chi(x_j)) \prod_{\sigma_j\text{-Imag}} (1 \pm \operatorname{Im} \chi(x_j))\, d\mu_f > 0$$

and the assertion follows.

REMARK. Theorem 2.3 may be used to argue as in [5, Corollary 1.3] that Γ_X is precisely the set of extreme points of B_X.

It follows from (2.1.1) that the semicharacters in Γ_S are bounded so that $|\chi(x)| \leq 1$ for all $\chi \in \Gamma_S$ and all $x \in S$. If the integral representation theorem above is compared with [7, Theorem 2.1] then the formally different notion of $*$-definite defined therein is seen to agree with the concept of positive S-definite here. Moreover if S is a Hermitian semigroup then $0 \leq \nabla f(1; \pm x)$ implies

$$|f(x)| = |\operatorname{Re} f(x)| \leq f(1) = 1 \qquad (2.3.2)$$

for all $f \in K_X$ and all $x \in X$. But if f is a semicharacter then (2.3.2) holds for all $x \in S$ so that again $\Gamma_S = \Gamma_X$ and hence the notions of positive X-definite and Hermitian definite introduced in [7] coincide for all choices of X. Finally if S is an inverse semigroup, cf. [2, §1.9], then $|\chi| = 1$ for all $\chi \in \Gamma_X$ so that $\Gamma_S = \Gamma_X$ also. In particular for S a group, the classical notion of positive definite is synonomous with that of positive X-definite.

COROLLARY 2.4. *A complex valued function f on an involution semigroup S is $*$-definite if and only if f is positive S-definite. If f is bounded in absolute value by $f(1)$ or if S is either an inverse semigroup or a Hermitian semigroup then f is positive X-definite if and only if f is positive S-definite.*

PROOF. The only unproved assertion is the case where f is bounded in absolute value by $f(1)$. If f is positive S-definite then the integral representation theorem implies $|f| \leqslant f(1)$. To prove the converse let K denote the closed cone of all $f \in K_X$ such that $|f| \leqslant f(1)$. Then $B = B_X \cap K$ is a compact base for K and the argument that the extreme points of B are contained in Γ_X follows as in Proposition 2.2. But since each extreme point is a bounded function the set of all extreme points of B is contained in Γ_S so that the integral representation of Theorem 2.3 holds. The converse assertion follows.

EXAMPLE 2.5. Consider the product $S = N \times N$ of the additive semigroup N of nonnegative integers with itself, and with involution $(m, n)^* = (n, m)$. If $X = \{(1,0),(0,1)\}$ Γ_X is the set of all maps χ_z defined by $\chi_z(m,n) = z^m(z^*)^n$ where $|\mathrm{Re}\, z| \leqslant 1$ and $|\mathrm{Im}\, z| \leqslant 1$ $(0^0 = 1)$ while $\Gamma_S = \{\chi_z \in \Gamma_X \mid |z| \leqslant 1\}$. Thus Γ_S and K_S are properly contained in Γ_X and K_X respectively. The cones K_X and K_S provide all solutions with nonnegative measures to the following respective moment problems

$$a_{m,n} = \int_{|\mathrm{Re}\, z| \leqslant 1;\, |\mathrm{Im}\, z| \leqslant 1} z^m (z^*)^n \, d\mu(z) \tag{2.5.1}$$

$$a_{m,n} = \int_{|z| \leqslant 1} z^m (z^*)^n \, d\mu(z), \qquad m, n = 0, 1, 2, \ldots . \tag{2.5.2}$$

3. **BV-functions.** Theorem 2.3 implies that the map $f \longrightarrow \mu_f$ of the real linear span of E_X onto its representing measure is an isomorphism. Let $|\mu|$ denote the variation of a measure μ; then the lattice properties of the regular Borel measures can be imposed on E_X by defining

$$f \vee (-f) \equiv |f|(x) = \int_{\Gamma_X} \hat{x} \, d|\mu_f|(\chi). \tag{3.0.1}$$

Also define $\|f\| = |f|(1) = \|\mu_f\|$.

An explicit description of those functions in E_X as well as a characterization of their variation will be given. First the total variation of regular Borel measures on Γ_X will be described in terms of certain partitions of unity on Γ_X. For this let P denote the collection of all partitions obtained upon expansion of products of the form

$$\prod_j \frac{1}{2^{n_j}} \left[\left(1 + \frac{\sigma_j}{2}\hat{x}_j + \frac{\sigma_j^*}{2}\hat{x}_j^*\right) + \left(1 - \frac{\sigma_j}{2}\hat{x}_j - \frac{\sigma_j^*}{2}\hat{x}_j^*\right)\right]^{n_j}$$

where $\sigma = 1, i$ and $x \in X$. A typical member of P is then a set

$$p = \{P_{i_1 \cdots i_k} \mid i_j = 0, \ldots, n_j, j = 1, \ldots, k\}$$

of functions.

$$P_{i_1 \cdots i_k} = \frac{1}{2^n} \prod_j \binom{n_j}{i_j} \left(1 + \frac{\sigma_j}{2}\hat{x}_j + \frac{\sigma_j^*}{2}\hat{x}_j^*\right)^{i_j} \left(1 - \frac{\sigma_j}{2}\hat{x}_j - \frac{\sigma_j^*}{2}\hat{x}_j^*\right)^{n_j - i_j}$$

where $n = \Sigma_j n_j$, $\sigma_j = 1$ or i and $x_j \in X$. If μ_f represents f then (2.3.1) implies

$$\text{(3.0.2)} \qquad \int_{\Gamma_X} P_{i_1 \cdots i_k}\, d\mu_f = \frac{1}{2^n} \prod_j \binom{n_j}{i_j} \nabla f(1;\ \{\sigma_j i_j x_j,\ \sigma_j(i_j - n_j)x_j\}).$$

Thus if

$$\text{(3.0.3)} \qquad \|f\|_{(\{x_j\},\ \{n_j\},\ k)} = \frac{1}{2^n} \sum_\sigma \prod_j \binom{n_j}{i_j} |\nabla f(1;\ \{\sigma_j i_j x_j,\ \sigma_j(i_j - n_j)x_j\})|,$$

then

$$\text{(3.0.4)} \qquad \|f\| \geqslant \sup \|f\|_{(\{x_j\},\ \{n_j\},\ k)}$$

where the x_j's form a finite and possible repetitious sequence of elements in X, the n_j's are nonnegative integers and $j = 1, 2, \ldots, k$ where k is allowed to vary.

Lemma 3.1. *If $f \in E_X$ then $\|f\| = \sup \|f\|_{(\{x_j\},\ \{n_j\},\ k)}$.*

Proof. The inequality (3.0.4) will be reversed by showing that $\hat{P}$ is rich enough to describe the total variation of the measures μ on Γ_X. That is

$$\text{(3.1.1)} \qquad \|\mu\| = \sup_{p \in P} \sum_{P \in p} \left| \int P\, d\mu \right|.$$

Let χ_1 and χ_2 be two distinct members of Γ_X. There exists $x \in X$ such that $\chi_1(x) \neq \chi_2(x)$. Suppose Re $\chi_1(x) \neq$ Re $\chi_2(x)$. Let G_1 and G_2 be disjoint open sets about $\chi_1(x)$ and $\chi_2(x)$ respectively. Since Re $\hat{x}$ is continuous on Γ_X their inverse images $(\text{Re}\ \hat{x})^{\leftarrow}[G_l]$ $(l = 1, 2)$ are open and disjoint subsets of Γ_X. By an appropriate change of variables, Bernstein polynomials on the interval $[-1, 1]$ take the form

$$(B_n g)(t) = \frac{1}{2^n} \sum_{i_1=0}^{n} \binom{n}{i_1} g\left(1 - 2\frac{i_1}{n}\right) (1 + t)^{i_1} (1 - t)^{n - i_1}.$$

From [4, p. 6] there exists a sufficiently large integer N such that if $n > N$ then the partition of unity

$$\left\{ \frac{1}{2^n} \binom{n}{i_1} (1 + t)^{i_1} (1 - t)^{n - i_1} \right\}_{i_1}$$

contains a subpartition which is arbitrarily close to unity on G_1 and arbitrarily close to zero on G_2. Thus

$$\left\{ \frac{1}{2^n} \binom{n}{i_1} \left(1 + \frac{\hat{x}}{2} + \frac{\hat{x}^*}{2}\right)^{i_1} \left(1 - \frac{\hat{x}}{2} - \frac{\hat{x}^*}{2}\right)^{n - i_1} \right\}_{i_1}$$

has the same separation property with respect to the open Re $\hat{x}^{\leftarrow}(G_1)$ and Re $\hat{x}^{\leftarrow}(G_2)$. If Im $\chi_1(x) \neq$ Im $\chi_2(x)$ then a similar argument yields a separating partition of the form

$$\left\{\frac{1}{2^n}\binom{n}{i_1}\left(1+\frac{i}{2}\hat{x}-\frac{i}{2}\hat{x}^*\right)^{i_1}\left(1-\frac{i}{2}\hat{x}+\frac{i}{2}\hat{x}^*\right)^{n-i_1}\right\}.$$

The arguments of Lemmas 3.1 and 3.2 of [8] can now be used to prove the lemma.

LEMMA 3.2. *Let $j = 2, 3, \ldots, k$ and $\sigma, \sigma_2, \ldots, \sigma_k$ be fourth roots of unity. Then*

$$\sum_{i_1=0}^{n}\binom{n}{i_1}|\nabla f(1;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)|$$
$$\leq \frac{1}{2}\sum_{i_1=0}^{n+1}\binom{n+1}{i_1}|\nabla f(1;\ \{\sigma_j x_j\}_j, \sigma i_1 x,\ \sigma(i_1-n-1)x)|.$$

PROOF.

$$\begin{aligned}
|\nabla f(1;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x|
&\leq \tfrac{1}{2}\Big|\nabla f(1;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)\\
&\quad+\frac{\sigma}{2}\nabla f(x;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)\\
&\quad+\frac{\sigma^*}{2}\nabla f(x^*;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)\Big|\\
&\quad+\tfrac{1}{2}\Big|\nabla f(1;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)\\
&\quad-\frac{\sigma}{2}\nabla f(x;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)\\
&\quad-\frac{\sigma^*}{2}\nabla f(x;\ \{\sigma_j x_j\}_j,\ \sigma i_1 x,\ \sigma(i_1-n)x)\Big|.
\end{aligned}$$

The assertion follows upon applying (2.0.2), regrouping and observing that

$$\binom{n}{i_1}+\binom{n}{i_1-1}=\binom{n+1}{i_1}\quad \text{if } n \geq i_1 > 0.$$

In particular the above lemma implies that $\|f\|_{(\{x_j\},\ \{n_j\},\ k)}$ is an increasing function of n_j and k. Moreover duplicate entries among $\{x_j\}$ may be eliminated as follows. Suppose $x_1 = x_2$ then

$$\|f\|_{(\{x_j\},\ \{n_j\},\ k)} = \|f\|_{(\{x_2,\ldots,x_k\},\ \{n_1+n_2,\ n_3,\ldots,n_k\},\ k)}$$

since

$$\sum_{s+t=i_2} \binom{n_1}{s}\binom{n_2}{t} = \binom{n_1+n_2}{i_2}.$$

When the x_j's are distinct with $X' = \{x_j\}_j$ and $n_j = n$ for all j the notation $(\{x_j\}, \{n_j\}, k)$ will be replaced by (X', n). Thus

$$\sup\|f\|_{(\{x_j\},\{n_j\})} = \lim\|f\|_{(X',n,k)} \tag{3.2.1}$$

where the set $\{(X',n)\}$ is directed by $(X', n) \geqslant (X'', n')$ whenever $X' \supset X''$, $n \geqslant n'$.

For arbitrary $f \in C$ the total variation $\|f\|$ is defined by (3.2.1). If $f(x) = f^*(x^*)$ and if $\|f\| < \infty$ then f will be called a BVX-*function* or equivalently $f \in$ BVX(S). Lemma 3.1 implies BVX(S) $\supset E_X$ and the main theorem, Theorem 3.7 of this section will show that equality holds. Lemmas 3.3–3.6 pave the way for this result.

LEMMA 3.3. *Let $f \in$ BVX, $x \in X$ and α be a fourth root of unity. Then*

$$\|f\| = ½\|\nabla f(\cdot\,;\alpha x)\| + ½\|\nabla f(\cdot\,;-\alpha^* x^*)\|.$$

PROOF. From (3.2.1),

$$\|f\| \leftarrow \frac{1}{2^{kn+1}}\sum_i \binom{1}{i_0}\binom{n}{i_1}\cdots\binom{n}{i_k}$$

$$\cdot\,|\nabla f(1;\{\sigma_j i_j x_j, \sigma_j(i_j-n)x_j^*\}, \alpha i_0 x, \alpha^*(i_0-1)x^*)|$$

$$= \frac{1}{2}\cdot\frac{1}{2^{kn}}\sum\binom{n}{i_1}\cdots\binom{n}{i_k}\left|\nabla\left(f+\frac{\alpha}{2}f_x+\frac{\alpha^*}{2}f_{x^*}\right)(1;\{\sigma_j i_j x_j, \sigma_j(i_j-n)x_j^*\})\right|$$

$$+\frac{1}{2}\cdot\frac{1}{2^{kn}}\sum\binom{n}{i_1}\cdots\binom{n}{i_k}\left|\nabla\left(f-\frac{\alpha}{2}f_x-\frac{\alpha^*}{2}f_{x^*}\right)(\{\sigma_j i_j x_j, \sigma_j(i_j-n)x_j^*\})\right|\cdot$$

$$\to ½\|\nabla f(\cdot\,;\alpha x)\| + ½\|\nabla f(\cdot\,;\alpha^* x^*)\|.$$

In particular $\nabla f(\cdot;\{\alpha_j x_j\}) \in$ BVX whenever $f \in$ BVX.

COROLLARY 3.4. *Let $f \in$ BVX, T_k be a set of distinct 4th root of unity-valued functions $\alpha_{(\cdot)}$ on $\{j \mid j = 1, \ldots, k\}$ and n_α a nonnegative integer for each α. If $x_1, \ldots, x_k \in X$ then*

$$\sum_{\alpha\in T_k} n_\alpha\|\nabla f(\cdot;\{\alpha_j x_j\})\| = \left\|\sum_{\alpha\in T_k} n_\alpha \nabla f(\cdot;\{\alpha_j x_j\})\right\|.$$

PROOF. Let T_k' denote the set of all 4th root of unity-valued functions on $\{1, 2, \ldots, k\}$ which are not in T_k and $n = \max_\alpha n_\alpha$. Then

$$4^k n\|f\| = \left\|\sum_{\alpha_1}\sum_{\alpha_2}\cdots\sum_{\alpha_k} n\nabla f(\cdot\,;\{\alpha_j x_j\})\right\| \qquad (\alpha\in T_k\cup T_k')$$

$$\leq \left\|\sum_{\alpha\in T_k'} n_\alpha \nabla f(\cdot\,;\{\alpha_j x_j\})\right\| + \left\|\sum_{\alpha\in T_k}(n-n_\alpha)\nabla f(\cdot\,;\{\alpha_j x_j\})\right\|$$

$$+ \left\|\sum_{\alpha\in T_k'} n\nabla f(\cdot\,;\{\alpha_j x_j\})\right\|$$

$$\leq \sum_{\alpha\in T_k} n_\alpha\|\nabla f(\cdot\,;\{\alpha_j x_j\})\| + \sum_{\alpha\in T_k}(n-n_\alpha)\|\nabla f(\cdot\,;\{\alpha_j x_j\})\|$$

$$+ \left\|\sum_{\alpha\in T_k'} n\nabla f(\cdot\,;\{\alpha_j x_j\})\right\|$$

$$\leq n \sum_{\alpha\in T_k\cup T_k'} \|\nabla f(\cdot\,;\{\alpha_j x_j\})\|$$

$$= 4^{k-1} n \sum_{\alpha_1}\cdots\sum_{\alpha_{k-1}} 4\|\nabla f(\cdot\,;\{\alpha_j x_j \mid j=1,\ldots,k-1\})\| = 4^k n\|f\|.$$

A second application of the triangle inequality verifies the assertion.

LEMMA 3.5. *Let f be a complex valued function on S, T_k be all fourth root of unity valued functions α defined on $\{1,\ldots,k\}$ and $x_1,\ldots,x_k\in S$. Then*

$$f\left(\prod_j x_j\right) = \frac{1}{2^k}\sum_{\alpha\in T_k}\left(\prod_j \alpha_j^*\right)\nabla f(1;\{\alpha_j x_j\}).$$

PROOF. Let A_k be as defined in the proof of Proposition 2.1. Then the above summation can be rewritten as

$$\frac{1}{2^k}\sum_\alpha \prod_j \alpha_j^* \sum_{\sigma\in A_k}\prod_j\left(\frac{\alpha_j}{2}\right)^{\sigma_j} f\left(\prod_j x_j^{\sigma_j}\right)$$

$$= f\left(\prod_j x_j\right) + \frac{1}{2^k}\sum_{\sigma_j\neq 1}\left(\sum_\alpha \prod_j \alpha_j^*\left(\frac{\alpha_j}{2}\right)^{\sigma_j}\right) f\left(\prod_j x^{\sigma_j}\right)$$

$$= f\left(\prod_j x\right) + \frac{1}{2^k}\sum_{\sigma_j\neq 1}\left(\sum_{\alpha\in T_{k-1}}\prod_{j=1}^{k-1}(\alpha_j^*\alpha_j^{\sigma_j})\sum_{\alpha_k}(\alpha_k^*\alpha_k^{\sigma_k})\frac{f(\Pi_j x^{\sigma_j})}{\Pi_j(2^{\sigma_j})}\right).$$

But a trivial computation shows that $\Sigma_{\alpha_k}\alpha_k^*\alpha_k^{\sigma_k} = 0$ if $\sigma_k\neq 1$ and the assertion is proved.

Let $f\in E_X$ with representing measure μ_f and $x_1,\ldots,x_k\in X\cup X^*$. Then if $x=\Pi_j x_j$, Lemma 3.5, and formulas (2.0.6) and (3.0.1) imply

$$|f|(x) = \frac{1}{2^k} \sum_{\alpha \in T_k} \left(\prod_j \alpha_j^*\right) \nabla |f|(1; \{\alpha_j x_j\}_j)$$

$$= \frac{1}{2^k} \sum \left(\prod_j \alpha_j^*\right) \int_{\Gamma_X} \prod_j \left(1 + \frac{\alpha_j}{2} \hat{x}_j + \frac{\alpha_j}{2} \hat{x}_j^*\right) d|\mu_f|$$

and hence the variation $|f|$ of f can be characterized by (3.5.1) below:

(3.5.1) $$|f|(x) = \frac{1}{2^k} \sum_\alpha \left(\prod_j \alpha_j^*\right) \| \nabla f(\cdot\, ; \{\alpha_j x_j\}_j) \|.$$

Since Lemma 3.3 implies $\|\nabla f(\cdot\, ; \{\alpha_j x_j\}_j)\| < \infty$ whenever $f \in BVX$, (3.5.1) will be taken as a definition for $|f|$ for all $f \in BVX$.

It remains to be seen that $|f|$ is well defined. For this let $x_1, \ldots, x_k \in X$ and x denote their product. Then (2.1.3) implies

(3.5.2) $$\operatorname{Re}|f|(x) = ½\|2^{k-1}f + ½f_x + ½f_{x^*}\| - ½\|2^{k-1}f - ½f_x - ½f_{x^*}\|$$

and

$$\operatorname{Im}|f|(x) = ½\|2^{k-1}f - ½if_x + ½if_{x^*}\| - ½\|2^{k-1}f + ½if_x - ½if_{x^*}\|$$

where $|f|(x)$ is the variation defined by the factors x_j.

Thus $|f|(x)$ is at worst dependent on the number of factors of x rather than the factors themselves. To see that f is also independent of this number observe that if x admits a factorization by $k + m$ elements from X and if $|f|_0(x)$ denotes the variation of f at x computed with respect to this latter factorization then

$$\operatorname{Re}|f|_0(x) = ½\|2^{k-1}(2^m - 1)f + (2^{k-1}f + ½f_x + ½f_{x^*})\|$$

$$- ½\|2^{k-1}(2^m - 1)f + (2^{k-1}f - ½f_x - ½f_{x^*})\|$$

$$= \frac{1}{2} \left\| \frac{2^m - 1}{2^{k+1}} \sum_\beta \nabla f(\cdot\, ; \{\beta_j x_j\}) + \frac{1}{2^{k-1}} \sum_{\Pi\alpha_j = 1} \nabla f(\cdot\, ; \{\alpha_j x_j\}) \right\|$$

$$- \frac{1}{2} \left\| \frac{2^m - 1}{2^{k+1}} \sum_\beta \nabla f(\cdot\, ; \{\beta_j x_j\}) + \frac{1}{2^{k-1}} \sum_{\Pi\alpha_j = -1} \nabla f(\cdot\, ; \{\alpha_j x_j\}) \right\|$$

where $\beta(\cdot) \equiv \beta$ denotes an arbitrary 4th root of unity valued function. Corollary 3.4 now implies that $\operatorname{Re}|f|_0(x)$ is the $\operatorname{Re}|f|(x)$ computed in 3.5.2.

Analogously, it can be shown that $\operatorname{Im}|f|_0(x) = \operatorname{Im}|f|(x)$ so that $|f|(x)$ is well defined whenever $x \neq 1$ or $x = 1$ admits a factorization by members of $X \cup X^*$. If 1 does not admit such a factorization then $|f|(1)$ is defined to be $\|f\|$. Note that formula (3.5.2) implies that $|f|(1) = \|f\|$ when 1 admits such a factorization.

LEMMA 3.6. *If β is a fourth root of unity and $f \in BVX$ then*

(3.6.1) $$½|\nabla f(\cdot\, ; \beta x)| + ½|\nabla f(\cdot\, ; -\beta^* x^*)| = |f|,$$

$$|\nabla f(\cdot;\beta x)| - |\nabla f(\cdot;-\beta^* x^*)| = \beta |f|_x + \beta^* |f|_{x^*}, \tag{3.6.2}$$

for all $x \in X$.

PROOF. Evaluation of (3.6.1) at 1 is just Lemma 3.3. If $x_1, \ldots, x_k \in X \cup X^*$ and $y = \Pi_j x_j$ then

$$|\nabla f(\cdot;\beta x)|(y) = \frac{1}{2^k} \sum_\alpha \prod_j \alpha_j^* \|\nabla f(\cdot;\ \{\alpha_j x_j\}, \beta x)\|,$$

so that equality holds in (3.6.1) for evaluation at y from Lemma 3.3. To verify (3.6.2) observe that both

$$|\nabla f(\cdot;\beta x)|(1) = \frac{1}{4} \sum_{\alpha^4=1} \|\nabla f(\cdot;\beta x, \alpha 1)\|, \tag{3.6.3}$$

$$|f|_1(x) = |f|(x) = \frac{1}{4} \sum_{\alpha^4=1} \alpha^* \|\nabla f(\cdot;\alpha x)\|. \tag{3.6.4}$$

To prove that (3.6.2) holds when evaluated at 1, substitute the above two equations into (3.6.2) and show that positive and negative real as well as positive and negative imaginary parts of each side of the equation agree respectively. This results in verifying sixteen equations which can be written succinctly as

$$\|f(\cdot;\alpha x)\| + \|\nabla f(\cdot;\alpha\beta^2 x)\| = \|\nabla f(\cdot;\alpha\beta^* x, \beta 1)\| + \|\nabla f(1;-\alpha\beta^* x, -\beta 1)\|. \tag{3.6.5}$$

That (3.6.5) holds for all fourth roots of unity α and β follows from Lemma 3.3 and (3.6.6) below.

$$\nabla g(\cdot;\beta 1) = \begin{cases} 0 & \text{if } \beta = -1, \\ g & \text{if } \beta^2 = -1, \\ 2g & \text{if } \beta = +1. \end{cases} \tag{3.6.6}$$

The right side of (3.6.2), when evaluated at y, expands to

$$\frac{1}{2^k} \sum_\alpha \prod_j \alpha_j^* \Bigg[\frac{\beta}{2} \sum_{\alpha_{k+1}} \alpha_{k+1}^* \|\nabla f(\cdot;\ \{\alpha_j x_j\}, \alpha_{k+1} x)\|$$

$$+ \frac{\beta^*}{2} \sum_{\alpha_{k+1}} \alpha_{k+1}^* \|\nabla f(\cdot;\ \{\alpha_j x_j\}, \alpha_{k+1} x^*)\| \Bigg] \quad (j = 1, \ldots, k)$$

$$= \frac{1}{2^k} \sum_\alpha \prod_j{}^* \alpha_j^* \left[\beta |\nabla f(\cdot;\ \{\alpha_j x_j\})|_x(1) + \beta^* |\nabla f(\cdot;\ \{\alpha_j x_j\})|_x(1)\right]$$

$$= \frac{1}{2^k} \sum \prod_j \alpha_j^* \left[|\nabla f(\cdot;\ \{\alpha_j x_j\}, \beta x)|(1) - |\nabla f(\cdot;\ \{\alpha_j x_j\}, -\beta^* x^*)|(1)\right]$$

$$= |\nabla f(\cdot;\beta x)|(y) - |\nabla f(\cdot;-\beta^* x^*)|(y).$$

THEOREM 3.7. $\mathrm{BVX} = E_X$.

PROOF. From the remarks following Lemma 3.1, the only part of the theorem yet to be proved is BVX $\subset E_X$. This will be done by showing that each BVX-function f is the difference of two positive X-definite functions. Adding half of (3.6.2) to (3.6.1) gives $\nabla|f|(\cdot; \beta x) = |\nabla f(\cdot; \beta x)|$ for all $x \in X$. Thus if $\{x_j\} \subset X$,

$$\nabla|f|(\cdot\,; \{\alpha_j x_j\}) = \left[\prod_j \left(I + \frac{\alpha_j}{2} E_{x_j} + \frac{\alpha_j^*}{2} E_{x_j^*}\right)\right] |f|$$

$$= \left|\prod_j \left(I + \frac{\alpha_j}{2} E_{x_j} + \frac{\alpha_j^*}{2} E_{x_j^*}\right) f\right| = |\nabla f(\cdot; \{\alpha_j x_j\})|.$$

In particular, $|f|$ is positive X-definite. Let $f^\pm = \frac{1}{2}(|f| \pm f)$. Then

$$\nabla f^\pm(1; \{\alpha_j x_j\}) = \tfrac{1}{2}\nabla|f|(1; \{\alpha_j x_j\}) \pm \tfrac{1}{2}\nabla f(1; \{\alpha_j x_j\})$$

$$= \tfrac{1}{2}\|\nabla f(1; \{\alpha_j x_j\}\| \pm \tfrac{1}{2}\nabla f(1; \{\alpha_j x_j\}).$$

But if g is any complex valued function on S such that $g(1)$ is real then

$$\|g\| \geqslant \tfrac{1}{2}|g(1) + \tfrac{1}{2}g(x) + \tfrac{1}{2}g(x^*)| + \tfrac{1}{2}|g(1) - \tfrac{1}{2}g(x) - \tfrac{1}{2}g(x^*)|$$

$$\geqslant |g(1)| \geqslant \pm g(1).$$

Since $f^*(x) = f(x^*)$ for all $x \in S$ and $1^* = 1$ it follows that $\nabla f(1; \{\alpha_j x_j\})$ is real so that $f^\pm$ is positive X-definite. The assertion follows since $f = f^+ - f^-$.

REMARK. The BVX-functions are precisely those functions which can be expressed as the difference of two positive X-definite functions. As in Corollary 2.4 if S is either a Hermitian semigroup or an inverse semigroup then BVS = BVX for all choices of generator sets X. If S is both square root closed and Hermitian then as remarked in [7, Proposition 4.2] the positive definite functions are just the completely monotonic (CM) function and in particular if S is a linearly ordered semilattice the CM-functions are just the nonincreasing nonnegative functions. For this classical case the usual notion of Bounded Variation agrees with that introduced here and the reader will recall that in the classical setting the BV-functions are those which can be expressed as the difference of two nonnegative, nonincreasing functions. Finally it should be noted that for arbitrary S, [7, Corollary 2.2] implies BVS is a Banach algebra under pointwise multiplications since the convolution of the representing measure of two BVS-functions is the representing measure of their product.

4. Moment problem with respect to characters. A semicharacter $\chi \in \Gamma_S$ will be called a *character* if $|\chi|^2 = |\chi|$. Let Γ_0 denote the set of all characters equipped with the topology of pointwise convergence and consider the moment problem

$$f(x) = \int_{\Gamma_0} \chi(x)\, d\mu_f(\chi), \tag{4.0.1}$$

where μ_f is nonnegative (or signed).

THEOREM 4.1. *A complex valued function f on S admits an integral representation of the form* (4.0.1) *if and only if f is positive S-definite* (*or* $f \in \mathrm{BV}(S)$) *and*

$$f[(xx^*)^2] = f(xx^*). \tag{4.1.1}$$

PROOF. If f admits such an integral representation then

$$f(xx^*) = \int_{\Gamma_0} |\chi(x)|^2\, d\mu_f(\chi) = \int_{\Gamma_0} |\chi(x)|^4\, du_f(\chi) = f[(xx^*)^2]$$

and the work of §§2 and 3 imply f is positive S-definite (or $f \in \mathrm{BV}(S)$).

As in §2 let $B_S = \{f \in K_S | f(1) = 1\}$ and set $B_0 = \{f \in B_S | f[(xx^*)^2] = f(xx^*)$. Since B_0 is convex and closed relative to the topology of simple convergence the converse assertion can be established by showing that B_0 is an extremal subset of B_S. For then it will follow from the remark following Theorem 2.3 that Γ_0 is the set of extreme points of B_0. Since Γ_0 is closed the results of [11] apply. To see that B_0 is extremal let $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$ where $f_j \in B(S)$ $(j = 1, 2)$ and $f \in B_0$. Then

$$f_1(xx^*) + f_2(xx^*) = 2f[(xx^*)^2] = f_1[(xx^*)^2] + f_2[(xx^*)^2]$$

for all $x \in S$. That $f = f_j$ follows since $f_j(xx^*) = f_j[(xx^*)^2] \geqslant 0$.

In particular if S is a group (with $x^* = x^{-1}$) or a semilattice (with $x^* = x$) then

$$(xx^*) = (xx^*)^2 \quad \text{for all } x \in S. \tag{4.1.2}$$

In general (4.1.2) holds whenever S is an inverse semigroup. Motivated by [2, §1.9] S will be called *-regular if $xx^*x = x$ for all $x \in S$. Note that every *-regular semigroup is invertible and as is well known admits enough bounded semicharacters to separate points. Clearly every bounded semicharacter is a character.

For arbitrary S consider the evaluation map $x \longrightarrow \hat{x}$ where $\hat{x}(\chi) = \chi(x)$ for all $\chi \in \Gamma_0$ and $x \in S$. Since $\hat{x}(\hat{x})^*\hat{x} = \hat{x}$, S is *-regular and *-isomorphism to S if and only if S admits enough characters to separate points. It can be shown that $\hat{S}$ is the maximal *-regular, *-homeomorphic image of S, cf. [2, §1.5]. But $\Gamma(\hat{S}) = \Gamma_0(S)$ so that the positive S-definite (BVS-) functions which satisfy (4.1.1) can be uniquely identified with the positive $\hat{S}$-definite (BV$\hat{\mathrm{S}}$) functions by the lifting map $f \longrightarrow \hat{f}$ where $\hat{f}(\hat{x}) = \int \chi(x)\, d\mu_f$.

5. BV-functions on Hermitian semigroups. Let S be Hermitian. It follows from (2.0.4) that

$$\|f\| = \lim \frac{1}{2^{kn}} \sum \binom{n}{i_1} \cdots \binom{n}{i_k} |\Delta f(1;\ \{-i_j x_j,\ (n - i_j) x_j\})| \tag{5.0.1}$$

where the limit is taken with respect to n and all finite subsets $\{x_1, \ldots, x_k\}$ of a fixed generator set X for S. Moreover (3.5.1) implies

$$|f|(x) = \frac{1}{2^k} \sum_\sigma \left(\prod_j \sigma_j\right) \|\Delta f(\cdot;\ \{-\sigma_j x_j\}_j)\| \tag{5.0.2}$$

where the summation is taken over all square roots of unity σ. In particular if $S = X$ then $|f|(x) = \frac{1}{2}\|f + f_x\| - \frac{1}{2}\|f - f_x\|$. For convenience the BV-functions defined in [8] will be denoted by BV(CM). Then BV(CM) $\subset$ BV with $\|f\|_{CM} = \|f\|$ and $|f|_{CM} = |f|$ whenever $\|f\|_{CM} < \infty$.

If S is a Hermitian group (i.e. a group such that $x^2 = 1$ for all $x \in S$) then the only CM-functions on S are constant functions thus the BV(CM) functions form a one-dimensional space while the BV-functions contain all of the characters. Also the equalities $(I - E_x)(I + E_x) = 0$ and $(I \pm E_x)^p = 2^{p-1}(I \pm E_x)$ reduce (5.0.1) quite simply to

$$\|f\| = \lim_k \frac{1}{2^k} \sum |\Delta f(1;\ \{\sigma_j x_j\})|.$$

In this case it is easy to see that S admits a *base*, i.e. a subset X which is maximal with respect to the property that $\Pi x_j^{\beta_j} = 1$, where β_j is a 0-1 valued function on the first k natural numbers. It follows that X is a minimal generator set.

Finally the moment problem $f(n) = \int_{-1}^{1} t^n\, d\mu(t)$ can be put into the setting of this work by selecting S to be the Hermitian semigroups of nonnegative additive integers. The $*$-semicharacters are then just the maps $n \longrightarrow t^n$ $(-1 \leqslant t \leqslant 1)$. Since the integer 1 is a generator set for S, (5.0.1) reduces to

$$\|f\| = \lim_n \frac{1}{2^n} \sum \binom{n}{i_1} |\Delta_n f(0;\ \underbrace{-1, \ldots, -1}_{i_1},\ \underbrace{1, \ldots, 1}_{i_1})|.$$

Recall that Hausdorff's solution to the little moment problem $f(n) = \int_0^1 t^n\, d\mu(t)$ is given by

$$\|f\| = \lim_n \sum_{i_1} \binom{n}{i_1} |\Delta_{n-i} f(i;\ \underbrace{-1, \ldots, -1}_{n-i})|,$$

cf. [8, Corollary 6.1].

It is interesting to note that the semicharacters $f\colon n \longrightarrow t^n$ $(-1 \leqslant t < 0)$ while positive-definite are not BV(CM). Indeed, if f is such a semicharacter then

$$\|f\|_{(\mathrm{CM})} = \lim_n \sum \binom{n}{i_1} |t(1-t)^{n-i_1}| = \lim_n \sum \binom{n}{i_1} (-t)^{i_1}(1-t)^{n-i_1}$$

$$= [(1-t)-t]^n = (1-2t)^n \longrightarrow \infty.$$

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