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INCLUSION MAPS OF 3-MANIFOLDS WHICH INDUCE MONOMORPHISMS OF FUNDAMENTAL GROUPS

BY

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ABSTRACT. The main result is the following "duality" theorem. Let M be a 3-manifold, P a compact and connected polyhedral 3-submanifold of int M, and X a compact and connected polyhedron in int P. If $\pi_1(X) \longrightarrow \pi_1(P)$ is onto, then $\pi_1(M - P) \longrightarrow \pi_1(M - X)$ is one-to-one. Some related results are proved, for instance: we can allow P to be noncompact if also X satisfies a certain noncompactness condition: if M lies in a 3-manifold W with $H_1(W) = 0$, then the condition that $\pi_1(X) \longrightarrow \pi_1(P)$ is onto can be replaced by the weaker one that $H_1(X) \longrightarrow H_1(P)$ is onto.

1. Definitions and notation.

SOME NOTATION. If X and Y are spaces (we are not going to consider spaces more general than polyhedra) and $f: X \longrightarrow Y$ a map, then by f_* and $f_\#$ we denote the homomorphisms of homology and homotopy groups, respectively, induced by f. If $X \subset Y$ and we write $H_n(X) \longrightarrow H_n(Y)$ or $\pi_n(X) \longrightarrow \pi_n(Y)$ without indicating the homomorphism, we mean the homomorphism induced by the inclusion $X \subset Y$.

If X is disconnected, then the statement $\pi_n(X) = 0$ means that $\pi_n(X, x) = 0$ for any $x \in X$. Analogously we interpret the statement $\pi_n(X, Y) = 0$ when Y is disconnected. If X is disconnected and $f: X \longrightarrow Y$ is a map, then the statement "$f_\#: \pi_n(X) \longrightarrow \pi_n(Y)$ is an iso- (mono-, epi-) morphism" means that for any $x \in X$, $f_\#: \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an iso- (mono-, epi-) morphism.

The symbols $\simeq$, $\sim$, $\approx$ mean "homotopic", "homologous", and "isomorphic", respectively.

$Z = Z_0$ = integers, Z_p (for p a positive integer) = integral residues modulo p, $Q = Z_\infty$ = rationals.

D is the standard 2-simplex, S^n is the boundary of the standard $(n + 1)$-simplex.

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MANIFOLDS. We work in the PL category. Each manifold is supposed to have a fixed PL structure. If M is a manifold, then by a submanifold of M or by a surface, arc, polyhedron, etc., in M we mean a respective object contained in M as a subpolyhedron. All maps are PL. Our manifolds are never automatically assumed to be without boundary, compact, connected, or orientable. However, by a *surface* we mean a compact and connected 2-manifold. A *closed manifold* is, as usual, a compact manifold without boundary. On the other hand, by a *relatively closed submanifold* of a manifold M we understand a submanifold that is a closed subset of M. A *polyhedron* is not necessarily compact, a *subpolyhedron* is not necessarily a closed subset.

If M is a manifold and P a closed polyhedron in M, then by a *regular neighborhood* of P in M we mean the simplicial neighborhood of P in the second barycentric subdivision of any triangulation of M (consistent with the chosen PL structure of M) in which P and all closed subpolyhedra of M which were constructed before P are contained as subpolyhedra.

A *cube with n handles* ($n \geqslant 0$) is a 3-manifold homeomorphic to a regular neighborhood of a connected finite linear graph of Euler characteristic $1 - n$ in the Euclidean 3-space.

We denote the interior of a manifold M by int M and the boundary by Bd M. However, if M is oriented, then ∂M is the manifold Bd M oriented coherently with M. If P is a codimension 0 submanifold of an oriented manifold M, then we always assume that P has the orientation inherited from M unless the contrary is said explicitly.

A manifold M is said to have *one end* if, for every compact subset $X \subset M$, $M - X$ has precisely one component with noncompact closure.

HOMOLOGY. When we use homology or cohomology without specifying the coefficients then integer coefficients are to be understood. Similarly, if we say "chain" or "cycle" without mentioning coefficients, we mean a chain or cycle over Z.

An oriented compact n-submanifold N of an m-manifold M determines a PL n-chain in M and the boundary of this chain corresponds in the same sense to ∂N. We shall make no distinction in notation between N and the chain that N determines. In some cases some kind of converse of the above assertion is true. For instance, if $m \geqslant 2$, any element of $H_1(M)$ can be represented by a closed oriented 1-manifold in int M; if $m = 3$, every element of $H_2(M, \text{Bd } M)$ can be represented by an oriented compact properly embedded 2-manifold in M; finally, if $m = 3$ and if $J \subset M$ is an oriented closed 1-manifold representing the zero of $H_1(M)$, then J is the boundary of an oriented compact 2-manifold in M.

A space X will be called *1-acyclic over G*, where G is an abelian group, if X is connected and $H_1(X; G) = 0$; *1-acyclic* means 1-acyclic over Z. It is well known that every 1-acyclic manifold is orientable.

GROUPS. Let p be either 0 or a prime. For a group G and a subgroup $H \subset G$ denote by $[G, H]^p$ the subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}k^p$, where $g \in G$ and $h, k \in H$ (cf. [8]). For any group G we define a sequence of subgroups $G_0^p \supset G_1^p \supset G_2^p \supset \cdots \supset G_\infty^p$ of G inductively as follows:

$$G_0^p = G, \qquad G_{n+1}^p = [G, G_n^p]^p \qquad (n \geq 0), \qquad G_\infty^p = \bigcap_{n=0}^{\infty} G_n^p.$$

If $f: G \longrightarrow H$ is a homomorphism of groups, then obviously $f(G_n^p) \subset H_n^p$; it follows that all G_n^p are normal subgroups of G and that f induces homomorphisms $f_n^p: G/G_n^p \longrightarrow H/H_n^p$ $(0 \leq n \leq \infty)$.

Let P be a property of groups. We say that a group G has the property P *residually* [3] if for each $x \in G$, $x \neq 1$, there exists a group H with property P and an epimorphism $f: G \longrightarrow H$ such that $f(x) \neq 1$.

2. Statements of theorems.

2.1. GENERAL SETTING. M is a 3-manifold. $P \subset \operatorname{int} M$ is a relatively closed 3-submanifold of M. X is a closed polyhedron in P intersecting each component of P in a nonempty connected set. If the boundary components of P are not all compact we also assume that X intersects every end of P which any component of Bd P intersects; by this we mean the following: for any compact polyhedron K in P, each component of Bd $P - K$ with noncompact closure can be joined to X by an arc in $P - K$.

Our main result is the following theorem, which is a generalization of 3.2 of [2] (although the latter result does not really follow from 2.2 but from 2.3).

2.2. THEOREM. *Assume* 2.1. *If* $\pi_1(P, X) = 0$, *then the inclusion* $M - P \subset M - X$ *induces monomorphisms on the following homotopy and homology groups*: π_1, π_2, H_1, H_2.

The other theorems are similar to 2.2. We only vary the conditions put on M, P, X.

2.3. THEOREM. *Let p be* 0, *a prime, or* ∞. *Assume* 2.1 *and suppose, in addition, that each component of P is compact and that P satisfies the following homological conditions*: *P is orientable over* Z_p, $H_1(P; Z_p)$ *is a free* Z_p*-module, and* $H_2(P; Z_p) = 0$. *If* $H_1(P, X; Z_p) = 0$, *then the inclusion* $M - P \subset M - X$ *induces monomorphisms on* $\pi_1, \pi_2, H_1(\cdot\,; Z_p), H_2(\cdot\,; Z_p)$.

2.4. ADDENDUM. Let $p \neq \infty$. Then under the hypotheses of 2.3 also the following is true:

(1) $\pi_1(M - P)/(\pi_1(M - P))_n^p \longrightarrow \pi_1(M - X)/(\pi_1(M - X))_n^p$ is a monomorphism for each n $(0 \leq n \leq \infty)$;

(2) if $0 \leqslant n < \infty$, then the monomorphism of (1) has a left inverse.

The Addendum is interesting because of its similarity to Milnor's Theorem 1.2′ in [5] and since 2.3 and 2.4 give in a sense the best possible result under the given hypotheses (see 2.11).

For an orientable 3-manifold P with compact components, the condition that $H_1(P; Z_p)$ is a free Z_p-module and $H_2(P; Z_p) = 0$ is equivalent to the following: each component of P has connected boundary and P can be embedded in a 3-manifold which is 1-acyclic over Z_p (see 3.1 of [9] and [10]). This shows the connection between 2.3 and our next theorem.

2.5. Theorem. *Assume* 2.1. *Suppose that M can be embedded into some* 1-*acyclic* 3-*manifold. If $H_1(P, X; Q) = 0$, then the inclusion $M - P \subset M - X$ induces monomorphisms on π_1, π_2, H_1, H_2.*

Assume 2.1 and let P satisfy the hypotheses of 2.3 with $p = 0$. If $H_1(P, X; Q) = 0$, then $H_1(\text{Bd } P) \longrightarrow H_1(P - X)$ is one-to-one; this is obtained from 2.3 by replacing the original M by a regular neighborhood of P in M (and taking $p = \infty$). The converse is also true: if $H_1(\text{Bd } P) \longrightarrow H_1(P - X)$ is one-to-one, then $H_1(P, X; Q) = 0$ and the conclusions of 2.3 follow. Theorem 2.5 admits a similar converse. These two converses are essentially the content of Theorem 2.6 below. X is not mentioned, though; we write P for $P - X$ and M for $M - X$. Note that (for P with compact components) the condition that the intersection of X with each component of P is nonempty and connected is equivalent to requiring that each component of $P - X$ (or P in the new notation) has one end.

2.6. Theorem. *Let M and P be as in* 2.1. *Suppose, in addition, that they satisfy either* (1′) *or* (1″), *and both* (2) *and* (3) *below* (*see also* 2.7):

(1′) *each component of P has compact connected boundary*; *P can be embedded in a* 1-*acyclic* 3-*manifold*;

(1″) *M can be embedded in a* 1-*acyclic* 3-*manifold*;

(2) *each component of P has one end*;

(3) $H_1(\text{Bd } P) \longrightarrow H_1(P)$ *is one-to-one.*

Then the inclusion $M - P \subset M$ induces monomorphisms on π_1, π_2, H_1, H_2.

2.7. Lemma. *Suppose that P can be embedded into some* 1-*acyclic* 3-*manifold. Then the conditions* (2) *and* (3) *are together equivalent to*: *no component of P is compact and $H_2(P, \text{Bd } P) = 0$.*

Remarks. 2.8. Consider the condition $H_1(P, X; Z_p) = 0$ of 2.3 and 2.5. Using standard homology tools we can prove that, under the hypotheses 2.1, the condition $H_1(P, X; Q) = 0$ is equivalent to the following:

(*) For every (integral) 1-cycle z in P there exists a positive integer n such that nz is homologous in P to a 1-cycle in X.

Similarly we can prove that for p a prime the condition $H_1(P, X; Z_p) = 0$ is equivalent to (*) with the additional requirement that n is not divisible by p. The condition $H_1(P, X) = 0$ is equivalent to (*) with $n = 1$.

The condition $\pi_1(P, X) = 0$ of 2.2 is clearly equivalent to $\pi_1(X) \longrightarrow \pi_1(P)$ being an epimorphism.

2.9. In the proofs of our theorems we can always assume that X lies in int P and, whenever convenient, that X is a 3-manifold.

2.10. The author does not know whether the conclusion of 2.5 is true without the condition that M can be embedded in a 1-acyclic 3-manifold nor whether the conclusions of 2.4 hold without the special conditions of 2.3 on P.

2.11. Suppose that M, P, X, p satisfy the hypotheses of 2.4 and that $X \subset$ int P. Assume for simplicity that P is connected. Then 2.3 and 2.4 "almost" imply that there exists a retraction $P - X \longrightarrow$ Bd P (clearly, the existence of such a retraction would imply 2.3 and 2.4). In fact, suppose that the monomorphism of 2.4(1) has always a left inverse also for $n = \infty$ (for any M). Choose, in particular, M to be the union of P and an outer collar on Bd P. Then $(\pi_1(M - P))_\infty^p = (\pi_1(\text{Bd } P))_\infty^p = 0$ (cf. the end of the proof of 2.3 in §4). Therefore our supposition that the homomorphism of 2.4(1) has a left inverse for $n = \infty$ implies that also $\pi_1(\text{Bd } P) \longrightarrow \pi_1(P - X)$ has a left inverse, h say. Now suppose that Bd P is not a sphere. Then Bd P is aspherical and this enables us to construct a retraction r: $P - X \longrightarrow$ Bd P such that, in dimension 1, $r_\# = h$. (If Bd P is a sphere, then there is always a retraction $P - X \longrightarrow$ Bd P.)

However, in general there is no retraction $P - X \longrightarrow$ Bd P. The picture shows a counterexample, in which P is a cube with 3 handles. Think of P as standardly embedded in S^3, i.e. so that $T = S^3 -$ int P is also a cube with 3 handles. A retraction $P - X \longrightarrow$ Bd P would yield a retraction $S^3 - X \longrightarrow T$, but Jaco and McMillan [4, Theorem 6] proved that $\pi_1(S^3 - X)$ admits no homomorphism onto the free group of rank 3.

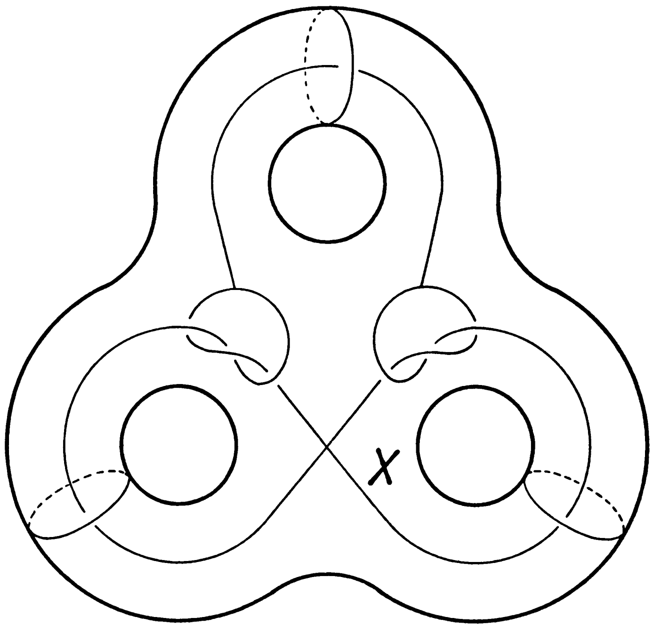
We will now prove the common easy part of our theorems.

2.12. LEMMA. *Let M be a connected 3-manifold and $P \subset$ int M a relatively closed 3-submanifold of M such that no component of P is compact. Let $T = M -$ int P and let i: $T \longrightarrow M$ be the inclusion. Then*

(1) $i_*: H_2(T; Z_p) \longrightarrow H_2(M; Z_p)$ *is one-to-one* ($p = 0, \infty$, *or a prime*);

(2) *if i induces a monomorphism on π_1, then so it does on π_2.*

PROOF. Since P has no compact components, $H_3(M, T; Z_p) \approx H_3(P, \text{Bd } P; Z_p) = 0$. This implies (1).



Suppose that $i_{\#}: \pi_1(T) \longrightarrow \pi_1(M)$ is one-to-one. Let $q: M' \longrightarrow M$ be the universal covering of M. Let $P' = q^{-1}(P)$, $T' = q^{-1}(T)$, $i': T' \subset M'$. Obviously each component of P' is mapped by q onto some component of P. It follows that P' has no compact components and thus $i'_*: H_2(T') \longrightarrow H_2(M')$ is one-to-one. The fact that M' is simply connected and the hypothesis taht $\pi_1(T) \longrightarrow \pi_1(M)$ is one-to-one easily imply that $\pi_1(T') = 0$.

Choose a component T_0 of T and a component T'_0 of $q^{-1}(T_0)$. Consider the following commutative diagram, in which h denotes the Hurewicz homomorphism.

$$\begin{array}{ccccc} H_2(T'_0) & \xleftarrow{h} & \pi_2(T'_0) & \xrightarrow{(q|T_0)_\#} & \pi_2(T_0) \\ \downarrow{\scriptstyle i'_*} & & \downarrow{\scriptstyle i'_\#} & & \downarrow{\scriptstyle i_\#} \\ H_2(M') & \xleftarrow{h} & \pi_2(M') & \xrightarrow{q_\#} & \pi_2(M) \end{array}$$

All horizontal maps are isomorphisms. Since i'_* is a monomorphism so is $i_\#$.

2.13. COROLLARY. *Suppose that M, P, X satisfy the hypotheses of one of the Theorems* 2.2, 2.3, 2.5. *Let $i: M - P \subset M - X$. Then*

(1) *i induces a monomorphism on $H_2(\cdot\,; Z_p)$ (for any p);*

(2) *if i induces a monomorphism on π_1, then i induces a monomorphsim on π_2.*

3. Two auxiliary results. It is well known that if S is a closed surface in the interior of a 1-acyclic 3-manifold W, then S separates W and consequently S is orientable. The following lemma will be used several times.

3.1. LEMMA ([9, 2.2]). *Let S be a closed surface in the interior of a 1-acyclic 3-manifold W. Denote by U and V the closures of the two components of $W - S$. Then there exists a cube with handles, V', and a homeomorphism h:* Bd $V' \longrightarrow S$ *such that*

(1) *the 3-manifold $W' = V' \cup_h U$ is 1-acyclic;*

(2) *if J is a closed oriented 1-manifold in S, then $J \sim 0$ in V if and only if $h^{-1}(J) \sim 0$ in V'.*

The following theorem provides the crucial step in the proofs of our theorems.

3.2. THEOREM [8, 5.1 and 3.2]. *Let $f: K \longrightarrow L$ be a map of connected polyhedra and let p be either 0 or a prime. Suppose that f induces an isomorphism on $H_1(\cdot\,; Z_p)$ and an epimorphism on $H_2(\cdot\,; Z_p)$. Then $(f_\#)^p_n: \pi_1(K)/(\pi_1(K))^p_n \longrightarrow \pi_1(L)/(\pi_1(L))^p_n$ is an isomorphism if $n < \infty$ and a monomorphism if $n = \infty$.*

4. Proof of 2.3 and 2.4. Throughout this section we assume that M, P, X, p satisfy the hypotheses of Theorem 2.3 and, in addition, that $X \subset$ int P, that X is a 3-manifold, and that $M - X$ is connected. We denote the closure of $M - P$ by T. Note that T is connected. Suppose that $H_1(P, X; Z_p) = 0$.

There exists a linear graph $Y \subset X$ such that the inclusion $Y \subset P$ induces isomorphisms on $H_0(\cdot\,; Z_p)$ and $H_1(\cdot\,; Z_p)$; specifically, for each component P_0 of P we let $Y \cap P_0$ be a bouquet of simple closed curves in X such that the homology classes of these curves constitute a basis for $H_1(P_0; Z_p)$. Exploiting the exact homology sequence of the pair (P, Y), the universal coefficient theorem, and duality we get $H_n(M - Y, M - P; Z_p) = 0$ for $n = 1, 2$.

PROOF OF 2.4. Suppose that p is 0 or a prime. Let $i: T \longrightarrow M - X$, $j: M - X \longrightarrow M - Y$, and $k = ji: T \longrightarrow M - Y$ be inclusions. Then k, in the role of f, satisfies the hypotheses of 3.2. Therefore $(k_\#)^p_n$ is an isomorphism for $n < \infty$ and a monomorphism for $n = \infty$, and hence $(i_\#)^p_n$ is a monomorphism for $n \leqslant \infty$. Let h^p_n be the inverse of $(k_\#)^p_n$ $(n < \infty)$. Then $h^p_n(j_\#)^p_n$ is a left inverse of $(i_\#)^p_n$. Addendum 2.4 is proved.

PROOF OF 2.3. We showed above that $H_2(M - Y, T; Z_p) = 0$. Therefore $H_1(T; Z_p) \longrightarrow H_1(M - Y; Z_p)$, and hence $H_1(T; Z_p) \longrightarrow H_1(M - X; Z_p)$, are monomorphisms. Because of 2.13 the only assertion of 2.3 still to be proved is

4.1. PROPOSITION. *$\pi_1(T) \longrightarrow \pi_1(M - X)$ is one-to-one.*

Suppose that this is false. Then there exists a map f: $(D, S^1) \longrightarrow$

$(M - X, T)$ such that $f|S^1 \not\simeq 0$ in T. In this case the quadruple (M, P, X, f) will be called a *counterexample to* 4.1. The following lemma shows that there exists no counterexample to 4.1 of a particularly simple kind (cf. 3.1(3) of [2]).

4.2. LEMMA. $\pi_1(\text{Bd } P) \longrightarrow \pi_1(P - X)$ *is one-to-one.*

PROOF. We can assume that P is connected. Suppose that the homomorphism of the lemma has nontrivial kernel. By the Loop Theorem [7] there exists a properly embedded disk E in $P - X$ such that $\text{Bd } E \not\simeq 0$ in $\text{Bd } P$. If $P - E$ is connected, we can find a simple closed curve J in int P such that, for appropriate orientations of P, J, and E over Z_p, J and E have intersection number 1 over Z_p. But this is impossible since J is homologous over Z_p to a 1-cycle in $X \subset P - E$. Thus E separates P. Let P', P'' be the closures of the two components of $P - E$; let P' be the one containing X. Consider the inclusions

$$(P'', \varnothing) \subset (P'', E) \subset (P, P') \supset (P, X).$$

The first two of them induce isomorphisms and the third an epimorphism on $H_1(\cdot\,; Z_p)$. Hence $H_1(P''; Z_p) = 0$. However, since $\text{Bd } P''$ is not a sphere we can find oriented simple closed curves $J, K \subset \text{Bd } P''$ which cross each other at exactly one point. It is well known that at least one of J, K represents a nonzero element of $H_1(P''; Z_p)$.

4.3. LEMMA. *Let M' be a* 3-*manifold and $P' \subset$ int M' a relatively closed* 3-*submanifold of M'. Suppose that $\pi_1(\text{Bd } P') \longrightarrow \pi_1(P')$ is one-to-one and that the kernel of $\pi_1(M' - P') \longrightarrow \pi_1(M')$ is nontrivial. Then there exist a finite union, E, of disjoint properly embedded disks in M' −* int *P' and a map f: $(D, S^1) \longrightarrow (E \cup P', \text{Bd } P')$ such that $f|S^1 \not\simeq 0$ in M' −* int *P'. (In particular, $f(D)$ meets only one component of P'.)*

PROOF. Denote $M' - \text{int } P'$ by T'. Choose a map g: $(D, S^1) \longrightarrow (M', \text{int } T')$ in general position with $\text{Bd } P'$ such that $g|S^1 \not\simeq 0$ in T' and $g^{-1}(\text{Bd } P')$ has the smallest possible number of components. Let C_i $(0 \leqslant i \leqslant n)$ be the components of $g^{-1}(\text{Bd } P')$ and let D_i be the subdisk of D bounded by C_i. By standard techniques we can show that $g|C_i \not\simeq 0$ in $\text{Bd } P'$ for each i, that for an appropriate choice of the numbering $D_1, \ldots, D_n$ are disjoint and lie in int D_0, and that g maps $(D - D_0) \cup \bigcup_{i=1}^{i=n} D_i$ into T' and $D_0 - \bigcup_{i=1}^{i=n} D_i$ into P'. Imitating the proof of Theorem (17.1) in [6] or using inductively the Loop theorem [7] we can now find a finite union, E, of disjoint properly embedded disks in T' such that $g|C_i \simeq 0$ in $E \cup \text{Bd } P'$ $(1 \leqslant i \leqslant n)$. Therefore we can replace g by a map f satisfying the assertion of the lemma.

PROOF OF 4.1. Suppose that 4.1 is false and let (M, P, X, f) be a counterexample to 4.1. By 4.2 and 4.3 we can assume that $f(D) \subset P \cup E$, where E is a properly embedded compact 2-manifold in T, each component of E is a disk, and Bd E lies in a single component P_0 of P. Let T_0 be a regular neighborhood of $E \cup \text{Bd}\, P_0$ in T, let $M_0 = P_0 \cup T_0$, and $X_0 = X \cap P_0$. Obviously (M_0, P_0, X_0, f) is a counterexample to 4.1 (with the same p as (M, P, X, f)).

Choose a prime q such that P_0 and X_0 satisfy the hypotheses of 2.3 with q taking the role of p. Such a prime q always exists. In particular, if p is a prime, let $q = p$; if $p = 0$, q can be any prime; if $p = \infty$, q can be any sufficiently large prime. Addendum 2.4 implies that

$$\pi_1(T_0)/(\pi_1(T_0))^q_\infty \longrightarrow \pi_1(M_0 - X_0)/(\pi_1(M_0 - X_0))^q_\infty$$

is a monomorphism. If we can prove that $(\pi_1(T_0))^q_\infty = 0$, it will obviously follow that also $\pi_1(T_0) \longrightarrow \pi_1(M_0 - X_0)$ is one-to-one and this will contradict our conclusion above that (M_0, P_0, X_0, f) is a counterexample to 4.1. Proposition 4.1 will be proved.

Consider $\pi_1(T_0) = \pi_1(E \cup \text{Bd}\, P_0)$. If we split each component of E into two disks by counting each point twice, we obtain from $E \cup \text{Bd}\, P_0$ a closed 2-manifold homeomorphic to Bd T_0 − Bd P_0. In other words, we can view $E \cup \text{Bd}\, P_0$ as the quotient space of Bd T_0 − Bd P_0 obtained by pairwise identifying the members of a collection of disjoint disks. Therefore it follows from van Kampen's theorem that $\pi_1(T_0)$ is isomorphic to a finite free product whose factors are the fundamental groups of the components of Bd T_0 − Bd P_0 and (possibly) a free group of finite rank.

Suppose that P_0 is orientable. It easily follows that T_0 is orientable. Therefore $\pi_1(T_0)$ is the free product of a free group and a finite number of fundamental groups of orientable closed surfaces. Obviously Z is residually a finite q-group. A theorem of Gruenberg [3, Corollary to Theorem 4.1] says that if two groups are residually finite q-groups then so is their free product. It follows that any free group of finite rank, and hence any free group, is residually a finite q-group. The fundamental group of a closed orientable surface is residually free (this is obvious if the surface is a torus; for other surfaces see [1], Theorem 1 and the discussion in §1.3 following after Theorem 1). It easily follows that the fundamental group of a closed orientable surface is residually a finite q-group. Again applying the above mentioned Gruenberg's result we obtain that $\pi_1(T_0)$ is residually a finite q-group. Then the fact that a finite q-group G has $G^q_\infty = 0$ [8, 4.2] easily implies that $(\pi_1(T_0))^q_\infty = 0$.

Now suppose that P_0 is not orientable. Then $q = p = 2$. We will prove that the fundamental group of a nonorientable closed surface is residually a finite 2-group; this is, as shown to the author by J. Hempel, a simple consequence of the fact that the assertion holds for orientable surfaces. Then the same proof as above will show that $(\pi_1(T_0))^2_\infty = 0$.

Let G be the fundamental group of a closed nonorientable surface and let H be the subgroup of G of index 2, corresponding to the orientable twofold covering of the surface. Take an $x \in G$, $x \neq 1$. Since H is residually a finite 2-group, as explained above, we can find a normal subgroup K of H of index 2^k, say, such that $x \notin K$. K may not be normal in G. However, if L is the intersection of all normal subgroups of H of index 2^k, then L is invariant under all automorphisms of H and hence L is normal in G. Clearly $x \notin L$. Since H, as a finitely generated group, has only finitely many normal subgroups of index 2^k, L has index a power of 2 in H and hence index a power of 2 in G.

5. **Proof of 2.5.** For easier reference we repeat the hypotheses, with some inessential additions.

5.1. HYPOTHESES. M is a connected open 3-manifold which can be embedded in some 1-acyclic 3-manifold. P is a relatively closed 3-submanifold of M. The closure of $M - P$ is denoted by T. $X \subset \text{int } P$ is a relatively closed 3-submanifold of P. The intersection of X with each component of P is nonempty and connected. X intersects every end of P which any component of Bd P intersects. $H_1(P, X; Q) = 0$.

The following lemma is an easy consequence of the fact that a closed surface lying in the interior of a 1-acyclic 3-manifold W separates W.

5.2. LEMMA. *Let M be a connected open 3-manifold lying in some 1-acyclic 3-manifold and let N be a 3-submanifold of M. Suppose that N is closed in M and that* Bd N *is compact. Then the boundary of each component of $M -$* int N *is connected.*

5.3. LEMMA. *Let P be a connected orientable 3-manifold and $X \subset$* int P *a connected closed subpolyhedron of P which intersects every end of P that any component of* Bd P *intersects and for which $H_1(P, X; Q) = 0$. Let G be an oriented nonclosed properly embedded surface in $P - X$. Then G separates P and ∂G bounds an oriented compact 2-submanifold of* Bd P.

PROOF. Suppose that $P - G$ is connected. Then there exists a simple closed curve $J \subset \text{int } P$ piercing G at exactly one point. For some orientations of J and P the intersection number of J and G over Q is 1. The condition $H_1(P, X; Q) = 0$ implies that J is homologous over Q to a 1-cycle in $X \subset P - G$ and hence the intersection number of J and G is 0. This contradiction shows that G separates P.

Let P' and P'' be the closures of the two components of $P - G$; call P' the one that contains X. Then $F = P'' \cap \text{Bd } P$ is compact, for otherwise F would intersect an end of P which X cannot intersect. Orient P'' incoherently with G and then F coherently with P''. Then $\partial F = \partial G$.

5.4. COROLLARY. *Assume* 5.1. *Then* $H_1(T) \longrightarrow H_1(M - X)$ *is one-to-one.*

Now, by 2.13, the following proposition remains to be proved.

5.5. PROPOSITION. *Assume* 5.1. *Then* $\pi_1(T) \longrightarrow \pi_1(M - X)$ *is one-to-one.*

Suppose that this is false. Then there exist M, P, X satisfying 5.1 and a map f: $(D, S^1) \longrightarrow (M - X, T)$ such that $f|S^1 \not\simeq 0$ in T. A quadruple (M, P, X, f) like this will be called a *counterexample to* 5.5. We shall start with an arbitrary counterexample to 5.5 and shall change it in three steps (5.6–5.8) into a counterexample to 2.3. This will prove 5.5.

5.6. LEMMA. *If* 5.5 *is false, then there exists a counterexample* (M, P, X, f) *to* 5.5 *such that* P *is connected and has connected boundary.*

PROOF. If we assume 5.1, then $\pi_1(\text{Bd } P) \longrightarrow \pi_1(P - X)$ is one-to-one; the proof of this assertion is exactly the same as for 4.2 (note that if we define P'' as in the proof of 4.2, then, as shown in the proof of 5.3, Bd P'' is compact; this implies the existence of simple closed curves J, K such as in the proof of 4.2). This means that $M' = M - X$ and $P' = P - X$ satisfy the hypotheses of 4.3. Therefore, if 5.5 is false, we can find a counterexample (M, P, X, f) to 5.5 such that $f(S^1) \subset \text{Bd } P$ and $f(D) \subset P \cup E$, where E is a finite union of disjoint properly embedded disks in T (all disks intersecting the same component of P).

Let P_0 be the component of P which intersects $f(D)$, M_1 the interior of a regular neighborhood of $P_0 \cup E$ in M, T_1 the component of $M_1 \cap T$ which contains $f(S^1)$, $P_1 = M_1 - \text{int } T_1$, and $X_1 = X \cap P_1$. Since $H_1(P_0, X_1; Q) = 0$ and $H_1(P_1, P_0; Q) = 0$ it follows from the exact homology sequence of the triple (P_1, P_0, X_1) that $H_1(P_1, X_1; Q) = 0$. Therefore (M_1, P_1, X_1, f) is a counterexample to 5.5 and it satisfies the conclusions of 5.6.

5.7. LEMMA. *If* 5.5 *is false, then there exists a counterexample* (M, P, X, f) *to* 5.5 *such that* P *is connected and* Bd P *compact and connected.*

PROOF. We start with a counterexample (M, P, X, f) to 5.5 which satisfies the conclusions of 5.6. Suppose that Bd P is not compact. Let L' be the union of $f(D)$ and an arc in Bd P such that $L' \cap \text{Bd } P$ is connected. Add to L' those components of Bd $P - L'$ which have compact closures and call the new polyhedron L. Let N be a regular neighborhood of L in $M - X$. Then N is a compact connected 3-submanifold of M and $N \cap \text{Bd } P$ is connected. Denote by V the component of $M - \text{int } N$ which contains X.

SUBLEMMA. (a) Bd $P - N \subset \text{int } V$, (b) $P \cap V$ *is connected.*

PROOF. Take an arbitrary point $x \in \text{Bd } P - N$. It follows from the definition of L and N that x belongs to a component of Bd $P - N$ with noncompact closure. Since X intersects every end of P which Bd P intersects, x can be connected to X by an arc $A \subset P - N$. Clearly $A \subset P \cap \text{int } V$ and this implies (a) of the Sublemma. Now take a point $y \in \text{int } P \cap \text{int } V$. Choose an arc $B \subset \text{int } V$ from y to a point in X. If B is not contained in P, let B_1 be the component of $B \cap P$ containing y; let x be the endpoint of B_1 distinct from y. As shown above x can be joined to X by an arc $A \subset P \cap \text{int } V$. We see that each point $y \in P \cap V$ lies in the component of $P \cap V$ containing X, hence (b) of the Sublemma is true.

Let $P_0 = P \cup V$. Then Bd $P_0 = (\text{Bd } P - \text{int } V) \cup (\text{Bd } V - \text{int } P)$ is compact and connected. Indeed, it follows from (a) of the Sublemma that Bd $P -$ int V is equal to $N \cap \text{Bd } P$, which is compact and connected; by 5.2, Bd V is a component of Bd N and therefore each component of Bd $V -$ int P is compact and intersects $N \cap \text{Bd } P$.

Let C be a collar on Bd V in V and let X_0 be the closure of $V - C$. Consider the following section of the Mayer-Vietoris sequence of $((P_0, X_0); (P, X), (V, X_0))$

$$H_1(P, X; Q) \oplus H_1(V, X_0; Q) \longrightarrow H_1(P_0, X_0; Q) \longrightarrow H_0(P \cap V, X; Q).$$

Since $H_1(P, X; Q)$, $H_1(V, X_0; Q)$, and $H_0(P \cap V, X; Q)$ are trivial (the first by hypothesis, the third by (b) of the Sublemma), also $H_1(P_0, X_0; Q)$ is trivial. Therefore (M, P_0, X_0, f) is a counterexample to 5.5 and it satisfies the conclusion of 5.7.

5.8. Lemma. *If 5.5 is false, then there exists a counterexample (M, P, X, f) to 5.5 such that P is compact and connected and has connected boundary.*

Proof. Choose a counterexample (M, P, X, f) to 5.5 which satisfies the conclusion of 5.7. Think of M as a submanifold of an open 1-acyclic 3-manifold W. Suppose that P is not compact. Let N be a regular neighborhood of $(f(D) \cap P) \cup \text{Bd } P$ in $P - X$. Let $S_0, \ldots, S_n$ be the components of $S = \text{Bd } N - \text{Bd } P$. Denote by V_i the component of $W -$ int N which contains S_i. By 5.2, $V_0, \ldots, V_n$ are all distinct. We can assume that the numbering has been chosen so that $X \subset V_0$.

Let $U_0 = W - \text{int } V_0$. By 3.1 we can attach a cube with handles, V_0', to U_0 along S_0 in such a manner that the 3-manifold $W_0 = U_0 \cup V_0'$ is 1-acyclic and that every 1-cycle in S_0 which bounds in V_0 also bounds in V_0'. Performing a similar surgery along all other S_i we replace each V_i by a cube with handles V_i' so that

(a) if we denote $V_0 \cup \cdots \cup V_n = V$, $V'_0 \cup \cdots \cup V'_n = V'$, then the 3-mani fold $W' = (W - \text{int } V) \cup V'$ is 1-acyclic;

(b) every 1-cycle in S which bounds in V also bounds in V'. Let $M' = (M - \text{int } V) \cup V'$, $P' = (P - \text{int } V) \cup V'$. Note that P' is compact and Bd $P' =$ Bd P.

We claim that $H_1(P', V_0'; Q) = 0$. Take an arbitrary element of $H_1(P')$.

Represent it by an oriented simple closed curve $J \subset \operatorname{int} P'$. Since $H_1(P', P' - V') \approx H_1(V', \operatorname{Bd} V') = 0$ we may assume that $J \subset P' - V' = P - V$. We have to show that some multiple of J is homologous in P' to a 1-cycle in V_0' (cf. 2.8). Since $J \subset P$ and since $H_1(P, X; Q) = 0$ there exists a positive integer k and an oriented compact 2-manifold $F \subset P$ such that $\partial F = kJ \cup K$, where K is an oriented closed 1-manifold in int $X \subset \operatorname{int} V_0$ and kJ denotes the union of k disjoint oriented simple closed curves lying in a regular neighborhood of J and homologous to J in that neighborhood. We can assume that F is in general position with S. Then $G = F \cap (V - V_0)$ is an oriented compact properly embedded 2-manifold in $V - V_0$. By (b) above, ∂G bounds in $V' - V_0'$; hence there exists an oriented compact properly embedded 2-manifold $G' \subset V' - V_0'$ such that $\partial G' = \partial G$. Then $F' = (F - \operatorname{int} V) \cup G'$ is an oriented compact 2-manifold in P' and $\partial F' = kJ \cup K'$, where $K' \subset \operatorname{Bd} V_0'$. This proves that $H_1(P', V_0'; Q) = 0$. We see that (M', P', V_0', f) is a counterexample to 5.5 and clearly it satisfies the conclusion of 5.8.

A compact connected 3-manifold P which has connected boundary and which can be embedded in a 1-acyclic 3-manifold satisfies the hypotheses of 2.3 with $p = \infty$. Therefore 5.5 follows from 5.8 and 2.3.

6. Proof of 2.2.

6.1. HYPOTHESES. M is a connected 3-manifold, $P \subset \operatorname{int} M$ is a relatively closed 3-submanifold of M, $X \subset \operatorname{int} P$ is a closed subpolyhedron of P intersecting each component of P in a nonempty connected set. X intersects every end of P which any component of Bd P intersects. $\pi_1(P, X) = 0$.

6.2. LEMMA. *Assume* 6.1. *Let* $q: M' \longrightarrow M$ *be an arbitrary covering and* $P' = q^{-1}(P)$, $X' = q^{-1}(X)$. *Then* M', P', X' *satisfy* 6.1.

PROOF. We will show that X' intersects every end of P' that any component of Bd P' intersects, which is the only nontrivial thing to prove. Let K' be a compact polyhedron in P'. Choose a component C' of Bd $P' - K'$ which has noncompact closure. Let $K = q(K')$, $C = q(C')$. Then K is a compact polyhedron and C is a connected open subset of Bd P. We consider two cases.

CASE 1. Suppose that $\bar{C}$ is not compact. Obviously $C - K$ has only finitely many components and therefore one of them, say C_0, has noncompact closure. It is easy to see that C_0 is also a component of Bd $P - K$. Thus there exist a point $c \in C_0$ and an arc $A \subset P - K$ from c to a point in X. Choose a point $c' \in q^{-1}(c) \cap C'$. Lift A to an arc $A' \subset M'$ that has c' as one of its endpoints. The other endpoint is then in X' and A' is contained in $P' - q^{-1}(K) \subset P' - K'$.

CASE 2. Suppose that $\bar{C}$ is compact. Let S' be the component of Bd P' containing C'. Obviously $S = q(S')$ is a component of Bd P and it contains C. Since S' is a component of $q^{-1}(S)$, $r = q|S'$: $S' \longrightarrow S$ is a covering map. For an arbitrary point $x \in S$ let $n(x)$ be the cardinality of $r^{-1}(x) \cap C'$.

Denote by L' the boundary of C' in S'. Obviously L' is a compact 1-dimensional polyhedron contained in K'. Therefore $L = r(L')$ is a compact 1-dimensional polyhedron in S. Our first objective is to show that there exists a $c \in C$ such that $n(c)$ is infinite. This will follow from our next assertion.

SUBLEMMA. *Suppose that $n(x)$ is finite for each $x \in C$. Then there is a number n_0 such that each $x \in S$ has a connected open neighborhood U in S with the following properties*

(a) *$\bar{U}$ is compact;*

(b) *r maps each component of $r^{-1}(\bar{U})$ homeomorphically onto $\bar{U}$;*

(c) *C' intersects at most n_0 components of $r^{-1}(U)$.*

PROOF. Take first an $x \in S - L$. Choose a connected open neighborhood U of x in $S - L$ which satisfies (a) and (b) of the Sublemma. Then each component of $r^{-1}(U)$ misses L' and therefore it either lies in C' or does not intersect C'. It follows that exactly $n(x)$ components of $r^{-1}(U)$ intersect C'. We see that $n(y) = n(x)$ for each $y \in U$, i.e. the function n is locally constant on $S - L$. Hence n is constant on each component of $S - L$. But $S - L$ has only finitely many components and therefore there exists a positive integer m such that $n(x) \leqslant m$ for any $x \in S - L$. We have thus proved that each $x \in S - L$ has a connected open neighborhoood U in S which satisfies (a), (b), and (c) of the Sublemma with $n_0 = m$.

Now take a point $x \in L$. Since L is a finite linear graph there is a number s such that each point of L has order $\leqslant s$ (i.e. each point has a neighborhood in L which is homeomoprhic to the cone over a discrete set of at most s points). Choose a connected open neighborhood U of x in S which satisfies (a) and (b) of the Sublemma and for which $U - L$ has $\leqslant s$ components. Let $U_1, \ldots, U_k$ be the components of $U - L$ $(k \leqslant s)$. Choose a component U' of $r^{-1}(U)$ and let $U_i' = U' \cap r^{-1}(U_i)$. If U' intersects C', then at least one of $U_1', \ldots, U_k'$ lies in C'. But, as shown above, for each i at most m components of $r^{-1}(U_i)$ lie in C'. Therefore at most mk components of $r^{-1}(U)$ intersect C'. We see that U satisfies (a), (b), and (c) of the Sublemma with $n_0 = ms$. This concludes the proof of the Sublemma.

Suppose that $n(x)$ is finite for each $x \in C$. Then we can cover the compact set $\bar{C}$ by open subsets $U_1, \ldots, U_k$ of S such that each of them satisfies (a), (b), and (c) of the Sublemma. Let V_i be the union of all components of $r^{-1}(U_i)$ which intersect C' and let $V = \bigcup V_i$. Then $\bar{V}$ is compact and $C' \subset V$, which contradicts the hypothesis that $\bar{C}'$ is not compact.

We have thus proved that there is a $c \in C$ such that $q^{-1}(c) \cap C'$ contains an infinite sequence $c_1', c_2', c_3', \ldots$ of distinct points. Choose an arc A in P

from c to a point in X. For each i let A_i' be the lift of A which has c_i' as an endpoint. The other endpoint of A_i' then lies in X'. Since the arcs A_i' are disjoint and form a discrete family, the compact set K' cannot intersect more than a finite number of them. Therefore, for some n, A_n' is an arc in $P' - K'$ joining C' to X'.

6.3. COROLLARY. *Assume* 6.1. *Then* $\pi_1(M - P) \longrightarrow \pi_1(M - X)$ *is one-to-one.*

PROOF. Let $q: M' \longrightarrow M$ be the universal covering and let $P' = q^{-1}(P)$, $X' = q^{-1}(X)$. Then it follows from 6.2 that M', P', X' satisfy the hypotheses of 2.5 and thus 6.3 is an easy consequence of 2.5.

By 2.13, to complete the proof of 2.2 we only have to prove

6.4. LEMMA. *Assume* 6.1. *Then* $H_1(M - P) \longrightarrow H_1(M - X)$ *is one-to-one.*

PROOF. It suffices to prove the following: if G is an oriented compact properly embedded surface in $P - X$, then ∂G bounds in Bd P. This will certainly be true if P is orientable, for then 5.3 applies and gives the desired result. Suppose that P is nonorientable. We can assume that P is connected. We show as in the proof of 5.3 (only using intersection numbers modulo 2) that G separates P. Hence, if N is a regular neighborhood of G in P, then N is homeomorphic to $G \times [0, 1]$ and therefore it is orientable. Let $q: M' \longrightarrow M$ be the orientable twofold covering of M and let $P' = q^{-1}(P)$, $X' = q^{-1}(X)$. Since N is orientable it can be lifted into P' and hence G can be lifted. Choose a lift G' of G and orient G' so that $q|G': G' \longrightarrow G$ preserves orientation. It easily follows from 6.2 that P', X', and G' satisfy the hypotheses of 5.3 and therefore $\partial G'$ bounds in Bd P'. It follows that $\partial G = q(\partial G')$ bounds in Bd $P = q(\text{Bd } P')$.

7. Proof of 2.6 and 2.7.

PROOF OF 2.7. Let P be a connected 3-manifold with nonempty boundary, embedded in some open 1-acyclic 3-manifold W. Suppose that P has one end and that $H_1(\text{Bd } P) \longrightarrow H_1(P)$ is a monomorphism. Then P is not compact. To prove that $H_2(P, \text{Bd } P) = 0$ consider an arbitrary oriented properly embedded surface $F \subset P$ with nonempty boundary. Since $H_1(\text{Bd } P) \longrightarrow H_1(P)$ is one-to-one, $\partial F \sim 0$ in Bd P. Therefore ∂F bounds an oriented compact 2-manifold G in a regular neighborhood of Bd P in $(W - \bar{P}) \cup \text{Bd } P$. Then $F \cup G$ is a closed surface and therefore it separates W. It obviously follows that F separates P. Let P', P'' be the closures of the two components of $P - F$. Since P has one end exactly one of P', P'', say P', is compact. If we orient P', coherently with F, then $\partial P' = F \cup G'$, where $G' \subset \text{Bd } P$. This shows that F represents the zero element of $H_2(P, \text{Bd } P)$.

Now, conversely, suppose that P is not compact and $H_2(P, \text{Bd } P) = 0$.

Then clearly $H_1(\text{Bd } P) \to H_1(P)$ is one-to-one. It remains to be proved that P has one end. Take a compact polyhedron K in P. We have to show that all components of $P - K$ but one have compact closures. Let A be an arc in P such that $A \cup K$ is connected and let N be a regular neighborhood of $A \cup K$ in P. Then N is a compact and connected 3-submanifold of P and $N \cap \text{Bd } P$, if nonempty, is a compact 2-submanifold of $\text{Bd } P$. Let $S_1, \ldots, S_n$ be the components of the closure of $\text{Bd } N - \text{Bd } P$. Each S_i is an orientable properly embedded surface in P. Since $H_2(P, \text{Bd } P) = 0$ there exists for each i a compact 3-submanifold V_i of P such that $\text{Bd } V_i - \text{Bd } P = \text{int } S_i$ (thus S_i is the relative boundary of V_i in P).

N has a common part of the boundary with each V_i, namely S_i. Since $\text{Bd } V_i \cap \text{int } N = \varnothing$ we have either $N \subset V_i$ or $N \cap V_i = S_i$. If $N \cap V_i = S_i$ for each i, then $N \cup \bigcup V_i$ is a compact open subset of P; but this is impossible since P is connected and not compact. Therefore $K \subset N \subset V_k$ for some k. Since $\text{Bd } V_k \cap \text{int } P = \text{int } S_k$ is connected, $P - V_k$ is connected. It follows that exactly one component of $P - K$, namely the one containing $P - V_k$, has noncompact closure.

Proof of 2.6. Denote $M - \text{int } P$ by T. By excision and by 2.7 we have $H_2(M, T) \approx H_2(P, \text{Bd } P) = 0$. Hence $H_1(T) \to H_1(M)$ is one-to-one. By 2.12 it remains to consider the homomorphism $\pi_1(T) \to \pi_1(M)$. We divide the proof into two parts corresponding to the cases when (1′) or (1″) of 2.6 are satisfied. As in the preceding sections we can show that it suffices to consider the case when P is connected.

7.1. Lemma. *Suppose that M and P satisfy* 2.1 *and the conditions* (1′), (2), *and* (3) *of* 2.6. *Then $\pi_1(T) \to \pi_1(M)$ is a monomorphism.*

Proof. We assume that P is connected. Take a map $f: (D, S^1) \to (M, T)$, such that $f(D) \cap P \neq \varnothing$. We have to show that $f|S^1 \simeq 0$ in T. Let N be a regular neighborhood of $(f(D) \cap P) \cup \text{Bd } P$ in P. Since N is compact and P has one end the closure of $P - N$ has exactly one noncompact component; denote this component by X. Obviously it follows from 2.6(3) that $H_1(\text{Bd } P) \to H_1(P - X)$ is one-to-one. By 5.2, X has connected boundary and hence $P - X$ has one end. Thus 2.7 implies that $H_2(P - X, \text{Bd } P) = 0$.

It easily follows from 3.1 that we can attach a cube with handles, X', to $P - \text{int } X$ along $\text{Bd } X$ in such a way that $P' = (P - \text{int } X) \cup X'$ can be embedded in some 1-acyclic 3-manifold. Let $M' = (M - \text{int } X) \cup X'$. Let P'' be a regular neighborhood of P' in M'. Then

$$H_2(P'' - X', P'' - P') \approx H_2(P' - X', \text{Bd } P') = H_2(P - X, \text{Bd } P) = 0.$$

P'' is orientable; therefore, by duality, $H^1(P', X') \approx H_2(P'' - X', P'' - P') = 0$ and hence $H_1(P', X'; Q) = 0$. From the fact that P' is a compact 3-submanifold with connected boundary of a 1-acyclic 3-manifold it easily follows that $H_2(P') = 0$. Therefore M', P', X' satisfy the hypotheses of 2.3 with $p = \infty$. Hence $f|S^1 \simeq 0$ in $M' - \text{int}\, P' = T$.

7.2. LEMMA. *Suppose that M and P satisfy* 2.1 *and the conditions* (1″), (2), *and* (3) *of* 2.6. *Then* $\pi_1(T) \longrightarrow \pi_1(M)$ *is a monomorphism.*

PROOF. Suppose that M is a connected open 3-submanifold of an open 1-acyclic 3-manifold W. We will reduce 7.2 to 7.1 using the same idea as in the proof of 2.5. As observed above, we can assume that P is connected.

We claim that $\pi_1(\text{Bd}\, P) \longrightarrow \pi_1(P)$ is one-to-one. If this is false, then there exists a properly embedded disk $F \subset P$ such that $\text{Bd}\, F \not\simeq 0$ in $\text{Bd}\, P$. Since $H_2(P, \text{Bd}\, P) = 0$ there exists a compact 3-submanifold P' of P such that $\text{Bd}\, P' - \text{Bd}\, P = \text{int}\, F$. $\text{Bd}\, P'$ has positive genus and thus the fact that P' lies in a 1-acyclic 3-manifold implies that $H_1(\text{Bd}\, P' - F) \longrightarrow H_1(P')$ cannot be one-to-one (cf. [9, 2.7]). But this contradicts 2.6(3). Our claim is proved.

Suppose that 7.2 is false. There then exists a map $f\colon (D, S^1) \longrightarrow (M, T)$ such that $f|S^1 \not\simeq 0$ in T. By 4.3 we can assume that $f(D) \subset P \cup E$, where E is a finite union of disjoint properly embedded disks in T; we can also assume that $E \subset f(D)$. Let T_1 be the component of T which contains $f(S^1)$ and let $E_0 = E - T_1$.

Let $A \subset T_1$ be an arc such that $A \cup (f(D) \cap T_1)$ is connected. Denote by L the union of $A \cup f(D)$ and all components of $P - f(D)$ and of $\text{Bd}\, P - f(D)$ which have compact closures. Let N be a regular neighborhood of L in M. Then N is a compact connected 3-manifold and so is $N \cap T_1$. No component of $P - \text{int}\, N$ nor of $\text{Bd}\, P - \text{int}\, N$ is compact. Since P has one end $P - \text{int}\, N$ is connected.

Let $P_1 = P \cup (N - T_1)$. Thus P_1 is essentially P plus a finite number of 2-handles attached to P. We assert that $H_2(P_1, \text{Bd}\, P_1) = 0$. It suffices to prove this for the case when P_1 is obtained by attaching one 2-handle to P. So let B be a 3-ball and let K be an annulus such that $P \cap B = \text{Bd}\, P \cap \text{Bd}\, B = K$ and suppose that $P_1 = P \cup B$. For each n, the inclusion induced homomorphism j_n: $H_n(K, \text{Bd}\, K) \longrightarrow H_n(P, P \cap \text{Bd}\, P_1)$ can be factored as

$$H_n(K, \text{Bd}\, K) \xrightarrow{\approx} H_n(\text{Bd}\, P, P \cap \text{Bd}\, P_1) \longrightarrow H_n(P, P \cap \text{Bd}\, P_1).$$

Therefore, and since $H_2(P, \text{Bd}\, P) = 0$, we see from the exact homology sequence of the triple $(P, \text{Bd}\, P, P \cap \text{Bd}\, P_1)$ that j_2 is an epimorphism and j_1 is a monomorphism. Considering the Mayer-Vietoris sequence of $((P_1, \text{Bd}\, P_1); (P, P \cap \text{Bd}\, P_1), (B, B \cap \text{Bd}\, P_1))$,

$$\cdots \longrightarrow H_2(K, \text{Bd}\, K) \longrightarrow H_2(P, P \cap \text{Bd}\, P_1) \longrightarrow H_2(P_1, \text{Bd}\, P_1)$$

$$\longrightarrow H_1(K, \text{Bd}\, K) \longrightarrow H_1(P, P \cap \text{Bd}\, P_1) \oplus H_1(B, B \cap \text{Bd}\, P_1) \longrightarrow \cdots,$$

we now see that $H_2(P_1, \text{Bd } P_1)$ is indeed trivial.

Let $\{S_i\}$ be the collection of the compact components of $\text{Bd } P_1 - T_1$. If this collection is nonempty, let U be the closure of the component of $W - \bigcup S_i$ which contains int P_1. It follows from 3.1 that we can attach a cube with handles, V_i, to U along each S_i in such a way that $W' = U \cup V$ is 1-acyclic, where $V = \bigcup V_i$. Let $M_2 = (M \cap U) \cup V$ and $P_2 = P_1 \cup V$. Since

$$H_2(V \cup \text{Bd } P_2, \text{Bd } P_2) = 0 \quad \text{and} \quad H_2(P_2, V \cup \text{Bd } P_2) \approx H_2(P_1, \text{Bd } P_1) = 0$$

it follows from the exact homology sequence of $(P_2, V \cup \text{Bd } P_2, \text{Bd } P_2)$ that $H_2(P_2, \text{Bd } P_2) = 0$. Therefore M_2 and P_2 satisfy the hypotheses of 7.2. (If $\text{Bd } P_1 - T_1$ has no compact components, let $M_2 = M$, $P_2 = P_1$.)

Denote by R the component of $M_2 - \text{int } N$ which contains $P - \text{int } N$. Then $\text{Bd } P_2 - N = \text{Bd } P - N \subset R$ and hence R contains all components of $M_2 - P_2$ except T_1. By 5.2, Bd R is a component of Bd N.

$P_2 \cup R$ is a connected relatively closed 3-submanifold of M_2 and it has compact boundary:

$$\text{Bd}(P_2 \cup R) = (\text{Bd } P_2 - \text{int } R) \cup (\text{Bd } R - \text{int } P_2) \subset N.$$

The 3-manifold $T_3 = M - \text{int}(P_2 \cup R)$ is the union of $N \cap T_1$ and (possibly) some components of $T_1 - N$. Since $N \cap T_1$ is connected so is T_3. Thus it follows from 5.2 that $\text{Bd } T_3 = \text{Bd}(P_2 \cup R)$ is a single closed surface.

Choose a regular neighborhood C of Bd R in R and let

$$S = \text{Bd } C - \text{Bd } R, \quad P_3 = (P_2 - R) \cup (C - S), \quad M_3 = P_3 \cup T_3.$$

Then $\text{Bd } P_3 = \text{Bd}(P_2 \cup R)$ is compact and connected. Since the pair $(P_3, \text{Bd } P_3)$ can be deformed onto $((P_2 - R) \cup \text{Bd } R, \text{Bd } P_3)$ we have by excision

$$H_2(P_3, \text{Bd } P_3) \approx H_2(P_2 - \text{int } R, \text{Bd } P_2 - \text{int } R) \approx H_2(P_2 - R, \text{Bd } P_2 - R).$$

We assert that $H_2(P_2 - R, \text{Bd } P_2 - R) = 0$. Take an arbitrary oriented properly embedded surface $G \subset P_2 - R$. Since $H_2(P_2, \text{Bd } P_2) = 0$ there exists an oriented compact 3-submanifold B of P_2 such that $\partial B \cap \text{int } P_2 = \text{int } G$. Since the closure of $R \cap \text{int } P_2$ is equal to $P - \text{int } N$, which is connected and not compact, B must lie in $P_2 - R$.

We have thus proved that $H_2(P_3, \text{Bd } P_3) = 0$. Therefore M_3 and P_3 satisfy the hypotheses of 7.1. Note that we can think of f as a map $(D, S^1) \longrightarrow (M_3, T_3)$ and that $f|S^1 \not\simeq 0$ in T_3. But this contradicts 7.1. Lemma 7.2 is proved.

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