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.INFINITE CONVOLUTIONS ON LOCALLY COMPACT ABELIAN GROUPS AND ADDITIVE FUNCTIONS[1]

BY

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ABSTRACT. Let $\mu_1, \mu_2, \ldots$ be regular probability measures on a locally compact Abelian group G such that $\mu = \mu_1 * \mu_2 * \cdots = \lim \mu_1 * \cdots * \mu_n$ exists (and is a probability measure). For arbitrary G, we derive analogues of the Lévy theorem on the existence of an atom for μ and of the "pure theorems" of Jessen, Wintner and van Kampen (dealing with discrete $\mu_1, \mu_2, \ldots$) in the case $G = R^d$. These results are applied to the asymptotic distribution μ of an additive function f: $Z_+ \to G$ after generalizing the Erdös-Wintner result ($G = R^1$) which implies that μ is an infinite convolution of discrete probability measures.

1. **Introduction.** Let $\mu_1, \mu_2, \ldots$ be regular probability measures on R^d such that

$$\mu = \lim_{n\to\infty} \mu_1 * \cdots * \mu_n = \mu_1 * \mu_2 * \cdots \tag{1.1}$$

is convergent. A result of P. Lévy [11, Theorem XIII, p. 150] states that μ is not continuous (i.e., has an atom) if and only if

$$\prod_{n=1}^{\infty} d_n \neq 0, \quad \text{where } d_n = \max_t \mu_n(\{t\}) \tag{1.2}$$

is the largest "jump" of μ_n. Also, a theorem of Jessen and Wintner [9, Theorem 35, p. 86] states that if μ_n is purely discontinuous (= discrete), then μ is purely discontinuous or absolutely continuous or (continuous) singular and, more generally (van Kampen [10, pp. 443–444]), μ is pure; cf. §2 below. In §§2 and 3, we discuss generalizations of these results when R^d is replaced by a locally compact Abelian group G. Our methods follow van Kampen's treatment [10] of infinite convolutions on R; cf. also Jessen and Wintner [9], and Wintner [16], [17].

For example, our results imply that in the case when G is the circle group

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$T = R/Z$, where every closed subgroup H ($\neq G$) is finite, the analogue of the Jessen-Wintner (and van Kampen) result is valid, but the analogue of the Lévy theorem has the following form: μ is not continuous if and only if there exists an integer $\kappa > 0$ such that

$$\prod_{n=1}^{\infty} d_{\kappa n} \neq 0, \quad \text{where } d_{\kappa n} = \max_{\theta} \sum_{j=0}^{\kappa-1} \mu_n(\{\theta + j/\kappa\}).$$

The motivation for dealing with (1.1) when $\mu, \mu_1, \mu_2, \ldots$ are probability measures on a group arises, for example, from the consideration of the asymptotic distribution functions of real-valued additive functions mod 1 or, more generally, of additive functions $f: Z_+ \longrightarrow G$, where $Z_+ = \{1, 2, \ldots\}$. A result of Erdös and Wintner [6, p. 720] states that if $G = R$, then f has an asymptotic distribution μ if and only if (1.1) converges, where $\mu_n = \sigma_p$ is purely discontinuous and has the Fourier-Stieltjes transform

$$\hat{\sigma}_p(u) = (1 - p^{-1})\left[1 + \sum_{j=1}^{\infty} p^{-j} \exp iuf(p^j)\right], \tag{1.3}$$

and $p = p_n$ is the nth prime. This is generalized in §4 to the case of arbitrary locally compact Abelian groups G. In particular, it follows in the case $G = T$ (as in the Erdös-Wintner case $G = R$) that when μ exists, it is pure (hence absolutely continuous or purely discontinuous or (continuous) singular). §4 depends heavily on results of Halasz [7], and their applications by Delange [3].

This article was suggested by the paper of Elliott [5] dealing with the question of the continuity of the asymptotic distribution of a real additive function mod 1 (using results of Halasz and Delange, but not involving convolutions).

2. **Cauchy-convergent convolutions on groups.** Let G be a (Hausdorff) locally compact Abelian group (written additively) and Γ the dual group of continuous characters. We write (g, γ) for the pairing of G and Γ, $g \in G$ and $\gamma \in \Gamma$. Let $P(G)$ be the set of regular probability measures μ on G. The Fourier-Stieltjes transform of μ is

$$\hat{\mu}(\gamma) = \int_G (g, \gamma)\, d\mu \quad \text{for } \gamma \in \Gamma.$$

For $\mu, \nu \in P(G)$, we have $(\mu * \nu)\hat{}(\gamma) = \hat{\mu}(\gamma)\hat{\nu}(\gamma)$; cf. [15, pp. 13–15].

The standard topology on $P(G)$ is equivalent to the following: for any net $\{\mu_n\}$ in $P(G)$ and $\mu \in P(G)$, $\mu_n \longrightarrow \mu$ in $P(G)$ is equivalent to

$$\int_G f(g)\, d\mu_n \longrightarrow \int_G f(g)\, d\mu \quad \text{for all } f \in C_0^0(G), \tag{2.1}$$

where $C_0^0(G)$ is the set of complex-valued continuous functions on G with compact support [1, p. 82]. Furthermore, (2.1) can be replaced by any of the following three equivalent conditions on Fourier-Stieltjes transforms, where $\mu_n, \mu \in P(G)$:

(i) $\hat{\mu}_n(\gamma) \longrightarrow \hat{\mu}(\gamma)$ uniformly on compacts of Γ;

(ii) $\hat{\mu}_n(\gamma) \longrightarrow \hat{\mu}(\gamma)$ for all $\gamma \in \Gamma$;

(iii) $\int_\Gamma f(\gamma)\hat{\mu}_n(\gamma)\,d\gamma \longrightarrow \int_\Gamma f(\gamma)\hat{\mu}(\gamma)\,d\gamma$ for all $f \in L^1(\Gamma)$,

and $L^1(\Gamma)$ refers to a Haar measure on Γ; cf. [1, p. 89] (where $G = \hat{\Gamma}$ and $\Gamma = \hat{G}$ are interchanged).

Also, if $\{\mu_n\}$ is a net in $P(G)$, then $\lim \mu_n$ exists in $P(G)$ if and only if

$$\lim \hat{\mu}_n(\gamma) \text{ exists for all } \gamma \in \Gamma \text{ and is continuous at } \gamma = 0. \tag{2.2}$$

In fact, the limit function is then continuous on Γ by the analogue of the Increments Inequality (cf. Loève [12, p. 195]),

$$|\hat{\mu}(\gamma) - \hat{\mu}(\gamma + \delta)|^2 \leqslant 2[1 - \operatorname{Re} \hat{\mu}(\delta)] \quad \text{for } \gamma, \delta \in \Gamma, \mu \in P(G), \tag{2.3}$$

which holds for $\hat{\mu} = \hat{\mu}_n$ and hence for $\hat{\mu} = \lim \hat{\mu}_n$. And $\lim \hat{\mu}_n$, being continuous and positive definite with the value 1 at $\gamma = 0$, is the Fourier-Stieltjes transform $\hat{\mu}$ of some $\mu \in P(G)$ (Bochner, cf. [15, p. 19]) and satisfies (ii) above.

We write $\{g\}$ for the subset of G consisting of the point g, so that g is an atom if $\mu(\{g\}) > 0$. We write $\omega_1 = \omega_{1G} \in P(G)$ for the unit measure (i.e., $\omega_1(\{0\}) = 1$) and ω_{0G} for [normalized] Haar measure on G [if G is compact], so that $\omega_{0G} \in P(G)$ if G is compact. Also

$$\hat{\omega}_1(\gamma) = 1 \text{ and if } G \text{ is compact, } \hat{\omega}_{0G}(\gamma) = 0 \text{ for } \gamma \neq 0. \tag{2.4}$$

PROPOSITION 2.1. *Let $\mu, \nu \in P(G)$. The set of atoms [or support] of $\mu * \nu$ may be obtained by adding arbitrary elements of the sets of atoms [or supports] of μ and ν [and forming the closure]. Also $(\mu * \nu)(\{g\}) = \Sigma\mu(\{x\})\nu(\{y\})$ for $x + y = g$. If $\mu_n \longrightarrow \mu$ in $P(G)$ as $n \longrightarrow \infty$ and $\Sigma(\mu)$ denotes the support of μ, then $\Sigma(\mu) \subset \lim \Sigma(\mu_n)$ as $n \longrightarrow \infty$.*

By $\lim \Sigma(\mu_n)$ as $n \longrightarrow \infty$, we mean the set of points $g \in G$ with the property that, for every neighborhood U of g, $U \cap \Sigma(\mu_n) \neq \varnothing$ for large n. The next proposition follows by considering Fourier-Stieltjes transforms.

PROPOSITION 2.2. *If $\mu_1, \mu_2, \ldots \in P(G)$ satisfy*

$$\mu_n * \cdots * \mu_N \longrightarrow \omega_1 \quad \text{as } N \geqslant n \longrightarrow \infty, \tag{2.5}$$

then

$$\lim_{n\to\infty} \mu_1 * \cdots * \mu_n = \mu \quad \text{exists in } P(G). \tag{2.6}$$

In contrast to the case of convolutions on R^d, (2.6) does not imply (2.5). This is clear if G is compact and ω_{0G} is its normalized measure for, by (2.4), $\omega_{0G} * \mu_1 * \cdots * \mu_n = \omega_{0G} \longrightarrow \omega_{0G}$ as $n \longrightarrow \infty$ for arbitrary $\mu_1, \mu_2, \ldots$.

DEFINITION. When (2.6) holds, we say that the infinite convolution $\mu = \mu_1 * \mu_2 * \cdots$ is *convergent.* If, in addition, (2.5) holds, we say that it is *Cauchy-convergent.*

PROPOSITION 2.3. *If $\mu = \mu_1 * \mu_2 * \cdots$ is Cauchy-convergent, then $\Sigma(\mu) = \lim \Sigma(\mu_1 * \cdots * \mu_n)$ as $n \longrightarrow \infty$.*

The proof can be obtained by a modification of that of Wintner [16, pp. 60–61] for R. We consider analogues of Lévy's theorem for Cauchy-convergent convolutions.

THEOREM 2.1. (i) *If $\mu = \mu_1 * \mu_2 * \cdots$ is convergent and*

$$\prod_{n=1}^{\infty} d_{0n} \neq 0, \quad \textit{where } d_{0n} = \max_g \mu_n(\{g\}), \tag{2.7}$$

then μ is not continuous (i.e., μ has at least one atom). (ii) *Conversely, if $\mu = \mu_1 * \mu_2 * \cdots$ is Cauchy-convergent and μ is not continuous, then* (2.7) *holds.*

The following is similar to the proof of P. Lévy [11, pp. 150–152] as simplified by Jessen; cf. van Kampen [10, pp. 445–446] or Wintner [16, pp. 16–18].

PROOF. *On* (i). Let $g_n \in G$ satisfy $\Pi\mu_n(\{g_n\}) = d > 0$, e.g., let $d_{0n} = \mu_n(\{g_n\})$. Let $\lambda_n = \mu_1 * \cdots * \mu_n$, so that

$$\lambda_n(\{h_n\}) \geqslant \prod_{k=1}^{n} \mu_k(\{g_k\}) \geqslant d,$$

where $h_n = g_1 + \cdots + g_n$. There exists a compact $K \subset G$ which contains all but a finite number of $h_1, h_2 \cdots$. For otherwise, if K is any compact, then $\lambda_n(K) \leqslant 1 - d$ for infinitely many n. Thus $\mu(K) \leqslant 1 - d$ for any compact K; so that, since K is arbitrary, we obtain the contradiction $\mu(G) \leqslant 1 - d < 1$. Thus $h_1, h_2, \cdots$ has a cluster point g. If U is any neighborhood of $0 \in G$, then $\lambda_n(g + U) \geqslant d$ for infinitely many n, and so $\mu(g + U + U) \geqslant d$. Thus $\mu(\{g\}) \geqslant d > 0$.

On (ii). Let $\mu(\{g_0\}) = d > 0$. Following the arguments of [10, (18), p. 445] we can obtain:

(a) *Let $0 < d \leqslant 1$, $0 < 6\epsilon < d$ and U be a symmetric compact neighborhood of $0 \in G$. Let $\lambda, \mu, \nu \in P(G)$ with the properties*

$$\mu = \lambda * \nu \quad \textit{and} \quad \mu(\{g_0\}) = d \quad \textit{for some } g_0 \in G,$$

$$\mu(g_0 + U + U) < d + \epsilon \quad \textit{and} \quad \nu(U) > 1 - \epsilon.$$

Then there exist $g, h \in G$ such that $g_0 = g + h$, $h \in U$,

$$d - \epsilon < \lambda(\{g\}) < (d + \epsilon)/(1 - \epsilon) \quad \textit{and} \quad \nu(\{h\}) > 1 - 6\epsilon/d.$$

For $n = 1, 2, \ldots$, put $\lambda_n = \mu_1 * \mu_2 * \cdots * \mu_n$. Also $\nu_n = \mu_{n+1} * \mu_{n+2} * \cdots$ is defined and Cauchy-convergent, and $\mu = \lambda_n * \nu_n$ and $\nu_n \longrightarrow \omega_1$ as $n \longrightarrow \infty$. We now verify the following assertion; cf. [10, (19), p. 446]. (*Curiously, no assumption of metrizability of G is required.*)

(b) *There exist* $g_1, g_2, \ldots$ *and* $h_1, h_2, \ldots$ *in G such that* $g_0 = g_n + h_n$ *and* $g_n \longrightarrow g_0$, $h_n \longrightarrow 0$,

$$(2.8) \qquad \lambda_n(\{g\}) \longrightarrow d \quad \textit{and} \quad \nu_n(\{h_n\}) \longrightarrow 1 \quad \textit{as } n \longrightarrow \infty.$$

Let $D \geq 2$ be an integer such that $Dd > 6$, and for $m = D, D + 1, \ldots$, choose symmetric compact neighborhoods $U_D, U_{D+1}, \ldots$ of $0 \in G$ such that $U_D \subset U_{D+1} \subset \cdots$ and $\mu(g_0 + U_m + U_m) < d + 1/m$, and choose $N_D < N_{D+1} < \cdots$ so that $\nu_n(U_m) > 1 - 1/m \geq 1/2$ for $n \geq N_m$, $m \geq D$. Then, by (a), there exist $g_{nm}, h_{nm} \in G$ for $n \geq N_m$ satisfying $g_0 = g_{nm} + h_{nm}$, $h_{nm} \in U_m$,

$$d - 1/m < \lambda_n(\{g_{nm}\}) < (d + 1/m)/(1 - 1/m) \quad \text{and} \quad \nu_n(\{h_{nm}\}) > 1 - 6/dm.$$

Thus, $\lambda_n(\{g_{nm}\}) \longrightarrow d$ and $\nu_n(\{h_{nm}\}) \longrightarrow 1$ for $n \geq N_m$, $m \longrightarrow \infty$. For $n \geq N_D$, let $m = m(n)$ satisfy $N_m \leq n < N_{m+1}$. Put $g_n = g_{n,m(n)}$ and $h_n = h_{n,m(n)}$, so that (2.8) holds. Suppose that the sequence $h_D, h_{D+1}, \ldots \in U_D$ has a cluster point $g \neq 0$. If U is a compact neighborhood of g, then $\nu_n(U) \geq \nu_n(\{h_n\}) \geq 1 - 6/dm(n)$ for infinitely many n. But if $0 \notin U$, then $\nu_n(U) \longrightarrow \omega_1(U) = 0$ as $n \longrightarrow \infty$. This contradiction gives $h_n \longrightarrow 0$ as $n \longrightarrow \infty$, and proves (b).

Assertion (ii) can be proved by the use of (b) in the same way that Lévy's case of $G = R$ is proved in [10] with the use of the equivalent [10, (19), p. 446]. We omit details.

Let $\mu \in P(G)$. If (X, Ω, σ) is a probability measure space, a map $\phi: X \longrightarrow G$ is called measurable if $\phi^{-1}(A) \subset X$ is measurable for every Borel set $A \subset G$. If, also $\phi: (X, \sigma) \longrightarrow (G, \mu)$ is a measure preserving map (i.e., $\sigma(\phi^{-1}(A)) = \mu(A)$ for all Borel sets $A \subset G$), then μ is called the distribution of ϕ. It is clear that if μ is purely discontinuous, then there exist probability spaces (X, Ω, σ) and maps $\phi: X \longrightarrow G$ such that ϕ has μ as its distribution. (More generally, this is the case if (G, μ) is a Lebesgue measure space; cf. [14].)

Let (X, Ω, σ) be a probability space and $s_1(x), s_2(x), \ldots$ a sequence of measurable maps $s_n: X \longrightarrow G$. We say that $s_1(x), s_2(x), \ldots$ is *Cauchy in measure* if, for every neighborhood U of $0 \in G$,

$$(2.9) \qquad \sigma\{x \in X: s_N(x) - s_n(x) \notin U\} \longrightarrow 0 \quad \text{for } N > n \longrightarrow \infty.$$

If this is the case and, in addition, G is metrizable, then standard proofs for $G = R$ show that there exists a measurable map $s: X \longrightarrow G$ such that $s_n(x) \longrightarrow s(x)$ in measure as $n \longrightarrow \infty$, i.e.,

$$\sigma\{x \in X: s_n(x) - s(x) \notin U\} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.10}$$

Also there exists a subsequence $s_{n(1)}(x), s_{n(2)}(x), \ldots$ satisfying

$$s_{n(j)}(x) \longrightarrow s(x) \quad \text{a.e. on } (X, \Omega, \sigma) \text{ as } j \longrightarrow \infty. \tag{2.11}$$

We adopt the conventions of [10], omitting details here. Let $X = X_1 \times X_2 \times \cdots$ be an infinite product measure space carrying a product measure $\sigma = \Pi\sigma_n$, each X_n is a probability measure space with measure σ_n. A point $x \in X$ is a sequence $x = (x_1, x_2, \ldots)$ with $x_n \in X_n$ and, for example, a function $\phi_n(x_n)$ on X_n is also considered a function of $x \in X$ independent of x_k, $k \neq n$. Let $\mu_1, \mu_2, \ldots \in P(G)$ and let $\phi_n: X_n \longrightarrow G$ be a function having μ_n as its distribution. Then

$$s_n(x) = \phi_1(x_1) + \cdots + \phi_n(x_n), \tag{2.12}$$

considered as a function on X, has $\mu_1 * \cdots * \mu_n$ as its distribution.

PROPOSITION 2.4. *Let $\mu_1, \mu_2, \ldots \in P(G)$ and $X = X_1 \times X_2 \times \cdots$, $\phi_1(x_1), \phi_2(x_2), \ldots$ as above.* (i) *Then $s_1(x), s_2(x), \ldots$ is Cauchy in measure on X if and only if $\mu = \mu_1 * \mu_2 * \cdots$ is Cauchy-convergent.* (ii) *If this holds and, in addition, G is metrizable, then $s_1(x), s_2(x), \ldots$ has a limit $s(x)$ in measure on X and μ is the distribution of $s(x)$.*

Part (i) is clear, for the distribution of $s_N(x) - s_{n-1}(x) = \phi_n(x_n) + \cdots + \phi_N(x_N)$ is $\mu_n * \cdots * \mu_N$ for $N \geqslant n$. Part (ii) follows from the remarks concerning (2.10).

Following van Kampen [10], we define a *pure* probability measure $\mu \in P(G)$. Let $\mathfrak{A}$ be a class of Borel sets on G which is closed under countable unions and with the property that if $A \in \mathfrak{A}$, then every translate $A + g \in \mathfrak{A}$. (Such classes are, for instance, the class of enumerable sets or the class of null sets with respect to Haar measure ω_{0G}.) $\mu \in P(G)$ is said to be *pure* if it has the following property with reference to *every* class $\mathfrak{A}$: If $\mu(A) > 0$ for some $A \in \mathfrak{A}$, then there exists an $A_0 \in \mathfrak{A}$ such that $\mu(A_0) = 1$.

A probability measure $\mu \in P(G)$ is called *continuous* if it has no atoms (i.e., $\mu(\{g\}) = 0$ for all $g \in G$), *purely discontinuous* if $\mu(A) = 1$ for some enumerable set A, *absolutely continuous* (with respect to Haar measure ω_{0G}) if $\mu(A) = 0$ whenever $\omega_{0G}(A) = 0$ and, finally, (*continuous*) *singular* if it is continuous and if $\mu(A_0) = 1$ for some set A_0 with $\omega_{0G}(A_0) = 0$. [Note that $\mu \neq 0$ is absolutely continuous *and* purely discontinuous if G is countable.]

THEOREM 2.2 (PURE THEOREM). *Let $\mu_n \in P(G)$ be purely discontinuous and $\mu = \mu_1 * \mu_2 * \cdots$ Cauchy-convergent. Then μ is pure* (*hence absolutely continuous or purely discontinuous or* (*continuous*) *singular*).

If G is metrizable, this result follows from Proposition 2.4(ii) and the 0-or-1 principle; cf. the proof of [9, Theorem 35, p. 86] or [10, Theorem VIII, p. 444]. We shall modify these arguments, using Proposition 2.4(i), avoiding a "limit a.e." or "limit in measure". (Roughly speaking, we consider an arbitrary, but fixed, symmetric neighborhood V of $0 \in G$, a sequence $V_0, V_1, \ldots$ of such neighborhoods with $V = V_0$ and $V_{k+1} + V_{k+1} \subset V_k$ and the pseudo-metric induced on G by the neighborhood "base" $V_0, V_1, \ldots$ of $0 \in G$.)

PROOF. We give the proof in several steps. We write $A(2) = A + A$, $A(3) = A + A + A$, etc. If $Y \subset X$, we write Y^c for the complement of Y in X.

Let $X = X_1 \times X_2 \times \cdots$ and $\phi_1(x_1), \phi_2(x_2), \ldots$ be as in Proposition 2.4(i). Since ϕ_n is purely discontinuous, there is no difficulty about the existence of X_n and ϕ_n. It can also be supposed that the range of $\phi_n(x_n)$ in G is countable. Let M be a countable subset of G containing the ranges of s_n and $s_n - s_m$ for $n, m = 1, 2, \ldots$.

(a) *Let V be a symmetric neighborhood of* $0 \in G$. *Then there exists a sequence of integers* $0 < n(1) < n(2) < \cdots$, *depending on V, with the following property*: *if $\epsilon > 0$, then there exist an integer $N_\epsilon = N_{\epsilon V}$ and a measurable set $X_\epsilon = X_{\epsilon V} \subset X$ such that $\sigma(X_\epsilon) > 1 - \epsilon$ and*

$$s_{n(j)}(x) - s_{n(k)}(x) \in V \quad \textit{for } x \in X_\epsilon \textit{ and } n(j), n(k) \geqslant N_\epsilon.$$

In order to see this, let $V_0, V_1, \ldots$ be a sequence of symmetric neighborhoods of $0 \in G$ such that $V = V_0$ and $V_{k+1}(2) \subset V_k$ for $k = 0, 1, \ldots$, so that $(V_1 + \cdots + V_k) + V_k \subset V$ for $k = 1, 2, \ldots$. Choose $0 < n(1) < n(2) < \cdots$, so that

$$\sigma(\{x \in X: s_n(x) - s_m(x) \notin V_k\}) < 1/2^k \quad \text{for } n, m \geqslant n(k).$$

If K is so large that $2/2^K < \epsilon$ and if

$$X_\epsilon = \left[\bigcap_{k=K}^{\infty} \{x \in X: s_{n(k+1)}(x) - s_{n(k)}(x) \notin V_k\}\right]^c,$$

then $\sigma(X_\epsilon^c) < 1/2^K + 1/2^{K+1} + \cdots = 2/2^K < \epsilon$. Also, if $N_\epsilon = n(K)$, then $x \in X_\epsilon$ and $n(j) > n(k) \geqslant N_\epsilon$ imply that

$$\pm(s_{n(j)}(x) - s_{n(k)}(x)) \in V_k + V_{k+1} + \cdots + V_{j-1} \subset V.$$

This gives (a).

(b) In the remainder of the proof, except for the last two sentences, V is fixed. *We can therefore suppose that $n(1), n(2), \ldots$ is the full sequence* 1, 2, $\ldots$, for otherwise we replace $\mu_1 * \mu_2 * \cdots$ by $[\mu_1 * \cdots * \mu_{n(1)}] * [\mu_{n(1)+1} * \cdots * \mu_{n(2)}] * \cdots$, $X_1 \times X_2 \times \cdots$ by $[X_1 \times \cdots \times X_{n(1)}] \times [X_{n(1)+1} \times \cdots \times X_{n(2)}] \times \cdots$, and $\phi_1, \phi_2, \ldots$ by $s_{n(1)}, s_{n(2)} - s_{n(1)}, \ldots$.

For a subset A of G, introduce the following subsets of X:

$$D_n(A, V) = \{x \in X: s_n(x) \in A + V\} = s_n^{-1}(A + V),$$

$$D(A, V) = \{x \in X: s_n(x) \in A + V \text{ for large } n\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} D_n(A, V).$$

(c) *For $j > 0$, $D(A, V(j)) \cap X_\epsilon \subset D_n(A, V(j+1))$, hence $D(A, V(j)) \subset D_n(A, V(j+1)) \cup X_\epsilon^c$, for $n \geq N_\epsilon$.* For if $x \in D(A, V(j))$, then $s_m(x) \in A + V(j)$ for large m, and if $x \in X_\epsilon$, then $s_n(x) - s_m(x) \in V$ for $n, m \geq N_\epsilon$. Thus, $x \in D(A, V(j)) \cap X_\epsilon$ implies that $s_n(x) \in s_m(x) + V \subset A + V(j) + V = A + V(j+1)$ for $n \geq N_\epsilon$ and large m; i.e., $x \in D_n(A, V(j+1))$.

(d) *For $j > 0$, $D_n(A, V(j)) \cap X_\epsilon \subset D(A, V(j+1))$, hence $D(A, V(j)) \subset D(A, V(j+1)) \cap X_\epsilon^c$, for $n \geq N_\epsilon$.* For if $x \in D_n(A, V(j))$, then $s_n(x) \in A + V(j)$. Thus, $x \in D_n(A, V(j)) \cap X_\epsilon$ implies that $s_m(x) \in s_n(x) + V \subset A + V(j+1)$ for $m \geq n \geq N_\epsilon$; i.e., $x \in D(A, V(j+1))$. This gives (d) which together with (a), (b), and (c) have the following consequences.

(e) Let $A \subset G$ be a Borel set and $\lambda_n = \mu_1 * \cdots * \mu_n$. Then $\sigma(D(A, V(2))) \leq \sigma(D_n(A, V(3))) + \epsilon = \lambda_n(A + V(3)) + \epsilon$ and $\lambda_n(A + V) = \sigma(D_n(A, V)) \leq \sigma(D(A, V(2))) + \epsilon$ for $n \geq N_\epsilon$.

(f) Completion of the proof. Let $\mathfrak{A}$ be an admissible class of Borel subsets of G and suppose that $\mu(A) > 0$ for some $A \in \mathfrak{A}$. Then $\lambda_n(A + V) \geq \mu(A)/2 > 0$ for large n. Let $A_0 = A + M = \bigcup(A + g)$ for $g \in M$, so that $A_0 \in \mathfrak{A}$ since M is countable. If $0 < \epsilon < \mu(A)/2$, then $\sigma(D(A, V(2))) \geq \sigma(D_n(A, V)) - \epsilon = \lambda_n(A + V) - \epsilon > 0$. Thus $A_0 \supset A$ implies that $\sigma(D(A_0, V(2))) > 0$. The definitions of A_0 and $D(A_0, V)$ make it clear that $x = (x_1, x_2, \ldots) \in D(A_0, V(2))$ if and only if the same is true when any finite number of coordinates of x is changed. Thus, by the 0-or-1 principle, $\sigma(D(A_0, V(2))) = 1$ and, by (e), $\lambda_n(A_0 + V(3)) \geq 1 - \epsilon$ for $n \geq N_\epsilon$. Consequently, $\mu(A_0 + V(4)) \geq 1 - \epsilon$ for every $\epsilon > 0$ and every symmetric neighborhood V of $0 \in G$. This implies that $\mu(A_0) = 1$, and completes the proof.

3. **Convergent convolutions.** In this section, we consider the analogues of Theorems 2.1 and 2.2, when $\mu = \mu_1 * \mu_2 * \cdots$ is convergent (i.e., (2.6) holds), but not necessarily Cauchy-convergent (i.e., (2.5) need not hold).

For any closed subgroup H of G and $\mu \in P(G)$, define $\mu^{G/H} \in P(G/H)$ by

$$\mu^{G/H}(A) = \mu(T_H^{-1}A) \quad \text{for any Borel set } A \subset G/H, \tag{3.1}$$

where $T_H: G \longrightarrow G/H$ is the canonical map $g \longmapsto H + g$. Then

$$(\mu_1 * \cdots * \mu_n)^{G/H} = \mu_1^{G/H} * \cdots * \mu_n^{G/H} \tag{3.2}$$

and

(3.3) $\nu_n \longrightarrow \nu$ in $P(G) \Rightarrow \nu_n^{G/H} \longrightarrow \nu^{G/H}$ in $P(G/H)$.

The relation (3.3) is clear from the equivalence of (2.1) and (2.2), where f is constant on cosets of H (i.e., $f \in C_0^0(G/H)$).

PROPOSITION 3.1. *Let $\mu \in P(G)$ and $H \subset G$ a closed subgroup.* (i) *If μ is pure, then $\mu^{G/H}$ is pure and, conversely, if $\mu^{G/H}$ is pure and H is countable, then μ is pure.* (ii) *If μ is purely discontinuous [or absolutely continuous], then $\mu^{G/H}$ is purely discontinuous [or absolutely continuous], and the converse is valid if H is countable.* (iii) *If $\mu^{G/H}$ is continuous, then μ is continuous and, conversely, if μ is continuous and H is countable, then $\mu^{G/H}$ is continuous.*

PROOF. *On* (i). If $\mathfrak{A}$ is an admissible class of Borel sets on G [or on G/H], then $T_H\mathfrak{A}$ [or $T_H^{-1}\mathfrak{A}$] is an admissible class of sets on G/H [or on G].

Let μ be pure and let $\mathfrak{A}$ be an admissible class of sets of G/H such that $\mu^{G/H}(A) > 0$ for some $A \in \mathfrak{A}$. Then $\mu(T_H^{-1}A) = \mu^{G/H}(A) > 0$, so that there is an $A_0 \in \mathfrak{A}$ such that $\mu^{G/H}(A_0) = \mu(T_H^{-1}A_0) = 1$. Thus, $\mu^{G/H}$ is pure.

Conversely, let $\mu^{G/H}$ be pure and $\mathfrak{A}$ an admissible class of Borel sets of G such that $\mu(A) > 0$ for some $A \in \mathfrak{A}$. Then $\mu^{G/H}(T_HA) \geqslant \mu(A) > 0$ since $T_H^{-1}(T_HA) \supset A$. Hence, there is an $A_0 \in \mathfrak{A}$ such that $1 = \mu^{G/H}(T_HA_0) = \mu(T_H^{-1}(T_HA_0))$. But if H is countable, then $T_H^{-1}(T_HA_0)$ is the countable union of the sets $A_0 + h$, $h \in H$, so that $T_H^{-1}(T_HA_0) \in \mathfrak{A}$. Thus μ is pure.

On (ii). The statement concerning "purely discontinuous" is clear. If $\mu^{G/H}$ is absolutely continuous and H is countable, then the absolute continuity of μ follows as in (i).

Let μ be absolutely continuous and let $\mu'(g)$ be its Radon-Nikodým derivative. (The Radon-Nikodým theorem is valid on G even though ω_{0G} need not be σ-finite; cf. [8, (7), p. 256].) Let $\mathcal{C}$ be the collection of Borel sets on G/H and $T_H^{-1}\mathcal{C}$ the corresponding collection of sets on G. Then $\mu^{G/H}$ is absolutely continuous and its Radon-Nikodým derivative is the conditional expectation $E(\mu'/T_H^{-1}\mathcal{C})$; cf. [4, pp. 17–18].

On (iii). This is clear in view of the proof of (i).

Recall that an infinite product Πa_n of complex numbers is said to be convergent if there is an integer K such that $a_Ka_{K+1} \cdots a_n$ has a nonzero limit as $n \longrightarrow \infty$; i.e., if and only if $a_na_{n+1} \cdots a_N \longrightarrow 1$ as $N \geqslant n \longrightarrow \infty$.

PROPOSITION 3.2. *If $\mu_1, \mu_2, \ldots \in P(G)$ and Λ is the set of $\gamma \in \Gamma$ for which $\Pi\hat{\mu}_n(\gamma)$ converges, then Λ is a subgroup of Γ. If $\mu = \mu_1 * \mu_2 * \cdots$ is convergent in $P(G)$, then Λ is an open-closed subgroup.*

This result is given in Loynes [13, p. 451], who points out that Λ is a group in view of the Increments Inequality (2.3) and that Λ contains a neighborhood of $0 \in \Gamma$ when (1.1) is convergent since $\hat{\mu}(0) = 1 \neq 0$ and $\Pi\hat{\mu}_n(\gamma)$ converges uniformly on a neighborhood of $\gamma = 0$. In the latter case, Λ is open-closed; cf., e.g., [8, pp. 250–251].

When Λ is open-closed, it is the annihilator of its annihilator

$$H = \{g \in G\colon (g, \gamma) = 1 \text{ for all } \gamma \in \Lambda\} \tag{3.4}$$

[15, p. 36]. Also H is a compact subgroup of G since its dual group Γ/Λ is discrete; furthermore, the dual group of G/H is Λ [15, pp. 59, 35].

THEOREM 3.1. *Let $\mu_n \in P(G)$ and $\mu = \mu_1 * \mu_2 * \cdots$ be convergent. Then there exists a (unique smallest) compact subgroup H of G such that*

$$\mu^{G/H} = \mu_1^{G/H} * \mu_2^{G/H} * \cdots \tag{3.5}$$

is Cauchy-convergent in $P(G/H)$, i.e.,

$$\mu_n^{G/H} * \cdots * \mu_N^{G/H} \longrightarrow \omega_1 \quad \text{as } N \geqslant n \longrightarrow \infty. \tag{3.6}$$

Furthermore, if ω_{0H} is the normalized Haar measure on H, considered as a measure in $P(G)$, then

$$\mu = \mu * \omega_{0H}. \tag{3.7}$$

PROOF. Let Λ be as in Proposition 3.2 and H as in (3.4). The convergence of (3.5) follows from (3.3). Relation (3.6) follows from the definition of Λ in Proposition 3.2, from $\hat{\omega} \equiv 1$, and the fact that Λ is the dual group of G/H. Also, $\gamma \notin \Lambda$ implies that $\hat{\mu}(\gamma) = 0$, while $\hat{\omega}_{0H}(\gamma)$ is 1 or 0 according as $\gamma \in \Lambda$ or $\gamma \notin \Lambda$ [15, p. 59]. Thus (3.7) is a consequence of the uniqueness of the Fourier-Stieltjes transform.

THEOREM 3.2. *Let $\mu = \mu_1 * \mu_2 * \cdots$ be convergent and H a closed subgroup of G.* (i) *If*

$$\prod_{n=1}^{\infty} d_{Hn} \neq 0, \quad \text{where } d_{Hn} = \max_{y \in G/H} \mu_n^{G/H}(\{y\}) = \max_{g \in G} \mu_n(g + H), \tag{3.8}$$

then $\mu^{G/H}$ is not continuous; in which case, μ is not continuous if H is countable. (ii) *Conversely, if $\mu^{G/H}$ is not continuous (e.g., if μ is not continuous) and H is as in Theorem* 3.1, *then* (3.8) *holds.* (iii) *If H is as in Theorem* 3.1 *and $\mu^{G/H}$ is not continuous (or, equivalently,* (3.8) *holds), then μ is not continuous if and only if H is finite.*

PROOF. Except for the assertion (iii), this theorem follows from Theorem 2.1, by virtue of Theorem 3.1. The assertion (iii) is a consequence of the following: on the one hand, μ has an atom if $\mu^{G/H}$ does and H is countable; on the other hand, $\mu = \mu * \omega_{0H}$ is continuous if ω_{0H} is, while ω_{0H} is continuous unless H is finite.

THEOREM 3.3 (PURE THEOREM). *Let* $\mu = \mu_1 * \mu_2 * \cdots$ *be convergent in* $P(G)$, μ_n *purely discontinuous, and* $H \subset G$ *the compact subgroup of Theorem* 3.1. *Then* $\mu^{G/H}$ *is pure* (*hence absolutely continuous or purely discontinuous or* (*continuous*) *singular*), *and the same is true of* μ *if* H *is finite. Also* $\mu = \omega_{0G}$ *if* G *is compact and* $H = G$.

The first part of this theorem follows from Theorem 2.2 by virtue of Theorem 3.1, and the last part from (3.7).

In the important case where G is the circle group $T = R/Z$ and every closed subgroup H is T or is finite, we have

THEOREM 3.4. *Let* $\mu_n \in P(T)$ *and* $\mu = \mu_1 * \mu_2 * \cdots$ *be convergent.* (i) *Then* μ *is not continuous if and only if, for some integer* $\kappa > 0$,

$$(3.9) \qquad \prod_{n=1}^{\infty} d_{\kappa n} \neq 0, \quad \textit{where } d_{\kappa n} = \max_{\theta} \sum_{j=0}^{\kappa-1} \mu_n(\{\theta + j/\kappa\}).$$

(ii) *If, in addition,* μ_n *is purely discontinuous, then* μ *is pure* (*hence absolutely continuous or purely discontinuous or* (*continuous*) *singular*); *in particular,* μ *is purely discontinuous if and only if* (3.9) *holds for some integer* $\kappa > 0$.

It is easy to see that (1.2) and (3.9) are not equivalent. For example, in the case that $\omega_{1/\kappa} = \mu_1 = \mu_2 = \cdots$, where $\omega_{1/\kappa}$ is the probability measure on T with the atoms $0, 1/\kappa, \ldots, (\kappa - 1)/\kappa$ (mod 1) assigned the equal probability $1/\kappa$, then $\omega_{1/\kappa} * \omega_{1/\kappa} = \mu_1 * \mu_2 * \cdots = \omega_{1/\kappa}$. But $d_n = 1/\kappa$ and $d_{\kappa n} = 1$.

REMARK. If $\mu_1, \mu_2, \cdots$ are regular probability measures on R and the result concerning (3.9) is applied to (3.5) with $G = R$, $H = \epsilon Z$ with $\epsilon > 0$, then it follows that μ is not continuous if and only if

$$(3.10) \qquad \prod_{n=1}^{\infty} d_{\epsilon n} \neq 0, \quad \text{where } d_{\epsilon n} = \max_{t} \sum_{j=-\infty}^{+\infty} \mu_n(\{t + j\epsilon\}),$$

and so (1.2) and (3.10) are equivalent by Lévy's theorem (when (1.1) is convergent).

4. **Additive functions** f: $Z_+ \longrightarrow G$. Let f: $Z_+ \longrightarrow G$ be a G-valued function on the positive integers $Z_+ = \{1, 2, \ldots\}$. The mean value $M(f)$ is said to exist if $M(f) = \lim N^{-1} [f(1) + \cdots + f(N)]$ exists as $N \longrightarrow \infty$. Let $\tau_N \in P(G)$ be the distribution of the finite sequence $\{f(1), \ldots, f(N)\}$, i.e.,

$$\hat{\tau}_N(\gamma) = N^{-1} [(f(1), \gamma) + \cdots + (f(N), \gamma)] \quad \text{for } \gamma \in \Gamma.$$

The function f is said to possess an asymptotic distribution μ if there exists a $\mu \in P(G)$ and $\tau_N \longrightarrow \mu$ as $N \longrightarrow \infty$ in $P(G)$.

In the remainder of this paper, we suppose that f: $Z_+ \longrightarrow G$ *is additive, i.e.,* $f(m + n) = f(m) + f(n)$ *if* m, n *are relatively prime.*

For fixed primes p and P, let $f_p(n)$ be the additive function determined by its values on powers of primes given by

$$f_p(p^j) = f(p^j) \quad \text{and} \quad f_p(q^j) = 0 \quad \text{if } q \neq p \text{ is a prime}$$

and let $f^P(n)$ be the additive function

$$f^P(n) = \sum_{p<P} f_p(n), \quad \text{so that } f^P(n) \longrightarrow f(n) = \sum_p f_p(n) \text{ as } P \longrightarrow \infty$$

for $n = 1, 2, \ldots$. The additive function f_p has an asymptotic distribution $\sigma_p \in P(G)$, where

$$\hat{\sigma}_p(\gamma) = (1 - p^{-1})\left[1 + \sum_{j=1}^{\infty} p^{-j}(f(p^j), \gamma)\right], \tag{4.1}$$

and f^P has the asymptotic distribution $\sigma^P = \sigma_2 * \sigma_3 * \cdots * \sigma_P$; cf. [6] for $G = R$.

For fixed $\gamma \in \Gamma$, define the complex-valued multiplicative functions

$$F_\gamma(n) = (f(n), \gamma) \quad \text{and} \quad F_\gamma^P(n) = (f^P(n), \gamma) \quad \text{for } n = 1, 2, \ldots.$$

Thus, f has an asymptotic distribution μ if and only if $M(F_\gamma)$ exists for $\gamma \in \Gamma$ and is continuous at $\gamma = 0$ (in which case, $\hat{\mu}(\gamma) = M(F_\gamma)$). Note that the convergence of

$$\sigma_2 * \sigma_3 * \cdots * \sigma_P * \cdots \tag{4.2}$$

to μ in $P(G)$ is equivalent to $\mu \in P(G)$ and $M(F_\gamma^P) \longrightarrow \hat{\mu}(\gamma)$, as $P \longrightarrow \infty$, for all $\gamma \in \Gamma$.

Halasz's definitive paper [7] concerns the existence of the mean value $M(F)$ of a complex-valued multiplicative function F, $|F(n)| \leqslant 1$. On the one hand, his results and proof (cf. [7, p. 380]) imply that

$$\sum_p p^{-1}[1 - \mathrm{Re}(f(p), \gamma)p^{-iu}] < \infty \tag{4.3}$$

holds for at most one real μ. In the case that (4.3) fails for all real μ,

$$\sum_p p^{-1}[1 - \mathrm{Re}(f(p), \gamma)p^{-iu}] = \infty \quad \text{for } -\infty < u < \infty, \tag{4.4}$$

then $M(F_\gamma)$ exists and is 0, and also

$$M(F_\gamma^P) \longrightarrow 0 = M(F_\gamma) \quad \text{as } P \longrightarrow \infty. \tag{4.5}$$

On the other hand, a result of Delange (cf. [2], [3]) and/or of Halasz [7] shows that if

$$\sum_p p^{-1}[1 - (f(p), \gamma)] \quad \text{converges,} \tag{4.6}$$

then $M(F_\gamma)$ exists and is the convergent product

$$M(F_\gamma) = \prod_p \left\{ (1 - p^{-1}) \left[1 + \sum_{j=1}^{\infty} p^{-j} (f(p^j), \gamma) \right] \right\}, \tag{4.7}$$

so that

$$M(F_\gamma^P) = \prod_{p \leq P} \{\cdots\} \longrightarrow M(F_\gamma) \quad \text{as } P \longrightarrow \infty. \tag{4.8}$$

As observed by Delange, Halasz's results imply [3, Theorem C, p. 218], which, in turn, has the following consequence.

PROPOSITION 4.1. *Let $f\colon Z_+ \longrightarrow G$ be additive and $\gamma \in \Gamma$ fixed. Then $M(F_\gamma)$ exists and $M(F_\gamma) = 0$ if and only if either* (4.4) *holds or both* (4.3) *and*

$$2^{-iju} (f(2^j), \gamma) = -1 \quad \text{for } j = 1, 2, \ldots \tag{4.9}$$

hold for some real u.

A particular case of Delange [2] (see [3, Theorem A, p. 217]) is the following.

PROPOSITION 4.2. *Let $f\colon Z_+ \longrightarrow G$ be additive and $\gamma \in \Gamma$ fixed. Then $M(F_\gamma)$ exists and $M(F_\gamma) \neq 0$ if and only if* (4.6) *holds and* (4.9) *fails to hold for $u = 0$.*

Using arguments of Delange [3], we can obtain the next three propositions.

PROPOSITION 4.3. *Let $f\colon Z_+ \longrightarrow G$ be additive. Let Λ_0 be the set of $\gamma \in \Gamma$ for which there is a (unique) $u = u(\gamma)$ satisfying* (4.3). *Then Λ_0 is a group and $u(\gamma + \delta) = u(\gamma) + u(\delta)$ for $\gamma, \delta \in \Lambda_0$.*

PROOF. It is convenient to write (4.3) as

$$\sum_p p^{-1} \sin^2 [\arg(f(p), \gamma) - u \log p]/2 < \infty. \tag{4.10}$$

Thus the assertion follows from $\arg(f(p), \gamma + \delta) = \arg(f(p), \gamma) + \arg(f(p), \gamma)$ and from the simple inequality $\sin^2(x + y) \leq 2 \sin^2 x + 2 \sin^2 y$; cf. [3, p. 219].

PROPOSITION 4.4. *Let $f\colon Z_+ \longrightarrow G$ be additive. Let Λ be the set of $\gamma \in \Gamma$ satisfying* (4.6). *Then Λ is a subgroup of Λ_0.*

PROOF. This is contained in Proposition 3.2 since the finite product in (4.8) is $\Pi \hat{\sigma}_p(\gamma)$ for $p \leq P$. (A direct proof follows by the arguments of Delange [3, pp. 228–229].)

PROPOSITION 4.5. *Let $f\colon Z_+ \longrightarrow G$ be additive and let both $M(F_\gamma)$ and $M(F_{\gamma+\gamma})$ exist. Then either* (4.4) *or* (4.6) *holds.*

PROOF. Suppose that neither (4.4) nor (4.6) holds. Then, by Propositions 4.1 and 4.2, $M(F_\gamma) = 0$ and (4.3), (4.9) hold for some u. By Proposition 4.3, (4.3) holds if (γ, u) is replaced by $(\gamma + \gamma, 2u)$. But (4.9) does not hold if (γ, u) is replaced by $(\gamma + \gamma, 2u)$. Thus $M(F_{\gamma+\gamma}) \neq 0$. Consequently, (4.6) is convergent if γ is replaced by $\gamma + \gamma$, by Proposition 4.2; so that $\gamma + \gamma \in \Lambda_0$ with $u(\gamma + \gamma) = 0$, and

$$\sum_p p^{-1} \sin[\arg(f(p), \gamma)] \quad \text{converges} \tag{4.11}$$

if γ is replaced by $\gamma + \gamma$, i.e., $\arg(f(p), \gamma)$ by $2 \arg(f(p), \gamma)$. Since $\gamma \in \Lambda_0$, we have $u(\gamma) = u(\gamma + \gamma)/2 = 0$. Thus the real part of (4.6), i.e., (4.10) with $u = 0$, is convergent. The imaginary part (4.11) also converges since $|\sin 2x - 2 \sin x| = 4|\sin x|\sin^2(x/2)$. This is contrary to the assumption that (4.6) does not hold, and completes the proof.

For the case $G = R$, Erdös and Wintner [6, (iii), p. 720] show that an additive function $f\colon Z_+ \longrightarrow R$ has an asymptotic distribution μ if and only if (4.2) converges. We have the following generalization.

THEOREM 4.1. *Let $f\colon Z_+ \longrightarrow G$ be additive.* (i) *If f has an asymptotic distribution μ, then* (4.2) *converges.* (ii) *Conversely, if* (4.2) *is Cauchy-convergent, then f has an asymptotic distribution μ, and μ is pure.*

REMARK 1. Theorem 4.3 below for $G = T$ shows that, in general, the convergence of (4.2) is not sufficient for the existence of an asymptotic distribution.

REMARK 2. Since σ_p is purely discontinuous, the theorems of §§2–3 are applicable to (4.2).

PROOF. *On* (i). Since f has an asymptotic distribution μ, it follows that $M(F_\gamma)$ exists and $\hat{\mu}(\gamma) = M(F_\gamma)$ for all $\gamma \in \Gamma$. By Proposition 4.5, we have, for every fixed $\gamma \in \Gamma$, either (4.4), hence (4.5), or (4.6), hence (4.8). Consequently

$$\lim_{P\to\infty} \prod_{p<P} \hat{\sigma}_p(\gamma) = \hat{\mu}(\gamma) \quad \text{for } \gamma \in \Gamma. \tag{4.12}$$

This implies the convergence of (4.2).

On (ii). If (4.2) is Cauchy-convergent, then the product in (4.7) is convergent and is $\hat{\mu}(\gamma)$ for all γ. Hence the series in (4.6) is convergent and (4.7) holds for all γ, so that f has the asymptotic distribution μ. Also μ is pure by Theorem 2.2.

THEOREM 4.2. *Let $f\colon Z_+ \longrightarrow G$ be additive and let* (4.2) *converge. Then the subgroup Λ of Γ in Proposition* 4.4 *is the same as the subgroup Λ in Proposition* 3.2 *and Theorem* 3.1, *where $\mu_n = \sigma_p$ and $p = p_n$; so that if H is the annihilator of Λ, then $\mu^{G/H} = \sigma_2^{G/H} * \sigma_3^{G/H} * \cdots$ is Cauchy-convergent. In* *particular, $T_H f$: $Z_+ \longrightarrow G/H$ has an asymptotic distribution $\mu^{G/H}$, and $\mu^{G/H}$ is pure.*

This is clear from the proof of Theorem 4.1. For the case of $G = T$, we have the following partial converse to Theorem 4.1(i).

THEOREM 4.3. *Let f: $Z_+ \longrightarrow T$ be additive and let* (4.2) *converge, say, to μ.* (i) *If $\mu \neq \omega_{0T}$, then f has the asymptotic distribution $\mu \neq \omega_{0T}$.* (ii) *But if $\mu = \omega_{0T}$, then f need not have an asymptotic distribution.*

REMARK. Assertion (i) depends on Delange [3, Theorem 2, p. 226], the proof of which implies that (for an arbitrary group G) if Λ_0 in Proposition 4.3 is κZ and $\Lambda \neq \{0\}$ in Proposition 4.4, then $\Lambda = \Lambda_0$. In general, (i) is false if T is replaced by another group G, even the torus $G = T \times T$. For let h: $Z_+ \longrightarrow T$ have an asymptotic distribution $\sigma \neq \omega_{0T}$ and let f: $Z_+ \longrightarrow T$ be as in (ii) above, so that (h, f): $Z_+ \longrightarrow T \times T$ has no asymptotic distribution, but the analogue of (4.2) converges and is $\sigma \times \omega_{0T} \neq \omega_{0,T\times T}$.

PROOF. *On* (i). The convergence of (4.2) implies that (4.12) holds. If $\mu \neq \omega_{0T}$, then $\hat{\mu}(\gamma) \neq 0$ for some $\gamma \neq 0$. This implies the convergence of the product in (4.7), hence, of the series in (4.6) for some $\gamma \neq 0$. It follows from [3, Theorem 2, p. 226] that f has an asymptotic distribution. This distribution is (4.2) by Theorem 4.1.

On (ii). It will be shown that if h: $Z_+ \longrightarrow R$ is additive,

$$h(p) = \log p \quad \text{and} \quad h(p^j) = 0 \quad \text{for } j > 1, \tag{4.13}$$

and f: $Z_+ \longrightarrow T$, where $f(n) = h(n) \bmod 1$, then (4.2) is convergent with $\mu = \omega_{0T}$, but f does not have an asymptotic distribution. In order to see this, note that if $F_m(n) = \exp 2\pi i m f(n) = \exp 2\pi i m h(n)$, then

$$\begin{aligned} M(F_m^P) &= \prod_{p \leq P} (1 - p^{-1})[1 + p^{-1} e^{2\pi i m \log p}] \\ &= \exp\left\{- \sum_{p \leq P} p^{-1}(1 - e^{2\pi i m \log p}) + O(1)\right\}, \end{aligned}$$

as $P \longrightarrow \infty$. Hence

$$|M(F_m^P)| = \exp\left\{- 2 \sum_{p \leq P} p^{-1} \sin^2 \pi m \log p + O(1)\right\}.$$

As shown by Delange [3, p. 221],

$$\sum_p p^{-1} \sin^2 \pi m \log p = \infty \quad \text{for } m = 1, 2, \ldots,$$

so that $M(F_m^P) \longrightarrow 0$ as $P \longrightarrow \infty$ for $m = 1, 2, \ldots$. This gives (4.2) with $\mu = \omega_{0T}$.

Thus, by Theorem 4.1, it follows that if f has an asymptotic distribution, then this distribution is ω_{0T}. But then Propositions 4.1 and 4.5 imply that (4.4) holds for all γ, i.e.,

$$\sum_p p^{-1}[1 - \operatorname{Re} F_m(p)p^{-iu}] = \infty \quad \text{for } -\infty < u < \infty \text{ and } m = 1, 2, \ldots, \tag{4.14}$$

[3, Theorem 1, p. 220]. However, if $u = 2\pi m$, then $F_m(p)p^{-iu} = 1$, by (4.16), which contradicts (4.14). This completes the proof.

Theorem 3.4 will be seen to have the following consequences.

THEOREM 4.4. *Let f: $Z_+ \longrightarrow R$ be a real, additive function such that f* mod 1: *$Z_+ \longrightarrow T$ has an asymptotic distribution μ. Then* (i) *μ is pure* (*hence purely discontinuous or absolutely continuous or singular*), *and* (ii) *μ is purely discontinuous if and only if there exists an integer $\kappa > 0$ such that*

$$\sum_{\{p:\kappa f(p) \neq 0 \bmod 1\}} p^{-1} < \infty. \tag{4.15}$$

Part (ii) contains the corrected version of the theorem in [5] giving a necessary and sufficient condition for μ to be continuous.

A prime p does not occur in the sum (4.15) if the number $f(p)$ is of the form $f(p) = (\text{integer}) + j/\kappa$ for $j = 0, 1, \ldots$, or $\kappa - 1$. When $\mu \neq \omega_{0T}$, then μ is purely discontinuous if and only if (4.15) holds, where μ is chosen so that $0, 1/\kappa, \ldots, (\kappa - 1)/\kappa$ (mod 1) is the subgroup H of Theorem 4.2; cf. [3, pp. 227–229], where $\kappa = q$.

The necessary condition (4.15) cannot be replaced by

$$\sum_{\{p:f(p) \neq 0 \bmod 1\}} p^{-1} < \infty. \tag{4.16}$$

In order to see this, it suffices to exhibit a real additive function f possessing an asymptotic distribution mod 1 satisfying (4.15) for some $\kappa > 1$ (so that μ is purely discontinuous), but not satisfying (4.16). To this end, let $\kappa > 1$ be fixed and let f be a real additive function defined by $f(p) = 1/\kappa$ and $f(p^j) = 0$ for $j > 1$ for every prime p. Then the analogue of (4.6),

$$\sum_p p^{-1}[1 - \exp 2\pi i m f(p)] \quad \text{converges}$$

for every m divisible by κ, so that f has an asymptotic distribution mod 1 [3, Theorem 2, p. 226]. Note that $f(p) \neq 0$ mod 1 for every p, so that (4.16) fails. But (4.15) holds since the sum is over an empty set of primes.

PROOF OF THEOREM 4.4. *On* (i). Part (i) is a consequence of Theorem 3.3

for σ_p is purely discontinuous, since the mth Fourier-Stieltjes coefficient of σ_p is (1.3) with $u = 2\pi m$.

On (ii). By (1.3), the jump $\sigma_p(\{0\})$ is at least $1 - p^{-1} \geqslant 1/2$. Thus the maximum jump $d_{\kappa p}$ occurs at $\theta = 0$ and is

$$(1 - p^{-1})[1 + \epsilon_{\kappa p}p^{-1} + O(p^{-2})] = 1 + (\epsilon_{\kappa p} - 1)p^{-1} + O(p^{-2}),$$

as $p \longrightarrow \infty$, where $\epsilon_{\kappa p}$ is 1 or 0 according as $\kappa f(p)$ is or is not 0 mod 1. Hence (4.15) is equivalent to $\Pi d_{\kappa p} \neq 0$, and so part (ii) follows from Theorem 3.4.

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