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THE BRACKET RING OF A COMBINATORIAL GEOMETRY. II: UNIMODULAR GEOMETRIES

BY

NEIL L. WHITE[1]

ABSTRACT. The bracket ring of a combinatorial geometry G is a ring of generalized determinants which acts as a universal coordinatization object for G. Our main result is the characterization of a unimodular geometry as a binary geometry such that the radical of the bracket ring is a prime ideal. This implies that a unimodular geometry has a universal coordinatization over an integral domain, which domain we construct explicitly using multisets. An ideal closely related to the radical, the coordinatizing radical, is also defined and proved to be a prime ideal for every binary geometry.

To prove these results, we use two major preliminary theorems, which are of interest in their own right. The first is a bracket-theoretic version of Tutte's Homotopy Theorem for Matroids. We then prove that any two coordinatizations of a binary geometry over a given field are projectively equivalent.

1. **Introduction.** The bracket ring B_G of the combinatorial pregeometry $G(S)$ is defined in [10]. In summary, we consider the commutative polynomial ring over the integers in the indeterminants $\{[X]: X \in S^n, n = \text{rank } G\}$, called *brackets*. The *bracket ring* B_G is this polynomial ring divided by the ideal generated by the relations:

(1) $[X]$, if X contains repeated elements or is dependent in G,

(2) $[X] - (\text{sgn } \sigma)[\sigma X]$ for any permutation σ of X,

(3) $[x_1, \ldots, x_n][y_1, \ldots, y_n] - \Sigma_{i=1}^n [y_i, x_2, \ldots, x_n][y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_n]$.

The *syzygies* are any of the relations in this ideal. For example, we see immediately that (2), (3), and commutativity imply that

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$$(3') \qquad [x_1, \ldots, x_n][y_1, \ldots, y_n] - \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_n] \cdot [y_1, \ldots, y_{j-1}, x_i, y_{j+1}, \ldots, y_n]$$

is a syzygy for all j, $1 \leqslant j \leqslant n$. We need only the ordinary syzygies such as (3) or (3′) as opposed to the multiple syzygies of [10]; the distinction is irrelevant to our current work except perhaps to the validity of the conjectures in §6.

The distinction between pregeometry and geometry is also largely irrelevant to our work. We shall state most of our results for geometries, leaving the obvious generalization for pregeometries for the reader. We assume henceforth that all geometries are finite. The reader is referred to [4] for details regarding the basic concepts of combinatorial geometries.

The Homotopy Theorem provides a relation in B_G between two given brackets by constructing a path between them. It is so named in honor of Tutte's Homotopy Theorem, which inspired it. It may also be regarded as a homotopy theorem in the sense that a necessary condition for coordinatization is that the resulting relation be satisfied independently of the particular path constructed.

The Homotopy Theorem and Theorem 4.7 (the projective equivalence of coordinatizations of binary geometries) first were proved in the author's thesis [9]. Theorem 4.7 has since been proved by Brylawski and Lucas [2] using elementary techniques. Our current work is organized so that the reader may skip §§3 and 4 if he is primarily interested in the later sections, and is willing to assume Theorem 4.7.

Unimodular geometries, called regular matroids in [7], are those geometries which may be coordinatized over every field. They have applications in a number of fields, including integer programming and electrical network theory (see references in [3]).

The bracket ring has many interesting connections with algebraic geometry and its forerunner, classical invariant theory. Where algebraic geometers consider the ring $k[X_1, \ldots, X_n]$, for some algebraically closed field k, we must work with $\mathbf{Z}[X_1, \ldots, X_n]$ in order to consider coordinatizations of G over any field. Thus we are studying a discrete, characteristic-free version of the Grassmann manifold (the algebraic variety determined by the syzygies (2) and (3)). An analog of the Second Fundamental Theorem of Invariant Theory occurs explicitly in our Conjecture 6.8C, which states that any algebraic relation holding among the brackets under all coordinatizations is a relation in the ideal generated by the syzygies (assuming G is unimodular).

Many avenues of investigation regarding the bracket ring remain open; we mention just a few. First there are the Conjectures 6.7 and 6.8. An obvious but unanswered problem is the determination of the Krull dimension of B_G, i.e., the maximal length of a chain of prime ideals of B_G (*note*: an upper bound is easily obtained). Lucas' work [6] is very relevant to this problem. It would be very interesting to study the class of nonbinary geometries G for which rad(B_G) is prime. Little is known in this direction, although a counterexample is provided in Example 6.11. In this regard, one may wish to consider B_G with coefficients from $\mathbf{Z}_p$ (see Vámos [8]). Finally, the noncommutative version of B_G, considered in [9], should prove useful in studying coordinatizations over skew fields.

2. **Binary and unimodular geometries.** A *coordinatization* of $G(S)$ over the integral domain D is a mapping $\zeta: S \longrightarrow V$, where V is a vector space over D of dimension $n = \text{rank } G$, such that if $A \subseteq S$, then A is independent in $G \Leftrightarrow \zeta$ is one-to-one on A and ζA is linearly independent. Equivalently, we may assume D is a field by replacing D by its quotient field K; we shall alternate between these two viewpoints. Any coordinatization ζ may be represented by a matrix $M(\zeta)$ whose columns are $\{\zeta s: s \in S\}$. A *coordinatizing homomorphism*, or *c-homomorphism*, into a field K, is a homomorphism $\eta: B_G \longrightarrow K$ (not necessarily onto), such that $\eta[X] \neq 0$ for every basis X of G.

PROPOSITION 2.1. *Let $\zeta: S \longrightarrow V$ be a coordinatization of G over K. Then*

$$\eta([X]) = \det(\zeta(x_1), \ldots, \zeta(x_n)) \tag{1}$$

determines uniquely a c-homomorphism $\eta: B_G \longrightarrow K$. Conversely, if Y is a basis of G, every c-homomorphism $\eta: B_G \longrightarrow K$ determines uniquely a coordinatization ζ satisfying (1) *such that $M(\zeta)$ is in echelon form with respect to Y,*

$$M(\zeta) = \begin{array}{c} \begin{array}{cc} \quad Y \quad & \quad S - Y \end{array} \\ \left\| \begin{array}{ccccc|c} \alpha & & & & 0 & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & * \\ 0 & & & & 1 & \end{array} \right\| \end{array}$$

where $\alpha = \eta([Y])$. Two coordinatizations ζ_1 and ζ_2 determine the same c-homomorphism $\eta \Leftrightarrow M(\zeta_1) = EM(\zeta_2)$ for some matrix E such that $\det(E) = 1$.

PROOF. See Propositions 4.1 and 4.2 of [10].

A geometry is *binary* if it may be coordinatized over $GF(2)$, the 2-element field.

PROPOSITION 2.2. *Let G be a geometry. The following five conditions are equivalent.*

(1) *G is binary.*

(2) *No minor of G is the four-point line.*

(3) *No coline of G is contained in four distinct copoints.*

(4) *Every syzygy in B_G involves an even number of nonzero terms.*

(5) *If X and Y are bases of G, and $x \in X$, then the cardinality of $\{y \in Y: (X - x) \cup y$ and $(Y - y) \cup x$ are both bases$\}$ is odd.*

PROOF. The equivalence of 1, 2, and 3 is well known, see [4, p. 15.10]. We have $1 \Leftrightarrow 4$ via the c-homomorphism $\eta[X] = 1 \in GF(2)$ for every basis X, and $4 \Leftrightarrow 5$ is immediate.

A geometry $G(S)$ is *unimodular* (*regular* in [7]) if there is a coordinatization $\zeta: S \longrightarrow V$, where V is a vector space over the integers $\mathbf{Z}$, such that the matrix $M(\zeta)$ is totally unimodular, i.e., every k-by-k minor of $M(\zeta)$ has determinant 0 or ± 1, for all k, $1 \leqslant k \leqslant n$. Such a coordinatization ζ is called a *u-coordinatization.* A *u-homomorphism* is a homomorphism $\eta: B_G \longrightarrow \mathbf{Z}$ such that $\eta[X] = \pm 1$ for any basis X of G.

PROPOSITION 2.3. *A geometry $G(S)$ is unimodular if and only if there exists a u-homomorphism of B_G.*

PROOF. This follows from Proposition 2.1 by noting that if $M(\zeta)$ is in echelon form and all n-by-n minors of $M(\zeta)$ have determinant 0 or ± 1, then so do all k-by-k minors, $1 \leqslant k \leqslant n$.

PROPOSITION 2.4. *A u-homomorphism of B_G induces a c-homomorphism of B_G into K, for every field K; i.e., if G is unimodular, then G may be coordinatized over K for every field K.*

PROOF. Let $\eta: B_G \longrightarrow \mathbf{Z}$ be a u-homomorphism. Then if $\alpha: \mathbf{Z} \longrightarrow K$ is the canonical homomorphism, $\alpha\eta$ is the required c-homomorphism.

COROLLARY 2.5. *If G is unimodular, then G is binary.*

Let ζ and ζ' be two coordinatizations of the geometry $G(S)$ over a field K. We say ζ and ζ' are *projectively equivalent* if there exist nonsingular matrices C and D, with D a diagonal matrix, such that $M(\zeta) = CM(\zeta')D$.

We define θ^λ and θ^λ_s: $\mathrm{Hom}(B_G, K) \longrightarrow \mathrm{Hom}(B_G, K)$ by $\theta^\lambda\eta[X] = \lambda\eta[X]$ and

$$\theta^\lambda_s\eta[X] = \begin{cases} \lambda\eta[X] & \text{if } s \in X, \\ \eta[X] & \text{otherwise.} \end{cases}$$

for any $\lambda \in K - \{0\}$, $s \in S$, $\eta \in \mathrm{Hom}(B_G, K)$, $[X] \in B_G$. We note that $\theta^\lambda\eta$ and $\theta^\lambda_s\eta$ may be extended to all of B_G in the obvious manner, and it is straightforward to check that both are well defined homomorphisms on B_G.

If $\eta, \eta' \in \mathrm{Hom}(B_G, K)$, we say η and η' are *projectively equivalent* and that θ is a *projective transformation* from η' to η if $\eta = \theta\eta'$ where θ is a composite of any finite number of the operations θ^λ and θ^λ_s. Clearly projective equivalence is an equivalence relation on $\mathrm{Hom}(B_G, K)$.

PROPOSITION 2.6. *If ζ and ζ' are two coordinatizations of G over K, let η and η' (respectively) be the corresponding c-homomorphisms of B_G. Then ζ and ζ' are projectively equivalent if and only if η and η' are projectively equivalent.*

PROOF. From Proposition 2.1, $\eta = \eta'$ if and only if $M(\zeta) = EM(\zeta')$ where $\det(E) = 1$. Thus $M(\zeta) = CM(\zeta')$ for arbitrary nonsingular C, if and only if $\eta = \theta^\lambda\eta'$ where $\lambda = \det(C)$. If $M(\zeta) = M(\zeta')D$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, and the columns of $M(\zeta)$ and $M(\zeta')$ correspond to $s_1, \ldots, s_N$, where $S = \{s_1, \ldots, s_N\}$, then $\eta = \theta^{\lambda_1}_{s_1} \cdots \theta^{\lambda_N}_{s_N}(\eta')$, and conversely. The proposition follows by elementary arguments.

3. **The homotopy theorem.** Let Y and Z be distinct copoints containing a coline W in the geometry $G(S)$. Let H, J, and L be bases of Y, Z, and W (respectively). Let $x \in S - (Y \cup Z)$, $y \in Y - W$, $z \in Z - W$.

PROPOSITION 3.1. *For every $e \in S$,*

$$[L, y, x]\,[L, z, x]\,([J, x]\,[H, e] - [H, x]\,[J, e]) = [J, x]\,[H, x]\,[L, z, y]\,[L, x, e] \quad \textit{in } B_G.$$

PROOF. Let $e \in S$. Then

$$\begin{aligned}
&[\mathbf{L}, \mathbf{y}, \mathbf{x}]\,[L, z, x]\,[J, x]\,[H, \mathbf{e}] - [L, y, x]\,[L, z, x]\,[H, x]\,[J, e] \\
&\quad = [L, y, e]\,[L, z, x]\,[J, x]\,[H, x] - [L, y, x]\,[\mathbf{L}, \mathbf{z}, \mathbf{x}]\,[H, x]\,[J, \mathbf{e}] \\
&\quad = [L, \mathbf{y}, e]\,[\mathbf{L}, \mathbf{z}, \mathbf{x}]\,[J, x]\,[H, x] - [L, y, x]\,[L, z, e]\,[H, x]\,[J, x] \\
&\quad = [L, x, e]\,[L, z, y]\,[J, x]\,[H, x],
\end{aligned}$$

as required. At each step we have applied a syzygy on the elements in bold face. For example, $[\mathbf{L}, \mathbf{y}, \mathbf{x}]\,[H, \mathbf{e}] = [L, y, e]\,[H, x]$, where e may be exchanged only for x to give nonzero brackets, since any element of $L \cup y$ is dependent on H.

Let $T \subseteq S$ and suppose that every circuit of $G(S)$ is contained either in T or in $S - T$. We then say that T (and likewise $S - T$) is a *separator* of G. We say G is *connected* if it has no separators except S and $\emptyset$. A *connected component* of G is a minimal nonnull separator of G.

Let $e \in S$. A *Tutte path missing* e in G is a sequence $X_1, \ldots, X_k$ of copoints of G such that $e \notin X_i$ for all i, $1 \leqslant i \leqslant k$; $X_i \cap X_{i+1}$ is a coline, and

$X_i \cup X_{i+1} \cup e \neq S$, for all i, $1 \leqslant i \leqslant k-1$.

PROPOSITION 3.2. *If $G - e$ is connected and Y and Z are copoints of G such that $e \notin Y \cup Z$, then there exists a Tutte path missing e, $Y = X_1, \ldots, X_k = Z$, from Y to Z.*

PROOF. See Tutte, [7, pp. 150–151].

If $X_1, \ldots, X_k$ is a Tutte path missing e, we may choose H_i to be a basis of X_i, L_i a basis of the coline $X_i \cap X_{i+1}$, $x_i \in S - (X_i \cup X_{i+1} \cup e)$, $y_i \in X_i - X_{i+1}$, and $z_i \in X_{i+1} - X_i$, for all appropriate values of i.

THEOREM 3.3 (Homotopy Theorem). *Let $X_1, \ldots, X_k$ be a Tutte path missing e, with H_i, L_i, x_i, y_i, z_i chosen as above. Suppose x_i is dependent on $L_i \cup e$ for $1 \leqslant i \leqslant k-1$. Then in B_G,*

$$\left(\prod_{i=1}^{k-1} [L_i, y_i, x_i]\,[L_i, z_i, x_i]\,[H_{i+1}, x_i]\right)[H_1, e]$$

$$= \left(\prod_{i=1}^{k-1} [L_i, y_i, x_i]\,[L_i, z_i, x_i]\,[H_i, x_i]\right)[H_k, e].$$

PROOF. From Proposition 3.1,

$$[L_i, y_i, x_i]\,[L_i, z_i, x_i]\left([H_{i+1}, x_i]\,[H_i, e] - [H_i, x_i]\,[H_{i+1}, e]\right) = 0$$

since $[L_i, x_i, e] = 0$. The theorem follows.

PROPOSITION 3.4. *Let G be a binary geometry such that $G - e$ is connected. Let Y and Z be ordered bases of G each having e as its last element. Then there exist $(n-1)$-tuples $H_1, \ldots, H_k$ and elements $x_1, \ldots, x_{k-1}$, none of which involve e, such that for every c-homomorphism η,*

$$\eta[Y] = \eta[Z] \prod_{i=1}^{k-1} \frac{\eta[H_i, x_i]}{\eta[H_{i+1}, x_i]}.$$

PROOF. By Proposition 3.2, there is a Tutte path missing e, $\mathrm{Cl}(Y - e) = X_1, X_2, \ldots, X_k = \mathrm{Cl}(Z - e)$, where Cl denotes closure in the geometry G. Choose $H_1 = Y - e$, $H_k = Z - e$, and choose all other H_i, L_i, x_i, y_i, z_i arbitrarily as in Theorem 3.3. Since G is binary, X_i, X_{i+1}, and $\mathrm{Cl}(L_i \cup e)$ are the only three copoints on the coline $X_i \cap X_{i+1}$ for all i, hence $x_i \in \mathrm{Cl}(L_i \cup e)$. Thus Theorem 3.3 holds, and we apply η, then cancel the common factors $\eta[L_i, y_i, x_i]$ and $\eta[L_i, z_i, x_i]$.

4. **Projective equivalence.** In this section we prove that if G is binary and K is any field over which G may be coordinatized, then any two coordinatizations

of G over K are projectively equivalent. The proofs of the following two lemmas are straightforward.

LEMMA 4.1. *Let T be a separator of $G(S)$. Then G has a coordinatization (respectively u-coordinatization) over the field K if and only if both of the subgeometries $G(T)$ and $G(S - T)$ do. Furthermore, any coordinatization ζ of G is represented, for an appropriate choice of a basis of V, by a matrix of the form*

$$M(\zeta) = \left\| \begin{array}{c|c} M(\zeta_1) & 0 \\ \hline 0 & M(\zeta_2) \end{array} \right\|$$

where ζ_1 and ζ_2 are coordinatizations of $G(T)$ and $G(S - T)$, and conversely. Thus every c-homomorphism η: $B_G \longrightarrow K$ induces c-homomorphisms $\eta_1: B_{G(T)} \longrightarrow K$ and $\eta_2: B_{G(S-T)} \longrightarrow K$, and conversely, with unimodularity preserved in both directions.

LEMMA 4.2. *If T is a separator of G, and η (respectively η') is a c-homomorphism of B_G inducing c-homomorphisms η_1 and η_2 (respectively η'_1 and η'_2) on $B_{G(T)}$ and $B_{G(S-T)}$, then η and η' are projectively equivalent if and only if η_1 and η'_1, as well as η_2 and η'_2, are projectively equivalent.*

LEMMA 4.3. *Let $G(S)$ and $G^*(S)$ be dual pregeometries. Then there is a bijection between c-homomorphisms of B_G into K and those of B_{G^*} into K, preserving unimodularity and projective equivalence in both directions.*

PROOF. From Theorem 8.1 of [10], we have an isomorphism $B_G \simeq B_{G^*}$, under which brackets correspond to brackets, up to sign. The lemma follows.

If $G(S)$ is a combinatorial pregeometry, the *associated geometry* $F(T)$ is obtained by deleting any element $s \in S$ such that $\{s\}$ is dependent, and identifying each two remaining elements s_1 and $s_2 \in S$ such that $\{s_1, s_2\}$ is dependent in G.

LEMMA 4.4. *A coordinatization (respectively u-coordinatization) of G over K induces a coordinatization (respectively u-coordinatization) of the associated geometry F over K. Two coordinatizations of G are projectively equivalent if and only if the two induced coordinatizations of F are.*

PROOF. A coordinatization ζ of G corresponds to a coordinatization ζ' of F obtained by deleting 0-columns from $M(\zeta)$ and deleting all but one of any family of columns which are nonzero scalar multiples of each other. Although the resulting matrix $M(\zeta')$ is well defined only up to projective equivalence, the statements of the lemma follow.

LEMMA 4.5. *Let $G(S)$ be a connected pregeometry and F a flat of G such that G/F is connected. Then there exist flats $S = F_0 \supset F_1 \supset \cdots \supset F_k = F$ forming a maximal chain in the lattice of flats of G such that G/F_i is connected for all i.*

PROOF. This is a restatement of the Corollary on p. 14.4 of [4]. We note that our use of the term "connected" is that which is referred to in the footnote on p. 14.1 of [4].

LEMMA 4.6. *Let $G(S)$ be a connected pregeometry such that $G^*(S)$ is a geometry (not just a pregeometry). Then there exists an element $e \in S$ such that $G - e$ is connected.*

PROOF. Since G is connected, G^* is connected. We apply Lemma 4.5 to G^* and the flat $F = \varnothing$ to obtain a flat F_{k-1} of rank 1 such that G^*/F_{k-1} is connected. But since G^* is a geometry, $F_{k-1} = \{e\}$ for some $e \in S$, and $G^*/e = (G-e)^*$ is connected. Therefore the dual pregeometry $G - e$ is also connected.

THEOREM 4.7. *Let $G(S)$ be a binary pregeometry and K a field. Then any two c-homomorphisms η and η' of B_G into K are projectively equivalent. Furthermore, if η and η' are unimodular, the operations θ^λ and θ_s^λ which comprise the projective transformation from η' to η may be chosen so that $\lambda = -1$ is the only scalar used.*

PROOF. We proceed by induction on $|S|$, the theorem being trivial for $|S| = 1$.

By the preceding lemmas and the induction hypothesis, we may assume that $G(S)$ is connected and that $G^*(S)$ is a geometry, hence that $G - e$ is connected for some $e \in S$. Let $F = G - e$.

Since B_F is isomorphic to the subring of B_G generated by the brackets not containing e, we identify B_F with that subring. Let η and η' be any two c-homomorphisms of B_G. By the induction hypothesis, $\eta|_{B_F} = \theta(\eta'|_{B_F})$, where θ is a projective transformation on $\mathrm{Hom}(B_F, K)$, and where $\lambda = -1$ is the only scalar used if η and η' are u-homomorphisms.

Let θ^* be the transformation on $\mathrm{Hom}(B_G, K)$ induced by θ in the obvious manner, and let $\eta'' = \theta^*\eta'$. Let $[Z]$ be a fixed nonzero bracket containing e as its last element. Let $\lambda = \eta[Z]/\eta''[Z]$, and $\eta^* = \theta_e^\lambda\eta''$. Note that if η and η' are unimodular, $\lambda = \pm 1$, and if $\lambda = 1$, $\theta_e^\lambda =$ identity.

We now have $\eta[Z] = \eta^*[Z]$ and $\eta|_{B_F} = \eta^*|_{B_F}$. We are done if $\eta = \eta^*$. Thus it suffices to show $\eta[Y] = \eta^*[Y]$ for every nonzero bracket $[Y]$ containing e. We may assume by the antisymmetry relation for brackets that e is the last element in $[Y]$. By Proposition 3.4,

$$(*) \qquad \eta[Y] = \eta[Z] \prod_{i=1}^{k-1} \frac{\eta[H_i, x_i]}{\eta[H_{i+1}, x_i]}$$

for certain brackets $[H_i, x_i]$ and $[H_{i+1}, x_i]$ in B_F. Likewise, equation (*) holds with η^* replacing η. Hence $\eta[Y] = \eta^*[Y]$, completing the proof.

5. **The coordinatizing radical.** An ideal P in B_G is a *coordinatizing prime,* or *c-prime,* if P is the kernel of a c-homomorphism of B_G. Thus P is a c-prime if and only if P is a prime ideal such that $[X] \notin P$ for every basis X of G. The *coordinatizing radical,* denoted $c\text{-rad}(B_G)$, is $\bigcap\{P: P \text{ is a } c\text{-prime of } B_G\}$.

We say that $\eta_0: B_G \longrightarrow D_0$ is a *universal c-homomorphism* of B_G if D_0 is an integral domain, η_0 is a surjective c-homomorphism, and for every c-homomorphism $\eta: B_G \longrightarrow K$, there exists a homomorphism $\alpha: D_0 \longrightarrow K$ such that $\eta = \alpha\eta_0$. If we demand only that there exists α such that η and $\alpha\eta_0$ are projectively equivalent, we say η_0 is *universal up to projective equivalence.*

PROPOSITION 5.1. *For every geometry G, $c\text{-rad}(B_G)$ is a c-prime if and only if there exists a universal c-homomorphism of B_G, namely $\eta_0: B_G \longrightarrow B_G/c\text{-rad}(B_G)$.*

PROOF. If $c\text{-rad}(B_G)$ is a c-prime, let $D_0 = B_G/c\text{-rad}(B_G)$, and $\eta_0: B_G \longrightarrow D_0$ the canonical homomorphism. Then η_0 is a universal c-homomorphism, since $\ker \eta_0 \subseteq \ker \eta$ for every c-homomorphism η.

Conversely, let $\eta_0: B_G \longrightarrow D_0$ be a universal c-homomorphism of B_G, and let $I_0 = \ker \eta_0$. If P is any c-prime, then $\eta: B_G \longrightarrow B_G/P$ is a c-homomorphism, and $I_0 \subseteq P$ by the universality of η_0. Thus $I_0 \subseteq c\text{-rad}(B_G)$, but I_0 is a c-prime, hence $I_0 = c\text{-rad}(B_G)$, completing the proof.

If we are given a nonzero product of brackets $[X_1] \cdots [X_k]$ in B_G and if a_i is the total number of occurrences of s_i in $X_1, \ldots, X_k$ for every $s_i \in S$, then the *degree* of $[X_1] \cdots [X_k]$ is the multiset $M = \Pi s_i^{a_i}$, the product being taken over all $s_i \in S$. We will henceforth use this multiplicative notation for multisets rather than the additive notation of [10]. The multisets arising in this manner form a monoid $\mathcal{M}$ under the operation of formal multiplication, and $\mathcal{M}$ is a submonoid of the monoid $\mathcal{M}_n$ of all multisets on S of size a multiple of n (counting multiplicities). We proved in [10] that B_G is a graded ring over $\mathcal{M}_n$; we could have used $\mathcal{M}$ as easily.

PROPOSITION 5.2. *Suppose that $\eta_0: B_G \longrightarrow D_0$ is a universal c-homomorphism of B_G up to projective equivalence. Then $c\text{-rad}(B_G)$ is a c-prime. Furthermore, $c\text{-rad}(B_G)$ is the ideal generated by $\{J: J \in B_G, J$ is homogeneous of degree M for some multiset $M \in \mathcal{M}$, and $\eta_0 J = 0\}$.*

PROOF. Let Q be the ideal generated by $\{J: J$ is homogeneous of degree M for some multiset $M \in \mathbb{M}$, and $\eta_0 J = 0\}$. Since Q is a homogeneous ideal, to prove that Q is prime it suffices to prove that if C and E are homogeneous elements not in Q, then $CE \notin Q$ (see [11, vol. II, pp. 152–153]). But $\eta_0 C \neq 0$ and $\eta_0 E \neq 0$ implies $\eta_0 CE \neq 0$, and hence Q is prime.

Let J be a generator of Q as above, and let $\eta: B_G \longrightarrow K$ be a c-homomorphism. By hypothesis, $\eta = \theta\alpha\eta_0$ for some $\alpha: D_0 \longrightarrow K$ and projective transformation θ. Then $\alpha\eta_0 J = 0$ and since J is homogeneous, $\eta J = 0$. Thus $Q \subseteq c\text{-rad}(B_G)$.

We now prove the reverse inclusion. Let $C \notin Q$. Then $C = \Sigma_{M \in \mathbb{M}} C_M \not\equiv 0 \pmod{Q}$ where C_M is the homogeneous component of C of degree $M \in \mathbb{M}$, and only a finite number of $C_M \neq 0$. Let $S = \{s_1, \ldots, s_N\}$, $K =$ the quotient field of D_0, and $L = K(z_1, \ldots, z_N)$, where $z_1, \ldots, z_N$ are algebraically independent transcendentals over K. Then the inclusion map $\beta: D_0 \longrightarrow L$ induces a c-homomorphism $\beta\eta_0$, and

$$\theta_{s_1}^{z_1} \cdots \theta_{s_N}^{z_N} \beta\eta_0 C = \sum z_1^{a_1} \cdots z_N^{a_N} \beta\eta_0 C_M \neq 0,$$

where the sum is over multisets $M = \Pi s_i^{a_i} \in \mathbb{M}$. Thus $C \notin c\text{-rad}(B_G)$, and $Q = c\text{-rad}(B_G)$. It follows immediately that $c\text{-rad}(B_G)$ is a c-prime.

PROPOSITION 5.3. *If G is binary, then c-rad(B_G) is a c-prime.*

Furthermore, if G is unimodular, c-rad(B_G) is the ideal generated by $\{J - L$: J and L are products of brackets of the same degree, and $\eta_0 J = \eta_0 L\}$, where η_0 is any fixed unimodular homomorphism. If G is binary but not unimodular, c-rad(B_G) is the ideal generated by $\{J + L$: J and L are arbitrary nonzero products of brackets of the same degree$\}$.

PROOF. If G is unimodular, let $\eta_0: B_G \longrightarrow \mathbf{Z}$ be a u-homomorphism. Then Proposition 2.4, Theorem 4.7, and Proposition 5.2 give the required results immediately.

If G is binary but not unimodular, then by [7, p. 169], G has a minor F which is isomorphic either to the Fano plane (see Figure 1, adjoining Proposition 6.7 below) or to the dual of the Fano plane. In either case, G may be coordinatized over K if and only if char $K = 2$. Since G is binary we have a c-homomorphism $\eta_0: B_G \longrightarrow GF(2)$, and the canonical injection $\alpha: GF(2) \longrightarrow K$ fulfills the hypotheses of Proposition 5.2 by applying Theorem 4.7 again. The proposition follows.

6. **The radical of the bracket ring.** A prime ideal P of B_G is *trivial* if $[X] \in P$ for all bases X of G. If F and G are geometries on the same set S, then F is a *rank-preserving-weak-map* image (or simply rpwm-image) of G if rank $G =$ rank F and for every $A \subseteq S$, A is dependent in G implies that A is dependent in F.

If F is a rpwm-image of G, there is a canonical homomorphism $\pi_F: B_G \longrightarrow B_F$ whose kernel is generated by $\{[X] : X$ is a basis of G which is dependent in $F\}$.

PROPOSITION 6.1. *P is a nontrivial prime of B_G if and only if P is the kernel of a homomorphism $\eta: B_G \longrightarrow K$, for some field K, such that $\eta = \eta'\pi_F$ for some rpwm-image F of G and some c-homomorphism $\eta': B_F \longrightarrow K$.*

PROOF. This is a restatement of Theorem 4.3 of [10]. Indeed, F is simply the geometry on S defined by: X is a basis of $F \Leftrightarrow X$ is a basis of G such that $[X] \notin P$.

We say that F is a *proper* rpwm-image of G if F is a rpwm-image of G and $F \neq G$. Furthermore, F is a *simple* rpwm-image of G if F is a proper rpwm-image of G and if there does not exist F' such that F is a proper rpwm-image of F' and F' a proper rpwm-image of G. The following is a generalization of a result of Lucas [6, Proposition 6.21].

PROPOSITION 6.2. *Let G be binary, and let J and L be nonzero homogeneous of B_G of the same degree $M \in \mathbb{M}$. Then for every rpwm-image F of G, $\pi_F J \neq 0$ if and only if $\pi_F L \neq 0$.*

PROOF. Proceeding by induction over the rpwm-images of G, we may assume that F is a simple rpwm-image of G, hence by [6, Theorem 6.17], F is of the form $F \simeq (G/T) \oplus T$ for some subgeometry T of G. Then $\pi_F J \neq 0$

$\Leftrightarrow M$ includes precisely k (rank T) elements of T, counting multiplicities, where k (rank G) is the total number of elements of M, counting multiplicities
$\Leftrightarrow \pi_F L \neq 0$.

PROPOSITION 6.3. *Let G be binary, and let F be any rpwm-image of G. Then $B_F \subseteq B_G$ up to isomorphism, and $\pi_F|_{B_F} = \mathrm{id}$.*

PROOF. By Proposition 6.2, $B_F \subseteq B_G$ as additive groups. Clearly B_F is closed under multiplication, hence B_F is a subring of B_G.

PROPOSITION 6.4. *Let $G(S)$ be unimodular, and let P be a prime ideal of B_G. Then there exists a c-prime P' such that $P' \subseteq P$.*

PROOF. We may assume that P is nontrivial; otherwise let P' be the kernel of any u-homomorphism of B_G. By Propositions 6.1 and 6.3, P is the kernel of a homomorphism $\eta: B_G \longrightarrow K$, for some field K, such that $\eta|_{B_F}$ is a c-homomorphism of B_F for some rpwm-image F of G. As above, we may assume by induction that F is a simple rpwm-image of G, and hence that $F = (G/T) \oplus T$. By Proposition 2.4, there is a c-homomorphism $\nu': B_G \longrightarrow K$. Then $\nu = \nu'|_{B_F}$ is a c-homomorphism, and there is a projective transformation θ on $\mathrm{Hom}(B_F, K)$ such that $\eta|_{B_F} = \theta\nu$, by Theorem 4.7, using the fact that F is binary by [6, Theorem 6.5]. Let θ' be the projective transformation on $\mathrm{Hom}(B_G, K(z))$ induced by θ by extending each of the compositie factors θ^λ or θ^λ_s of θ in the obvious manner. Let

$$\theta^* = \theta^{z^k}\theta_{t_1}^{z^{-1}}\theta_{t_2}^{z^{-1}} \cdots \theta_{t_m}^{z^{-1}}$$

where z is transcendental over K, $T = \{t_1, \ldots, t_m\}$, and $k = \operatorname{rank} T$. Let $i: K \longrightarrow K(z)$ be the canonical injection, let $\eta' = \theta'\theta^*(i\nu')$, and $P' = \ker \eta'$.

Clearly η' is a c-homomorphism, since $i\nu'$ is, hence P' is a c-prime. If X is a basis of F, then X contains a basis of T, hence $\theta^* i\nu'([X]) = i\nu'([X])$, and $\eta'|_{B_F} = i\eta|_{B_F}$. Furthermore, $|Y \cap T| \leqslant k$ for every basis Y of G, with equality holding only if Y is a basis of F. Hence η' is a c-homomorphism of B_G into the polynomial domain $K[z]$, with z dividing $\eta'[Y]$ if Y is not a basis of F and $\eta'[Y] \in K$ if Y is a basis of F. Thus $\eta = \alpha\eta'$ where $\alpha: K[z] \longrightarrow K$, $\alpha(z) = 0$. Therefore $P' \subseteq P$, completing the proof.

We denote by $\mathrm{rad}(B_G)$ the radical of B_G, defined to be the intersection of all primes of B_G, or equivalently, the ideal of nilpotent elements [11, Vol. I, pp. 151–152].

COROLLARY 6.5. *If G is unimodular, then c-$\mathrm{rad}(B_G) = \mathrm{rad}(B_G)$. Thus $\mathrm{rad}(B_G)$ is a c-prime.*

REMARK 6.6. The preceding corollary means that for a unimodular geometry G, the universal coordinatization over the integral domain D_0 of Proposition 5.1 is also universal with respect to all coordinatizations of rpwm-images of G. We may explicitly construct D_0 as follows.

Fix a unimodular homomorphism $\eta_0: B_G \longrightarrow \mathbf{Z}$. Let J and L be any two nonzero products of brackets of the same degree M, for any $M \in \mathcal{M}$. Then by Proposition 5.3, $J \pm L \in \mathrm{rad}(B_G)$ if and only if $\eta_0 J \pm \eta_0 L = 0$. But $\eta_0 J = \pm 1$ and $\eta_0 L = \pm 1$, hence precisely one of $J + L$ and $J - L$ is in $\mathrm{rad}(B_G)$. Let R denote the free additive group $\bigoplus_{M \in \mathcal{M}} \mathbf{Z} \cdot M$ generated by $\mathcal{M}$. We define a product in R to be that induced by the ordinary product of two multisets in the monoid $\mathcal{M}$. Then R is a commutative ring, and there is a surjective homomorphism β: $B_G \longrightarrow R$ induced by $\beta J = \eta_0 J \cdot M$ for J homogeneous of degree M, with $\ker \beta = \mathrm{rad}(B_G)$. Thus $D_0 \simeq R$.

The problem remains to give a complete description of $\mathrm{rad}(B_G)$. Computational evidence to date supports the following conjectures.

CONJECTURE 6.7. If G is unimodular, and if J and L are products of brackets in B_G of the same degree M, and both are nonzero, then there exists a sequence of ordinary syzygies $J \equiv J_0 = J_1 = \cdots = J_k \equiv \pm L$, where each syzygy $J_{i-1} = J_i$ involves only the two nonzero products of brackets, J_{i-1} and J_i. (The computation in the proof of 6.9 provides an example of such a sequence of syzygies, even though the geometry involved is not unimodular.)

The conjecture may also be formulated as a basis exchange property for unimodular geometries, which the author plans to examine in a forthcoming paper. Furthermore, this conjecture implies the three following weaker statements, which are equivalent.

CONJECTURE 6.8A. If $\eta J = \eta L$ for J and L products of brackets of the same degree M and η a u-homomorphism, then $J = L$ in B_G if G is unimodular.

CONJECTURE 6.8B. If G is unimodular then $\mathrm{rad}(B_G) = 0$.

CONJECTURE 6.8C. If G is unimodular, and if $C, E \in B_G$ such that $\eta C = \eta E$ for all c-homomorphisms η of B_G, then $C = E$ in B_G.

We now complete the characterization of unimodular geometries in terms of the bracket ring.

THEOREM 6.9. *Let G be binary. Then G is unimodular if and only if* $\mathrm{rad}(B_G)$ *is a prime ideal.*

PROOF. We already have $\mathrm{rad}(B_G)$ prime if G is unimodular. Suppose that G is binary but not unimodular. By [7], G has a minor F which is isomorphic either to the Fano plane (illustrated in Figure 1, together with a coordinatization ζ over $GF(2)$) or to the dual of the Fano plane. In the latter case, G^* has a minor F^* isomorphic to the Fano plane, and since $B_G \simeq B_{G^*}$ from Theorem 8.1 of [10], we may simply assume that F is isomorphic to the Fano plane.

Letting $F = (G/T) - U$ where T is independent and $S - U$ spans $G(S)$, we have $B_F \simeq B_G(T, U) \subseteq B_G$, from Theorem 8.2 of [10], where $B_G(T, U)$ is the subring of B_G generated by all brackets $[X]$ such that $T \subseteq X \subseteq S - U$. Since $\mathrm{rad}(B_G(T, U)) = B_G(T, U) \cap \mathrm{rad}(B_G)$, we need only show that $\mathrm{rad}(B_F)$ is not prime.

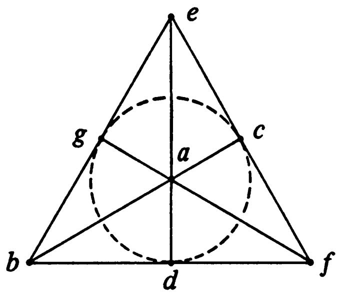
In B_F, let $C = [b, e, a]$, $E_1 = [e, f, d]\,[b, f, g]\,[b, e, c]$, and $E_2 = [b, e, d]\,[e, f, g]\,[b, f, c]$. Then

$$\begin{aligned} CE_1 &= [b, e, \mathbf{a}]\,[\mathbf{e}, \mathbf{f}, \mathbf{d}]\,[b, f, g]\,[b, e, c] \\ &= [b, e, d]\,[e, f, \mathbf{a}]\,[\mathbf{b}, \mathbf{f}, \mathbf{g}]\,[b, e, c] \\ &= [b, e, d]\,[e, f, g]\,[b, f, \mathbf{a}]\,[\mathbf{b}, \mathbf{e}, \mathbf{c}] = CE_2, \end{aligned}$$

where we have indicated the syzygies used by boldface, as in the proof of Proposition 3.1. Thus $C(E_1 - E_2) = 0 \in \mathrm{rad}(B_F)$.

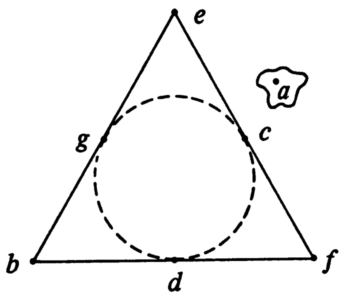
However, $C \notin \mathrm{rad}(B_F)$, for if η is the c-homomorphism corresponding to the given coordinatization ζ of F,

$$\eta[b, e, a] = \det \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1.$$



$$M(\zeta) = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left\| \begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right\| \end{array}$$

FIGURE 1



$$M(\zeta') = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left\| \begin{array}{ccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right\| \end{array}$$

FIGURE 2

On the other hand, we may find a coordinatization ζ' over $\mathbf{Z}$ of a rpwm-image of F (see Figure 2) such that if η' is the corresponding homomorphism of B_F, $\eta' E_1 = -1$ and $\eta' E_2 = 1$. Thus $E_1 - E_2 \notin \operatorname{rad}(B_F)$ and $\operatorname{rad}(B_F)$ is not prime.

COROLLARY 6.10. *If G is binary, then G is unimodular if and only if c-$\operatorname{rad}(B_G) = \operatorname{rad}(B_G)$.*

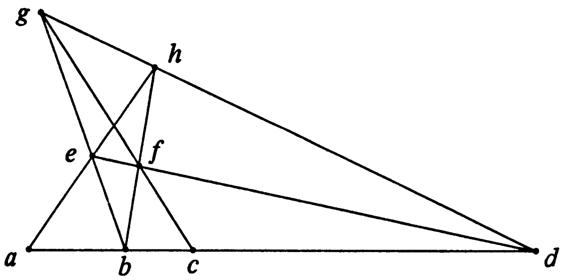
Little is known about $\operatorname{rad}(B_G)$ for nonbinary geometries. However, the following example, in which c-$\operatorname{rad}(B_G)$ is prime but $\operatorname{rad}(B_G)$ is not (as with binary nonunimodular geometries), may be of interest.

EXAMPLE 6.11. The geometry G of Figure 3 is coordinatizable over a field K if and only if char $K \neq 2$. For every field K such that char $K \neq 2$, the given matrix $M(\zeta_0)$ defines a coordinatization ζ_0 over K, and furthermore every coordinatization of G over K is projectively equivalent to ζ_0 (the proof of this fact is left as an exercise. Hint: Show that the columns a, d, f, and h may always be put into the form given in $M(\zeta_0)$ via a projective transformation). Thus the c-homomorphism $\eta_0: B_G \longrightarrow \mathbf{Z}$ induced by ζ_0 satisfies the hypothesis of Proposition 5.2, and c-$\operatorname{rad}(B_G)$ is prime.

Now consider the elements

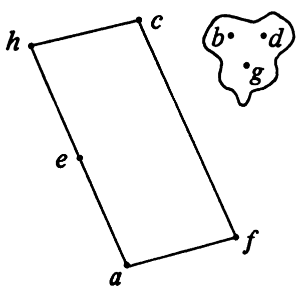
$$C = 2[a, c, h][c, e, f] - [a, c, f][c, e, h] \quad \text{and} \quad E = [b, d, g]$$

in B_G. For any field K, let $\eta: B_G \longrightarrow K$ be the c-homomorphism induced by η_0.



$$M(\zeta_0) = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \left\| \begin{matrix} 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{matrix} \right\| \end{array}$$

FIGURE 3



$$M(\zeta') = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \left\| \begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \end{matrix} \right\| \end{array}$$

FIGURE 4

Then $\eta C = 0$, and since every c-homomorphism into K is projectively equivalent to η and since C is homogeneous, $C \in c\text{-rad}(B_G)$. Thus $CE \in P$ for every c-prime P.

Therefore, to prove $CE \in \text{rad}(B_G)$, it suffices to prove $CE \in P$ for every prime P corresponding to a proper rpwm-image F of G. Suppose first that F is a simple rpwm-image of G. It may be verified that F must either be isomorphic to $(G/T) \oplus T$ for some subgeometry T of G, or to the Fano plane with the doubled point $\{a, c\}$, obtained from G by making $\{a, c\}$, $\{a, f, g\}$ and $\{c, e, h\}$ dependent. In any case, some bracket in each term of CE is on a set which is dependent in F. Hence the same is true for every proper rpwm-image F, and $CE \in \text{rad}(B_G)$.

On the other hand, $E \notin \text{rad}(B_G)$ since $\eta_0 E \neq 0$. Furthermore, there exists a rpwm-image F' of G, shown with a coordinatization $M(\zeta')$ over $\mathbf{Z}$ in Figure 4, such that if η' is the homomorphism of B_G corresponding to ζ', $\eta' C = 3 \neq 0$. Hence $C \notin \text{rad}(B_G)$ and $\text{rad}(B_G)$ is not prime.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611