TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 214, 1975

NORMAL STRUCTURE OF THE ONE-POINT STABILIZER OF A DOUBLY-TRANSITIVE PERMUTATION GROUP. II

BY

MICHAEL E. O'NAN[1]

ABSTRACT. The main result is that the socle of the point stabilizer of a doubly-transitive permutation group is abelian or the direct product of an abelian group and a simple group. Under certain circumstances, it is proved that the lengths of the orbits of a normal subgroup of the one point stabilizer bound the degree of the group. As a corollary, a fixed nonabelian simple group occurs as a factor of the socle of the one point stabilizer of at most finitely many doubly-transitive groups.

Introduction. This paper is a continuation of our study of the normal structure of the one-point stabilizer of a doubly-transitive permutation group, a study which we began in part I [9]. Here we prove the following theorems:

THEOREM A. *Let G be a doubly-transitive permutation group on a finite set X. Suppose that x is an element of X. Then either*

(i) *G_x has an abelian normal subgroup $\neq 1$, or*

(ii) *G_x has a unique minimal normal subgroup, and this minimal normal subgroup is simple.*

Actually, we prove slightly more than this. In fact, it is shown that the socle of G_x is either an abelian group or the direct product of an abelian group (possibly 1) and a simple group. Using this and a recent theorem of Aschbacher [11], it follows that in an unknown doubly-transitive group of minimal degree the socle of G_x is either abelian or simple.

THEOREM B. *Let G be a doubly-transitive permutation group on a finite set X and x an element of X. Suppose N^x is a normal subgroup of G_x. Let $|X| = n$ and suppose that on $X - x$ all orbits of N^x have length s. Then either*

(i) *N^x is semiregular on $X - x$, or*

(ii) *G is a normal extension of $L_n(q)$ (in its natural doubly-transitive representation), or*

(iii) $n < 2(s-1)^2$.

Presented to the Society, January 26, 1973; received by the editors May 6, 1974.
AMS (MOS) subject classifications (1970). Primary 20B20; Secondary 20B10, 20B25.
Key words and phrases. Doubly-transitive, permutation group, one-point stabilizer.
(1) This research was supported in part by NSF Grant GP-32319.

In both cases (i) and (ii), it is not possible to bound n in terms of s. Indeed, even if $s = 2$, n can be arbitrarily large. The bound of (iii), however, does not seem to be the best possible. There is a condition on N^x under which it can be improved. If $N^x \lhd G_x$, for other $y \in X$, we define N^y to be the unique conjugate of N^x in G_y. Then, we say N^x is balanced if $N_y^x = N_x^y = N^x \cap N^y$. Equivalently, N^x is a strongly closed subgroup of G_x in G.

THEOREM C. *Let G be a doubly-transitive group on a finite set X, and N^x, a normal subgroup of G_x. Suppose $|X| = n$ and the orbits of N^x on $X - x$ are of length s. Also, suppose that N^x is a permutation group of rank r on each of its orbits.*

Then if N^x is balanced and $N_y^x \neq 1$ if $y \in X - x$, it follows that

(i) $n \leqslant (s-1)^2$, *and*

(ii) *the number of orbits of N^x on $X - x$ is no more than* $r(s-1)(s-2)/s(s-r)$.

The bounds given here are much sharper. They hold exactly for an infinite number of groups G. (See the remark following the proof of Theorem C in §8.)

In the case $r = 2$, we are able to determine the group G without assuming that N^x is balanced. Indeed, we have:

THEOREM D. *Let G be a doubly-transitive group on a finite set X. Suppose N^x is a normal subgroup of G_x and N^x is doubly-transitive on each of its orbits on $X - x$. Then either*

(i) *N^x is transitive on $X - x$ and G is triply-transitive, or*

(ii) *G is a normal extension of $L_n(q)$, or*

(iii) $|N^x| = 2$ *and G has a regular normal subgroup.*

By combining Theorem C of Part I with Theorem A and Theorem B above, we obtain:

COROLLARY. *Let M be a fixed nonabelian finite simple group. Then there are only finitely many doubly-transitive groups G such that M is a factor of the socle of G_x.*

The following result is a consequence of Theorem A above and Theorem A of [8]. However, it is obtained preliminary to the proof of Theorem A:

PROPOSITION 4. *Let G be a doubly-transitive permuation group on a finite set X and suppose N^x is a normal subgroup of G_x. Then, either*

(i) *N^x restricts faithfully to its orbits on $X - x$, or*

(ii) *G is a normal extension of $L_n(q)$.*

The next result may also be of some independent interest:

PROPOSITION 8. *Let G be a doubly-transitive group on a finite set X and suppose that G_x admits a system of imprimitivity on $X - x$ having imprimitive block Δ. Suppose $|X| = n$ and $|\Delta| = s$. Suppose that no nonidentity element of G fixes all points of $x \cup \Delta$.*

Then, if $g \in G$, $g \neq 1$, g fixes at most $s\sqrt{n/2}$ points of X.

The bulk of the paper (the first seven sections) is connected with the proof of Theorem A.

In §1 we prove several counting lemmas. As a consequence we obtain the following result which is often of use:

LEMMA 1.3. *Let G be a doubly-transitive group on a set X, and N^x a normal subgroup of G_x having orbits of length s on $X - x$.*

Let $f \in G$, $f \neq 1$.

Set $S(f) = \{y \mid f \in N^y\}$.

Then, $|S(f)| \leq s - 1$.

The results of this section are crucial to the proofs of all our theorems.

In § 2 we begin the proof of Theorem A. As a first step, we show

LEMMA 2.3. *If G is a doubly-transitive group on X, $x \in X$, and G_x has a nonsolvable minimal normal subgroup, this nonsolvable minimal normal subgroup is unique.*

The proof of this result is, in fact, quite elementary and depends only on the use of Corollary B of [8].

In §3, we state several results about direct products of simple groups. These results are more or less standard.

In §4, we prove Proposition 4, stated above. Here, again the proof is not too complicated. At one point, we use a result of Glauberman which asserts that if M is a simple group and P a Sylow 2-subgroup of M the subgroup of Aut(M) which centralizes P is solvable.

In §§5, 6, and 7, N^x is a minimal normal subgroup of the point stabilizer G_x of a doubly-transitive group G. We assume $N^x = M_1^x \times \ldots \times M_k^x$, where the M_i^x's are isomorphic nonabelian simple groups. Our goal is to show that $k = 1$. This is done in three cases, according as:

(i) For all $x, y \in X$, $x \neq y$, and for all i, $1 \leq i \leq k$, $|(M_i^x)_y|$ is odd, i.e., all involutions of M_i^x fix only x.

(ii) For all $x, y \in X$, $x \neq y$, and for all i, j, $1 \leq i \leq k$, $1 \leq j \leq k$, we have $M_i^x \cap M_j^y = 1$, i.e., the subgroups M_i^x are T.I. sets.

(iii) $M_i^x \cap M_j^y \neq 1$ for some $x, y \in X$, $x \neq y$, and some i, j.

In §5, we treat case (i), in §6 case (ii), and so forth.

In the first case, our first step is to prove that N_y^x is of even order. Then, if j is an involution in N_y^x, and $S(j)$ is defined as in Lemma 1.3 above, we show that $C_G(j)$ is transitive on $S(j)$. The rank of j is defined to be the number of factors of N^x upon which j projects nontrivially. We then show that if j is an involution in N_y^x of minimal rank that $C_G(j) \mid S(j)$ is a permutation group whose point stabilizer has a semiregular normal 2-subgroup. By invoking a theorem of Shult [10], the precise structure of $C_G(j) \mid S(j)$ is obtained. The latter yields enough information to contradict the simplcity of the groups M_i^x.

We also note that the special case of Shult's result which we use is implied by some recent theorems of Aschbacher and also some of Goldschmidt.

The second case is more difficult. The methods used resemble closely those used in the proof Theorem A of Part I. Our first goal is to prove that those groups $(M_i^x)_y$ with $(M_i^x)_y \neq 1$ normalize the groups $(M_j^y)_x$ with $(M_j^y)_x \neq 1$. In proving this we utilize many of the theorems on (H, K, L)-configurations of Part I. When the latter result is obtained, it is not too difficult to complete the second case. In the latter part of the proof of the second case, we use the fact that, by the first case, $|(M_i^x)_y|$ is even.

In the third case, the counting methods of §1 come to the fore. First we obtain an inductive reduction to the case $k = 2$. The analysis for $k = 2$ is divided into two cases according as G_{xy} does or does not normalize M_1^x.

When G_{xy} normalizes M_1^x, we construct a block design and obtain a contradiction by counting. When G_{xy} does not normalize M_1^x, after some analysis, we force the group $(M_1^x)_y$ to be semiregular on the complement of its fixed point set and $|M_1^x : (M_1^x)_y|$ to be an odd number. Then, Bender's theorem is applicable to give the precise structure of M_1^x. At this point, formulas for $|X|$ and the lengths of the orbits of the groups M_i^x are obtained. Calculating explicitly with these formulas yields the desired contradiction.

In §8, we prove Theorems B and C, and in §9, we prove Theorem D. The proofs involve only the counting arguments of §1, Theorem A of [9], and Proposition 8, stated above.

1. The notations we use are the same as those of part I.

In this section we prove a counting lemma. In subsequent sections we apply this lemma in the proofs of Theorems A, B, and C.

We begin by studying the following general situation: G is a doubly-transitive group on a set X, and $\mathcal{G}_x$ is a graph on $X - x$ preserved by G_x.

Then, for other points $y \in X$, we may define a graph $\mathcal{G}_y$ on $X - y$, preserved by G_y, in such a way that if $f \in G$, $f(\mathcal{G}_y) = \mathcal{G}_{f(y)}$.

We suppose $|X| = n$ and the valence of $\mathcal{G}_x$ is k. If $a, b \in X$, $a \neq b$, we take $\tau(a, b) = \{x \mid \{a, b\} \in \mathcal{G}_x\}$. By the double-transitivity of G, $|\tau(a, b)|$ is independent of the pair $\{a, b\}$.

LEMMA 1.1. $|\tau(a, b)| = k$.

PROOF. Let $Y = \{(\{a, b\}, c) | \{a, b\} \in \mathcal{G}_c\}$. We determine in two ways the size of Y.

First, there are n possible choices for c. Since $\mathcal{G}_c$ is a graph of valence k on a set of $n - 1$ elements, $|\mathcal{G}_c| = (n - 1)k/2$. Thus, for each choice of c, there are $(n - 1)k/2$ choices for $\{a, b\}$. Thus, $|Y| = n(n - 1)k/2$.

Secondly, there are $n(n - 1)/2$ choices for $\{a, b\}$, and for each choice of $\{a, b\}$, there are $|\tau(a, b)|$ choices for c. Thus, $|Y| = n(n - 1)|\tau(a, b)|/2$.

Next we determine the number of edges in $\mathcal{G}_x \cap \mathcal{G}_y$, i.e., $|\mathcal{G}_x \cap \mathcal{G}_y|$ (with $x \neq y$). Again, by the double-transitivity of G, this number does not depend on the pair $\{x, y\}$.

LEMMA 1.2. *If* $x, y \in X$, $x \neq y$, $|\mathcal{G}_x \cap \mathcal{G}_y| = k(k - 1)/2$.

PROOF. Let $P = \{(\{a, b\}, \{x, y\}) | \{a, b\} \in \mathcal{G}_x$ and $\{a, b\} \in \mathcal{G}_y\}$. We determine in two ways the size of P.

There are $n(n - 1)/2$ choices for $\{x, y\}$, and for each $\{x, y\}$, there are $|\mathcal{G}_x \cap \mathcal{G}_y|$ choices for $\{a, b\}$. So $|P| = n(n - 1)|\mathcal{G}_x \cap \mathcal{G}_y|/2$.

There are $n(n - 1)/2$ choices for $\{a, b\}$, and for a fixed $\{a, b\}$, $\{x, y\}$ can be any two element subsets of $\tau(a, b)$. Thus, for each $\{a, b\}$, there are $k(k - 1)/2$ choices for $\{x, y\}$. Thus, $|P| = (n(n - 1)/2)(k(k - 1)/2)$.

Most frequently we shall use these lemmas in the following situation. Suppose G_x is imprimitive on $X - x$ and $\Delta(x, y)$ is the predesign function associated with the system of imprimitivity of G_x or $X - x$. Thus, the set $\Delta(x, y) - x$ is an imprimitive block of G_x.

We shall define a graph $\mathcal{G}_x$ on $X - x$ by $\{a, b\} \in \mathcal{G}_x$ if $\{a, b\}$ belongs to some $\Delta(x, y) - x$. Then, if $|\Delta(x, y) - x| = s$, $\mathcal{G}_x$ is a graph of valence $s - 1$.

In particular we note the following:

LEMMA 1.3. *Let G be a doubly-transitive group on X, and N^x a normal subgroup of G_x having orbits of length s on $X - x$.*

Let $f \in G$, $f \neq 1$.

Set $S(f) = \{y | f \in N^y\}$, when N^y is the unique conjugate of N^x in G_y.

Then $|S(f)| \leq s - 1$.

PROOF. We let $\Delta_N(x, y)$ be the predesign function associated with the orbits of N^x on $X - x$. We let $\mathcal{G}_x$ be the graph associated as above. Then, $\{a, b\} \in \mathcal{G}_x$ if and only if there is a $g \in N^x$ such that $g(a) = b$.

Then, with $\tau(a, b)$ as above, $\tau(a, b) = \{x | \text{there is a } g \in N^x \text{ such that } g(a) = b\}$.

Since $f \neq 1$, $f(a) = b$, with $a \neq b$, for some $a, b \in X$. Then, $S(f) \subseteq \tau(a, b)$. By Lemma 1.1, $|\tau(a, b)| = s - 1$.

2. In this section we prove that if the point stabilizer of a doubly-transitive group has a nonsolvable minimal normal subgroup, this minimal normal subgroup is unique.

Then, let G be a doubly-transitive group on X. Let M^x and N^x be minimal normal subgroups of G_x with $M^x \neq N^x$. We shall show that at least one of these groups is solvable. Assuming that this is not the case, M^x is a direct product of isomorphic nonabelian simple groups, as is N^x. Moreover, $M^x \cap N^x = 1$ and $[M^x, N^x] = 1$.

As usual, we define M^y so that $fM^y f^{-1} = M^{f(y)}$. Likewise, for N^y. Also, M^x_y is the subgroup of M^x which fixes $y \in X$.

By the Feit-Thompson theorem [3], M^x and N^x are both of even order. By Theorem C of part I, if M^x is semiregular on $X - x$, $O_2(M^x) \neq 1$. Since we are assuming M^x is nonsolvable, it follows that $M^x_y \neq 1$ if $y \in X - x$. Likewise, $N^x_y \neq 1$ if $y \in X - x$.

We take $\mathcal{B}$ and $\mathcal{C}$ to be the families of fixed point sets $\mathcal{B} = \{F_{M^x_y} \mid x, y \in X, x \neq y\}$ and $\mathcal{C} = \{F_{N^x_y} \mid x, y \in X, x \neq y\}$. By Corollary B1 of [8], $\mathcal{B}$ and $\mathcal{C}$ are block designs with $\lambda = 1$. Also, as $M^x_y \neq 1$ and $N^x_y \neq 1$, if $B \in \mathcal{B}$, $|B| < |X|$, and if $C \in \mathcal{C}$, $|C| < |X|$.

LEMMA 2.1. (i) *M^x fixes all blocks of $\mathcal{C}$ which contain x.*
(ii) *N^x fixes all blocks of $\mathcal{B}$ which contain x.*

PROOF. Let C be a block of $\mathcal{C}$ which contains x. If $y \in C - x$, then C is the fixed point set of N^x_y. Now, as $[M^x, N^x] = 1$, $[M^x, N^x_y] = 1$. Thus, M^x centralizes N^x_y, and so M^x fixes C, the fixed point set of N^x_y. Since C was any block of $\mathcal{C}$ containing x, (i) follows, and (ii) is the same.

LEMMA 2.2. (i) *$M^x \cap M^y = 1$ if $x \neq y$;*
(ii) *$N^x \cap N^y = 1$ if $x \neq y$.*

This is immediate from Lemma 2.1 and Lemma 2.8 of [8].

At this point, it is possible to quote Theorem A of part I to conclude that G is a normal extension of $L_n(q)$, in contradiction to the nonsolvability of M^x. Instead, we draw a direct contradiction.

Let C be the fixed point set of N^x_y, $x, y \in X$, $x \neq y$, (which is also the fixed point set of N^y_x, as $\mathcal{C}$ is a block design). By Lemma 2.1, M^x fixes C. Also, N^y_x fixes x and normalizes M^x. Thus, $M^x \cdot N^y_x$ is a subgroup of G which fixes the set C.

Let L be the subgroup of M^x which fixes all points of C. Then, $L \lhd M^x$, and as M^x is a direct product of isomorphic simple groups, $M^x = L \times K$. Moreover, as $M^x \lhd G_x$, and G_x is transitive on $X - x$, M^x has no orbit of length 1 on $X - x$; and so $K \neq 1$.

Now as N_x^y normalizes M^x and fixes C, N_x^y normalizes L. Since M^x is a direct product of simple groups, N_x^y normalizes K. Thus, $K \cdot N_x^y$ is a subgroup of G.

Moreover, $K \cdot N_x^y$ fixes the set C and N_x^y fixes all points of C. Since $K \cap L = 1$, it follows that N_x^y is the subgroup of $K \cdot N_x^y$ which fixes all points of C. Therefore, K normalizes N_x^y.

Now K does not fix y (or if so $M^x = K \times L$ would fix y), so there is an $f \in K$ with $f(y) \neq y$. Since f normalizes N_x^y, $N_x^y \leqslant N^y \cap N^{f(y)}$; but $N^y \cap N^{f(y)} = 1$, by Lemma 2.2. Hence, $N_x^y = 1$, contrary to hypothesis.

We have proved:

LEMMA 2.3. *If G is a doubly-transitive group and G_x has a nonsolvable minimal normal subgroup, then this nonsolvable minimal normal subgroup is unique.*

It follows from this that the point stabilizer of a doubly-transitive group is either local or has a unique minimal normal subgroup. In the next several sections, our goal is to prove the simplicity of this minimal normal subgroup.

3. In this section we note a few lemmas on direct products which we use in later sections. If $G = G_1 \times \ldots \times G_k$ is a direct product of groups, we let π_i be the projection of G on G_i. If S is a subset of $\{1, \ldots, k\}$, we let $G_S = \Pi_{i \in S} G_i$ and we take π_S to be the projection of G onto G_S. We say that a subgroup D of G is a diagonal if $\pi_i | D \colon D \longrightarrow G_i$ is an isomorphism for all i.

The following is easily proved:

LEMMA 3.1. *Let D be a diagonal of $G = G_1 \times \ldots \times G_k$. Then $N_G(D) = D(Z(G_1) \times \ldots \times Z(G_k))$.*

From this it follows quickly:

LEMMA 3.2. *If $G = G_1 \times \ldots \times G_k$ has a normal diagonal, $G_1, \ldots, G_k$ are abelian.*

LEMMA 3.3. *If $Z(G_1) = \ldots = Z(G_k) = 1$ and D is a diagonal of $G = G_1 \times \ldots \times G_k$, then $N_G(D) = D$.*

If $f \in G = G_1 \times \ldots \times G_k$, we take the support of f to be $\{i \mid \pi_i(f) \neq 1\}$.

LEMMA 3.4. *Suppose $G_1, \ldots, G_k$ are simple groups and H is a subgroup of $G = G_1 \times \ldots \times G_k$ such that $\pi_i(H) = G_i$ for all i. Then:*

(i) *There is a partition $P_1, \ldots, P_l$ of $\{1, \ldots, k\}$ such that H contains a diagonal H_j of $\Pi_{i \in P_j} G_i = G_{P_j}$.*

(ii) $H = H_1 \times \ldots \times H_l$.

(iii) $N_G(H) = H$.

(iv) *If $f \in H$, the support of f is a union of P_i's.*

PROOF. We say S is a set of support of H if S is the set of support of some $f \in H$. If S is a set of support of H, we let $H_S = H \cap G_S$.

First we claim: If S is a set of support of H, then $H_S = \pi_S(H)$.

Indeed, if $S = \{1, \ldots, k\}$, this is clear. So we may suppose $S \subset \{1, \ldots, k\}$. Now $\pi_i(H_S) \lhd \pi_i(H)$, as $H_S \lhd H$. But as S is the set of support of some $f \in H$, $\pi_i(H_S) \neq 1$ if $i \in S$. By hypothesis, $\pi_i(H) = G_i$. Since G_i is simple, $\pi_i(H_S) = G_i$. By induction in G_S, $N_{G_S}(H_S) = H_S$. Since $H_S \lhd \pi_S(H)$, it follows that $H_S = \pi_S(H)$.

Now if $\{1, \ldots, k\}$ is the only set of support of H, H is a diagonal of G, and the lemma follows. Otherwise, choose S a maximal set of support of H, properly contained in $\{1, \ldots, k\}$.

We claim: $H = H_S \times H_{\{1,\ldots,k\}-S}$.

For take $f \in H$ with $\pi_i(f) \neq 1$ for some $i \notin S$. Then $f = \pi_S(f)\pi_{\{i\}-S}(f)$. Since $\pi_S(f) \in H_S$, $g = \pi_{\{i\}-S}(f) \in H$. If the support of g is properly contained in $\{i\} - S$, S is not a maximal set of support. Thus, $H \subseteq \pi_S(H) \times \pi_{\{i\}-S}(H) = H_S \times H_{\{i\}-S}$.

Now (i) and (ii) follows by induction. (iii) follows by Lemma 3.3 and (iv) is clear.

It is well known that if $G = G_1 \times \ldots \times G_k$ is a direct product of simple groups and $N \lhd G$, then $N = G_S$ for some subset S of $\{1, \ldots, k\}$.

LEMMA 3.5. *Suppose $G_1, \ldots, G_k$ are simple groups and H is a subgroup of $G = G_1 \times \ldots \times G_k$. Suppose that H contains no normal subgroup $\neq 1$ of G. Then $|G:H| \geqslant 5^k$.*

PROOF. It suffices to show that if G admits a faithful transitive permutation representation on X, then $|X| \geqslant 5^k$.

Let G_1 be the factor of G of least order. Let $\Delta_1, \ldots, \Delta_l$ be the orbits of G_1 on X. Since G is transitive on X and $G_1 \lhd G$, there is an m such that $|\Delta_i| = m$ for all i.

Now if G_1 is the largest subgroup of G which fixes all of the sets $\Delta_1, \ldots, \Delta_l$, then G/G_1 acts faithfully and transitively on the l sets $\Delta_1, \ldots, \Delta_l$. By induction, $l \geqslant 5^{k-1}$. Since G is simple, $|\Delta_i| \geqslant 5$, and so $|X| \geqslant 5^k$.

If some other factor, say G_2, of G, fixes $\Delta_1, \ldots, \Delta_l$, since G_1 is transitive on Δ_i and $[G_2, G_1] = 1$, G_2 is semiregular on Δ_i. Since $|\Delta_i| \leqslant |G_1| \leqslant |G_2|$, $|G_2| = |\Delta_i|$. Thus, also G_1 is regular on Δ_i. We may identify the action of G_1 as the right regular representation of G_1 and that of G_2 as the left regular representation. Thus, $G_1 \times G_2$ is the largest subgroup of G fixing $\Delta_1, \ldots, \Delta_l$.

By induction, $l \geqslant 5^{k-2}$ and $|\Delta_i| = |G_i| \geqslant 5^2$.

We also note:

LEMMA 3.6. *Let $G = G_1 \times \ldots \times G_k$ and ϕ be an automorphism of G of prime order p.*

If $G_1, \ldots, G_p$ is an orbit of ϕ, then the centralizer of ϕ on $G_1 \times \ldots \times G_p$ is a diagonal of $G_1 \times \ldots \times G_p$.

4. In this section we prove

PROPOSITION 4. *Let G be a doubly-transitive group on X and suppose $N^x \lhd G_x$. Then either*

(i) *N^x restricts faithfully to each of its orbits on $X - x$, or*

(ii) *G is a normal extension of $L_n(q)$ (in its usual doubly-transitive representation).*

We take G a doubly-transitive group on X, $x \in X$, $N^x \lhd G_x$, such that N^x does not restrict faithfully to one of its orbits on $X - x$. Then, there is a minimal normal subgroup L of N^x which fixes $y \in X - x$. The characteristic closure K^x of L in N^x is either elementary abelian or semisimple. Moreover, K^x does not restrict faithfully to its orbits on $X - x$. If K^x is abelian, by Theorem A of [8], G is a normal extension of $L_n(q)$. Thus, it suffices to obtain a contradiction when K^x is the direct product of isomorphic simple groups. Clearly, we may take $K^x = N^x$.

We let $\Delta(x, y) = x \cup \{f(y) \mid f \in N^x\}$ be the predesign function associated with the orbits of N^x. We let $|\Delta(x, y)| = 1 + s$. We let $K(x, y)$ be the kernel of the homomorphism obtained by restricting N^x to $\Delta(x, y)$. By hypothesis, $K(x, y) \neq 1$. By double-transitivity, all the groups $K(x, y)$, with $x, y \in X$, $x \neq y$, are conjugate. $K(x, y)$ may be described as the largest normal subgroup of N^x which fixes the point y. From this definition, it is clear that if $K(x, y)$ and $K(x, y')$ fix a common point y'', then $K(x, y) = K(x, y')$. Thus, if we define $\Gamma(x, y) = F_{K(x,y)}$, $\Gamma(x, y)$ is a predesign function. It is clear from the definition that $\Delta(x, y) \subseteq \Gamma(x, y)$.

We note first a general lemma:

LEMMA 4.1. *Let G be a doubly-transitive group on a set X and suppose N^x is a normal subgroup of G_x and that N^x is a direct product of simple groups.*

Then, if $x, y \in X$, $x \neq y$, $\langle N^x, N^y \rangle$ is transitive on X.

PROOF. Suppose $\langle N^x, N^y \rangle$ is intransitive on X, for some $x, y \in X$, $x \neq y$. By the double-transitivity of G, this is true for all pairs $x, y \in X$, $x \neq y$.

Let $B(x, y) = \{z \mid N^z \subseteq \langle N^x, N^y \rangle\}$. Clearly, $\{x, y\} \subseteq B(x, y)$. We claim that the family of sets $\mathcal{B} = \{B(x, y) \mid x, y \in X, x \neq y\}$ form a block design on X (with $\lambda = 1$). Since $g(B(x, y)) = B(g(x), g(y))$, all the sets $B(x, y)$ have the same size. To prove that every two element subset of X belongs to exactly one $B(x, y)$, it suffices to show that if $\{a, b\} \subseteq B(x, y)$, $B(a, b) = B(x, y)$.

But if $\{a, b\} \subseteq B(x, y)$, $\langle N^a, N^b\rangle \subseteq \langle N^x, N^y\rangle$, and by double-transitivity, $\langle N^a, N^b\rangle = \langle N^x, N^y\rangle$, and the claim follows.

It is clear that if $y \in X - x$, N^x fixes the set $B(x, y)$. Thus, N^x fixes all blocks of $\mathcal{B}$ which contain x. By Theorem B of part I, G is a normal extension of $L_n(q)$. Since N^x is a direct product of simple groups, the latter is a contradiction to the structure of the one-point stabilizer in $L_n(q)$.

By hypothesis, $N^x = M_1^x \times \ldots \times M_k^x$, when the M_i^x's are isomorphic simple groups.

LEMMA 4.2. (i) $N_G(K(x, y)) \subseteq G_x$; (ii) $N_G(M_i^x) \subseteq G_x$.

PROOF. It suffices to show that if $L \lhd N^x$, $L \neq 1$, then $N_G(L) \subseteq G_x$. If $N_G(L) \nsubseteq G_x$, then also $L \lhd N^y$ for some $y \neq x$. Then, $\langle N^x, N^y\rangle$ fixes F_L. Since $F_L \subset X$, as $L \neq 1$, we have a contradiction by Lemma 4.1.

LEMMA 4.3. *If*, $x, y \in X$, $x \neq y$, (i) $\Delta(x, y) \neq \Delta(y, x)$; (ii) $\Gamma(x, y) \neq \Gamma(y, x)$.

PROOF. If $\Delta(x, y) = \Delta(y, x)$, $\langle N^x, N^y\rangle$ fixes the set $\Delta(x, y)$. $\Delta(x, y) \subset X$, as $K(x, y) \neq 1$ fixes all points of $\Delta(x, y)$. By Lemma 4.1, this is a contradiction. A similar contradiction results if $\Gamma(x, y) = \Gamma(y, x)$.

LEMMA 4.4. $K(y, x) \cap N^x = 1$ *if* $x \neq y$.

PROOF. Let $D = K(y, x) \cap N^x$ and suppose $D \neq 1$. Since $K(y, x)$ normalizes N^x, $D \lhd K(y, x)$. Since N^y is a direct product of simple groups and $K(y, x) \lhd N^y$, $D \lhd N^y$.

Let $T = \{t \mid D \subseteq N^t\}$. Then, $\{x, y\} \subseteq T$. Clearly, $N_G(D)$ fixes T. Thus, N^y fixes T. Since $\{x, y\} \subseteq T$, $\Delta(y, x) \subseteq T$. Therefore, $1 + s \leqslant |T|$.

On the other hand, by Lemma 1.3, $|T| \leqslant s - 1$. This contradiction shows $D = 1$.

LEMMA 4.5. $[K(y, x), N_y^x] = 1$.

PROOF. $K(y, x) \lhd G_{xy}$ and $N_y^x \lhd G_{xy}$. By Lemma 4.4, $K(y, x) \cap N_y^x = 1$.

LEMMA 4.6. *Let M_i^x be a simple factor of N^x. Then all orbits of M_i^x are of length* 1 *or of even length.*

PROOF. Let $y \in X - x$ and let Y be the orbit of y under M_i^x. Suppose $|Y|$ be odd. We show $|Y| = 1$.

Let L be the stabilizer of y in M_i^x. Then, as $|M_i^x|$ is even and $|Y|$ is odd, $L \neq 1$. Also, $L \subseteq N_y^x$. By Lemma 4.5, $K(y, x)$ centralizes L. Since $K(y, x)$ normalizes N^x and M_i^x is the unique simple factor of N^x which contains L, $K(y, x)$ normalizes M_i^x.

Since $|Y| = |M_i^x : L|$ is odd, $K(y, x)$ acts on M_i^x and centralizes a Sylow

2-subgroup of M_i^x. But by a theorem of Glauberman [5], the subgroup of $\mathrm{Aut}(M_i^x)$ which centralizes a Sylow 2-subgroup of M_i^x is solvable. Consequently, $K(y, x)$ centralizes M_i^x. Therefore, by Lemma 4.2(i), M_i^x fixes y. Hence, $|Y| = 1$.

We now complete the proof of the proposition.

Since $|\Gamma(x, y)| = |\Gamma(y, x)|$ and $\Gamma(x, y) \neq \Gamma(y, x)$ by Lemma 4.3, we may choose $z \in \Gamma(y, x)$, $z \notin \Gamma(x, y)$. Thus, $K(y, x)$ fixes z, but $K(x, y)$ does not fix z. Therefore, $K(y, x)$ normalizes $K(x, z)$.

Now $K(y, x)$ and $K(x, z)$ are both products of r isomorphic simple groups. Since $K(y, x)$ acts as a permutation group on the r simple factors of $K(x, z)$, it follows by Lemma 3.5, that some simple factor, say M_1^y, of $K(y, x)$ fixes all simple factors of $K(x, z)$.

Now if M_1^y centralizes all simple factors of $K(x, z)$, M_1^y centralizes $K(x, z)$, and by Lemma 4.2, $K(x, z)$ fixes y, contrary to hypothesis.

Thus, there is some simple factor, M_1^x, of $K(x, z)$ such that M_1^y normalizes but does not centralize M_1^x.

Let Y be the orbit of y under M_1^x. If $Y = \{y\}$, $M_1^x \leqslant N_y^x$. Since $K(y, x)$ centralizes N_y^x by Lemma 4.5, M_1^y centralizes M_1^x, contrary to choice. Therefore, $|Y| > 1$. By Lemma 4.6, $|Y|$ is even.

Since M_1^y normalizes M_1^x and M_1^y fixes y, M_1^y fixes Y. We consider the action of $M_1^x \cdot M_1^y$ on Y. If L is the stabilizer of y in M_1^x, by Lemma 4.5, $[L, M_1^y] = 1$. Moreover, in the group $M_1^x \cdot M_1^y$ the stabilizer of y is $L \times M_1^y$.

Since $|Y| > 1$, $L \subset M_1^x$. Therefore, M_1^y is the unique subgroup of its isomorphism type in $L \times M_1^y$. Therefore, M_1^y is a weakly closed subgroup of $L \times M_1^y$ in $M_1^x \cdot M_1^y$.

By Witt's theorem, the normalizer of M_1^y in $M_1^x M_1^y$ is transitive on the fixed point set of M_1^y in Y. But by Lemma 4.2, the normalizer of M_1^y fixes y. Thus, the fixed point set of M_1^y in Y is precisely $\{y\}$.

Thus, all orbits of M_1^y on $Y - y$ are of length greater than 1. By Lemma 4.6, $|Y - y|$ is even. Hence, $|Y|$ is odd, a contradiction, in so far as we have already proved that $|Y|$ is even.

This completes the proof of the proposition.

5. In the next three sections we complete the proof of Theorem A. We assume G is a doubly-transitive group on a set X and N^x a normal subgroup of G_x. We suppose $N^x = M_1^x \times \ldots \times M_k^x$, with the M_i^x isomorphic nonabelian simple groups. We shall prove that $k = 1$.

By the proposition of §4, N^x is faithful on its orbits on $X - x$. Hence no M_i^x fixes points in $X - x$.

Our analysis will be in three cases according as:

1. For all i, no involutions of M_i^x fix points in $X - x$, i.e., $|(M_i^x)_y|$ is odd for all $i, x, y \in X, x \neq y$.

2. $M_i^x \cap M_j^y = 1$ for all $x, y \in X$, $x \neq y$, and all i, j, but $|(M_i^x)_y|$ is even for some $i, x, y \in X$, $x \neq y$.

3. $M_i^x \cap M_j^y \neq 1$ for some $x, y \in X$, and some i, j.

In this section we assume $|(M_i^x)_y|$ is odd for all i and all $x, y \in X$, $x \neq y$. We assume $k \geq 2$ and obtain a contradiction. We take π_i to be the projection of N^x on M_i^x.

By the Feit-Thompson Theorem [3] and the Brauer-Suzuki Theorem [2] the 2-rank of M_i^x is at least two.

LEMMA 5.1. *$|N_y^x|$ is even.*

PROOF. If $|N_y^x|$ is odd, by Lemma 2.2 of part I, all involutions of N^x are conjugate in G_x. But this is not possible if $N^x \lhd G_x$ and N^x has at least two simple factors.

LEMMA 5.2. *If j is an involution, let $S(j) = \{z \mid j \in N^z\}$. Then $C_G(j)$ is transitive on $S(j)$.*

PROOF. Clearly $C_G(j)$ fixes the set $S(j)$. To prove the lemma, it suffices to show that if $z \in S(j)$, the orbits of $C_{G_z}(j)$ on $S(j) - z$ are of even length.

Now if $z \in S(j)$, $j \in N^z$, and there is an i such that $\pi_i(j) = j_i \neq 1$. Then $[j, j_i] = 1$ and j_i is semiregular on $X - x$, as $j_i \in M_i^x$.

Now if j is an involution of N^x, we define the rank of j to be the size of the support of j in the direct product $M_1^x \times \ldots \times M_k^x$. By Lemma 5.2, if $j \in N^x$ and $j \in N^y$, the rank of j in N^x and the rank of j in N^y are the same. Thus, the rank of j does not depend upon x, so long as $x \in S(j)$.

LEMMA 5.3. *Let r be the minimal rank of the involutions in N_y^x. Then, $N^x \cap N^y$ contains an involution of rank r.*

PROOF. We suppose that $N^x \cap N^y$ contains no involution of rank r. Then, by double-transitivity no $N^a \cap N^b$ contains an involution of rank r, if $a, b \in X$, $a \neq b$. Take $j \in N_x^y$, an involution of rank r. Since j does not belong to N^z for $z \neq y$, $C_G(j) \subseteq G_y$.

Now j permutes the factors M_i^x of N^x by conjugation. If j fixes some factor M_i^x of N^x, then $C_{M_i^x}(j) \subseteq (M_i^x)_y$ is of even order contrary to hypothesis. Therefore, j fixes no factor of N^x.

Then we may suppose j interchanges M_{2i-1}^x and M_{2i}^x for $i = 1, \ldots, k/2$. By Lemma 3.6, j centralizes a diagonal D_i of $M_{2i-1}^x \times M_{2i}^x$. Hence, $D_i \subseteq N_y^x$. Thus, $H = D_1 \times \ldots \times D_{k/2} \subseteq N_y^x$.

Now $\pi_i(H) = M_i^x$ for all i. By Lemma 3.4, $N_y^x = D_1 \times \ldots \times D_{k/2}$, as if $N_y^x \supset D_1 \times \ldots \times D_{k/2}$, N_y^x contains a factor of N^x, a contradiction.

Therefore, N_y^x is a direct product of isomorphic simple groups, as is N_x^y.

By Theorem A of part I $N^x \cap N^y \neq 1$. Since $N^x \cap N^y \lhd N^x_y$, $N^x \cap N^y$ contains some D_i.

Therefore, $N^x \cap N^y$ contains an involution of rank 2. As N^x_y contains no involution of rank 1, $r = 2$, in contradiction to our original assumption, that $N^x \cap N^y$ contains no involution of minimal rank r.

We now take j to be an involution in $N^x \cap N^y$ of rank r. Then we choose M^x_i, $1 \leqslant i \leqslant r$, so that $\pi_i(j) = j_i \neq 1$.

LEMMA 5.4. (i) *$C_{G_x}(j)$ in its action by conjugation permutes $\{j_1, \ldots, j_r\}$.*
(ii) *$\langle j_1, \ldots, j_r\rangle \lhd C_{G_x}(j)$.*
(iii) *$\langle j_1, \ldots, j_r\rangle | S(j) - x$ is semiregular.*

PROOF. (i) $C_{G_x}(j)$ permutes $M^x_1, \ldots, M^x_r$ and centralizes j, so $C_G(j)$ permutes $j_1, \ldots, j_r$. (ii) is clear. (iii) If $t \in \langle j_1, \ldots, j_r\rangle^\#$ fixes $y \in S(j) - x$, then $t \in N^x_y$ and t is of rank $\leqslant r$. If the rank of t is $< r$, t cannot fix y, as r is the smallest rank of an involution of N^x_y. If the rank of t is r, then $t = j$. Since $\langle j\rangle$ fixes all points of $S(j)$, the result follows.

It follows from Lemma 5.4 that $C_G(j) | S(j)$ is a permutation group whose point stabilizer has a semiregular normal 2-subgroup. By a theorem of Shult [10], either $C_G(j) | S(j)$ has a regular normal subgroup, or is a normal extension of $SL(2, 2^\alpha)$, $U_3(2^\alpha)$, or $Sz(2^\alpha)$. In particular, all involutions in $\langle j_1, \ldots, j_r\rangle/\langle j\rangle$ are conjugate under the action of $C_{G_x}(j)$.

LEMMA 5.5. $r = 3$.

PROOF. All involutions of $\langle j_1, \ldots, j_r\rangle/\langle j\rangle$ are conjugate under $C_{G_x}(j)$ by inspection. On the other hand, by Lemma 5.4, $\{j_1, \ldots, j_r\}$ is an orbit of $C_{G_x}(j)$. Hence, $2^{r-1} - 1 \leqslant r$. Thus, $r \leqslant 3$.

If $r = 2$, $C_G(j) | S(j)$ has a regular normal subgroup. Moreover, the Sylow 2-subgroup of $C_{M^x_1}(j) | S(j)$ is semiregular on $S(j) - x$, as all involutions of $C_{M^x_1}(j)$ are of rank 1. Hence, $C_{M^x_1}(j)$ has cyclic or generalized quaternion Sylow 2-subgroup. Hence, M^x_1 has cyclic or generalized quaternion Sylow 2-subgroup, a contradiction, by the Brauer-Suzuki Theorem.

Now we obtain a final contradiction. Since $\langle j_1, j_2, j_3\rangle/\langle j\rangle$ is elementary of order 4, $C_G(j) | S(j)$ is a normal extension of A_5 or $U_3(4)$. Also, if P_1 is a Sylow 2-subgroup of $C_{M^x_1}(j_1)$ and P_2 that of $C_{M^x_2}(j_2)$, $P_1 \times P_2$ is semiregular on $S(j) - x$, as no involution of N^x of rank $\leqslant 2$ fixes points on $X - x$.

But then by the structure of the Sylow 2-subgroups of A_5 and $U_3(4)$, P_i is cyclic or generalized quaternion. Again, a contradiction results, as M^x_1 has cyclic or generalized quaternion Sylow 2-subgroup.

6. As in the last section, G is a doubly-transitive group on a set X, $N^x \lhd G_x$, and $N^x = M^x_1 \times \ldots \times M^x_k$ with the M^x_i isomorphic simple groups.

In this section we assume $M_i^x \cap M_j^y = 1$ for all $x, y \in X$, $x \neq y$, and all i, j. By the last section $|(M_i^x)_y|$ is even for some $x, y \in X$, $x \neq y$, and some i. Again, we derive a contradiction if $k \geqslant 2$.

LEMMA 6.1. *If* $f \in M_i^x$, $f \neq 1$, *then* $C_G(f) \subseteq G_x$.

PROOF. If $g \in C_G(f)$, $f \in M_i^x \cap M_j^{g(x)}$. As $f \neq 1$, it follows that $g(x) = x$.

We choose $M_1^x, \ldots, M_r^x$ so that $(M_1^x)_y \neq 1, \ldots, (M_r^x)_y \neq 1$, and $(M_{r+1}^x)_y = 1, \ldots, (M_k^x)_y = 1$. Likewise, we choose $M_1^y, \ldots, M_r^y$ so that $(M_1^y)_x \neq 1, \ldots, (M_r^y)_x \neq 1$, and $(M_{r+1}^y)_x = 1, \ldots, (M_k^y)_x = 1$. We let $H^i = (M_i^y)_x$ and $L^j = (M_j^x)_y$.

Since H^i normalizes $M_1^x \times \ldots \times M_r^x$ and permutes the factors $M_1^x, \ldots, M_r^x$, H^i normalizes $L^1 \times \ldots \times L^r$ and permutes the factors $L^1, \ldots, L^r$. Likewise, L^j normalizes $H^1 \times \ldots \times H^r$ and permutes the factors $H^1, \ldots, H^r$.

We set

$$H_j^i = N_{H^i}(M_j^x) = N_{H^i}(L^j) \quad \text{and} \quad L_i^j = N_{L^j}(M_i^y) = N_{L^j}(H^i).$$

Thus, H_j^i is the subgroup of H^i fixing L^j, and vice versa.

Our first goal is to show that $H^i = H_j^i$ and $L^j = L_i^j$, i.e., that H^i's and L^j's normalize each other.

LEMMA 6.2. *If* $f \in H^i$ *and* $|f|$ *is prime, then for any* j, $f \in H_j^i$.

PROOF. We may take $j = 1$. Then if f does not fix M_1^x, M_1^x has an orbit $M_1^x, \ldots, M_p^x$ under f, where $p = |f|$. Then, f centralizes a diagonal D of $M_1^x \times \ldots \times M_p^x$. By Lemma 6.1, $D \subseteq N_y^x$. Therefore, $\pi_1(N_y^x) = M_1^x$. But $(M_1^x)_y \lhd \pi_1(N_y^x)$. Since $(M_1^x)_y \neq 1$ and M_1^x is simple $(M_1^x)_y = M_1^x$, contradicting Proposition 4.

Thus, f fixes M_j^x and $f \in H_j^i$.

It follows also from Lemma 6.2 that $H_j^i \neq 1$ and $L_i^j \neq 1$.

LEMMA 6.3. $[H_j^i, L_i^j] = 1$.

PROOF. $[H_j^i, L^j] \subseteq L^j$ and $[H^i, L_i^j] \subseteq H^i$. As $H^i \cap L^j = 1$, $[H_j^i, L_i^j] = 1$.

LEMMA 6.4. *If* $f \in H_j^i$, $f \neq 1$, *then* $C_{L^j}(f) = L_i^j$.

PROOF. By Lemma 6.3, $L_i^j \subseteq C_{L^j}(f)$. Suppose $g \in L^j$ and g centralizes f. By Lemma 6.1, g fixes y. Since g centralizes f and normalizes N^y, g fixes that factor of N^y which contains f, namely M_i^y. So $g \in L_i^j$.

Of course, a similar result holds with H and L interchanged.

LEMMA 6.5. *If* $f \in H_j^i$, $f \neq 1$, *then* $C_{M_j^x}(f) = L_i^j$.

PROOF. As before, $L_i^j \subseteq C_{M_j^x}(f)$. If $g \in C_{M_j^x}(f)$, then by Lemma 6.1, $g \in (M_j^x)_y$, and the rest follows by Lemma 6.4.

LEMMA 6.6. (i) $|H_j^i| = |L_i^j|$.

(ii) *For a given prime p, all Sylow p-subgroups of H_j^i and L_i^j are isomorphic.*

PROOF. In the language of §4 of part I, using Lemma 6.5, we see that we have a (H_j^i, M_j^x, L_i^j) configuration and a (L_i^j, M_i^y, H_j^i) configuration. By applying Lemma 4.2 of part I to each of the prime divisors of $|H_j^i|$ and $|L_i^j|$, (i) and (ii) follow.

LEMMA 6.7. *Suppose H^i does not fix some L^j, then*

(i) $|H^i : H_j^i| = 2$ *and H_j^i is abelian,*

(ii) *H^i is fixed by all L^k, $1 \leqslant k \leqslant r$.*

PROOF. Suppose some element of H^i moves L^j to L^k with $k \neq j$. Then some element of H^i moves L_i^j to L_i^k. Since L_i^j normalizes H^i, $L_i^k \subseteq H^i L_i^j$.

Now $H^i \cap L^j = 1$ and, by Lemma 6.4, $N_{H^i L_i^j}(L_i^j) = H_j^i L_i^j$. Thus, $L_i^k \cdot L_i^j \subseteq H_j^i \cdot L_i^j$. Since $|L_i^k| = |L_i^j| = |H_j^i|$ (by Lemma 6.6), $L_i^k \times L_i^j = H_j^i L_i^j$. Since $H_j^i \lhd L_i^k \times L_i^j$ and $H_j^i \cap L_i^k = H_j^i \cap L_i^j = 1$, H_j^i, L_i^k, L_i^j are isomorphic abelian groups.

Now if $|H^i : H_j^i| > 2$, there is some $L^l \neq L^j, L^k$, such that $L_i^l \times L_i^j \times L_i^k \subseteq H^i L_i^j$, a contradiction to the order of the normalizer of L_i^j in $H^i L_i^j$. Thus, (i) follows.

Next we prove (ii).

Take $L^l \neq L^j, L^k$. Then, $[H^i, L^j] \subseteq L^j \times L^k$ and so $[[H^i, L^j], L^l] = 1$. Now $[H^i, L_i^j] \neq 1$, as if $[H^i, L_i^j] = 1$, H^i fixes L_i^j and so H^i fixes L^j, the unique factor of $L^1 \times \ldots \times L^r$ containing L_i^j. This is contrary to the fact that H^i moves L^j. Hence, $[H^i, L_i^j] \neq 1$. Since $[H^i, L_i^j] \subseteq H^i$ and L^l centralizes $[H^i, L_i^j]$, L^l fixes H^i.

Thus, to prove (ii), it suffices to show that L^j and L^k fix H^i. So, we suppose L^j moves H^i. By (i) of this lemma, $|L^j : L_i^j| = 2$ and $|L^k : L_i^k| = 2$.

Let P be the Sylow 2-subgroup of H_j^i, Q^j a Sylow 2-subgroup of L^j normalized by P and R^j, the Sylow 2-subgroup of L_i^j, with similar definitions for Q^k, and R^k. Then $|Q^j : R^j| = |Q^k : R^k| = 2$. Also, L_i^j contains all involutions of L^j by Lemma 6.2, and as $|L^j|$ is even, $|R^j| \neq 1$. Since $|L_i^j| = |H_j^i|$, $P \neq 1$.

By Lemma 6.4, we have, in the terminology of part I, a constrained (P, Q^j, R^j) configuration and a constrained (P, Q^k, R^k) configuration. By Lemma 4.6 of part I, $\Omega_1(R^j) \subseteq Z(Q^j)$ and $\Omega_1(R^k) \subseteq Z(Q^k)$. But $H_j^i \subseteq L_i^k \times L_i^j$. So $P \subseteq R^j \times R^k$, and $\Omega_1(P) \subseteq Z(Q^k \times Q^j)$. Thus, if $x \in \Omega_1(P)$, $x \neq 1$, x centralizes Q^k, a contradiction, by Lemma 6.4.

Therefore, L^j does not move H^i.

We shall show next that all H^i's fix all the L^j's, and vice versa. Now if j is an involution interchanging $x, y \in X$, j normalizes $(H^1 \times \dots \times H^r)\cdot(L^1 \times \dots \times L^r)$ and interchanges H^i's and L^j's. Thus, if all groups H^i fix all the groups L^j, all the groups L^j fix all the groups H^i, and we obtained the desired conclusion. Thus, we may suppose, say H^1, does not fix all the L^j's, and $L^1 \cong H^1$, does not fix all the H^i's.

By Lemma 6.7, all the L^j's fix H^1 and all the H^i's fix L^1.

LEMMA 6.8. *H^1 is an abelian group.*

PROOF. Since H^1 fixes L^1, H^1 normalizes M_1^x, and, by Lemma 6.5, we have a constrained (H^1, M_1^x, L^1) configuration. By Proposition 4.9 and Proposition 4.15 of part I, as M_1^x is a simple group, H^1 is abelain.

LEMMA 6.9. *$H^1 = Z_4 \times A$, where A is an elementary abelian 2-group.*

PROOF. Suppose H^1 moves L^2 to L^3. By Lemma 6.7, $|H^1 : H_2^1| = 2$. Also, by the proof of Lemma 6.7, $L_1^2 \cong H_2^1$. As L^2 fixes H^1, $L^2 \cong H_2^1$. Thus, by Lemma 6.4, we have a constrained (L^2, H^1, H_2^1) configuration. Since $|H^1 : H_2^1| = 2$, $[L^2, H^1] \subseteq H_2^1$. By Lemma 4.17 of part I, and as H^1 is abelian, it follows that H^1 is a 2-group. By Lemma 4.6(i) of part I, H_2^1 is elementary abelian. By Lemma 6.2, H_2^1 contains all involutions of H^1, and so $H^1 = Z_4 \times A$, with A an elementary abelian 2-group.

We now obtain a final contradiction to the assumption that not all H^i's fix the L^j's by studying the constrained (H^1, M_1^x, L^1) configuration.

The following lemma, which we use again shortly, handles the case in which A of Lemma 6.9 is the identity.

LEMMA 6.10. *There is no (Z_{2^n}, G, Z_{2^n}) configuration in which G is a simple group, and $n \geqslant 1$.*

PROOF. Assume such a configuration exists and let Q be a Sylow 2-subgroup of G. Since G is simple $Q \supset Z_{2^n}$. By Lemma 4.6 of part I, Q is dihedral, cyclic, or generalized quaternion. By the Brauer-Suzuki Theorem, Q is dihedral. By Gorenstein and Walter [6], $G = PSL(2, q)$, q odd, or A_7.

We let P be the group of automorphisms of G such that the centralizer of each nonidentity element of P is R, with $P = R = Z_{2^n}$. We let j be the involution of P.

If $G = PSL(2, q)$, then j induces a field automorphism or a graph automorphism of G. In the first case, $C_G(j) = PSL(2, \sqrt{q})$, which is not a 2-group. In the second case, $C_G(j)$ is dihedral of order $q + 1$ or $q - 1$, and never cyclic if $q > 3$.

If $G = A_7$, an outer automorphism of A_7 centralizes an element of order 3.

We now begin the proof that the hypothesis of this section leads to a contradiction.

It follows from the last lemma, as M_1^x is simple, that A has order greater than 1.

Now let $T = C_{M_1^x}(L^1)$. By Lemma 4.6(i) of part I, L^1 is a Sylow 2-subgroup of its centralizer. Hence, $T = L^1 \times O(T)$. Since H^1 is noncyclic, $O(T) = \langle C_{O(T)}(j) \mid j \in H^1 \rangle$, $j \neq 1$. Since $C_{M_1^x}(j) \subseteq L^1$ if $j \in H^1$, $j \neq 1$, $O(T) \subseteq L^1$ and $O(T) = 1$. Thus $T = L^1$.

By Lemma 4.1 of part I, $[N_{M_1^x}(L^1), H^1] \subseteq L^1$. Thus, in the terminology of part I, we have a constrained (H, K, L)-configuration of type A. By Proposition 4.26 of part I, $L^1 \lhd M_1^x$, a contradiction, or L^1 is elementary abelian, again a contradiction.

Thus, we may assume that all H^i's normalize L^j's, and vice versa.

LEMMA 6.11. *$H^1, \ldots, H^r, L^1, \ldots, L^r$ are isomorphic abelian groups.*

PROOF. By choosing an involution t which interchanges $x, y \in X$, we may number the groups so that $H^1 \cong L^1, \ldots, H^r \cong L^r$. By Lemma 6.4 we then have a constrained (H^i, M_i^x, L^i) configuration. By part I, Propositions 4.9 and 4.15, H^i and L^i are abelian. Since $H^i = H_j^i$ and $L^j = L_i^j$, and by Lemma 6.6, all Sylow subgroups of H^i and L^j are isomorphic, H^i and L^j are isomorphic, for all i and j.

LEMMA 6.12. *For all i, $1 \leqslant i \leqslant k$, $(M_i^x)_y \neq 1$. (In other words, $r = k$.)*

PROOF. Assume the lemma is false, and $(M_{r+1}^x)_y = \ldots = (M_k^x)_y = 1$. We study the action of $H^1 = (M_1^y)_x$ on $K = M_{r+1}^x \times \ldots \times M_k^x$. We take $\Lambda = \{r+1, \ldots, k\}$. If $f \in H^1$, we speak of the action of f on Λ to mean the action of f on the corresponding factors of K.

First we claim that H^1 acts semiregularly on Λ. It suffices to verify this if $f \in H^1$ is of prime order p. Indeed, if f fixes M_t^x, $t \in \Lambda$, as $p \mid |M_t^x|$, $C_{M_t^x}(t) \neq 1$. But then, by Lemma 6.1, $(M_t^x)_y \neq 1$, a contradiction. Thus, the claim follows.

By the hypothesis of this section, some $(M_i^x)_y$ is of even order. So by Lemma 6.11, H^1 is of even order. We claim next that H^1 is a cyclic 2-subgroup.

Take $f \in H^1$ of prime order and let $\Gamma_1, \ldots, \Gamma_s$ be the orbits of f on Λ. By Lemma 3.6, f centralizes a diagonal D_t of $\Pi_{l \in \Gamma_t} M_l^x$. By Lemma 6.1, $D_t \subseteq K_y$. Thus, if $i \in \Lambda$, $\pi_i(K_y) = M_i^x$. Then, by Lemma 3.4, there is a partition $\Delta_1, \ldots, \Delta_u$ of Λ such that K_y is a product of diagonals of $\Pi_{l \in \Delta_t} M_l^x$.

Moreover, each orbit of f on Λ is a union of sets Δ_i, where f may be any element of H^1 of prime order.

Now if it is possible, choose two elements $f, g \in H^1$, f, g of prime order, and $\langle f \rangle \neq \langle g \rangle$. Then, $\langle f \rangle$ has orbits $\Gamma_1, \ldots, \Gamma_s$ on Λ and $\langle g \rangle$ has orbits $\Gamma'_1, \ldots, \Gamma'_{s'}$. Since $\langle f, g \rangle$ is semiregular on Λ, $|\Gamma_i \cap \Gamma'_{i'}| \leqslant 1$, for all choices of Γ_i and $\Gamma'_{i'}$. Since both Γ_i and $\Gamma'_{i'}$ are the union of sets Δ_i, it follows that $|\Delta_1| = \ldots = |\Delta_u| = 1$.

But then if $l \in \Lambda$, M_l^x fixes y, a contradiction. Thus, the choice of the previous paragraph is not possible. Therefore, H^1 is a cyclic 2-group.

Next we consider the constrained (H^1, M_1^x, L^1) configuration. Since $H^1 = L^1 = Z_{2^n}$, by Lemma 6.10, we obtain a final contradiction proving Lemma 6.12.

The final contradiction of §6 will follow immediately from the next lemma.

LEMMA 6.13. *For all i and j, $F_{H^i} = F_{L^j}$.*

PROOF. Take $z \in F_{H^i}$. We show that $z \in F_{L^j}$. Now $H^i = (M_i^y)_x$ and $L^j = (M_j^x)_y$. Since all H^i's normalize L^j's, H^i normalizes L^j and fixes M_j^x. By Lemma 6.12 and double-transitivity, $(M_j^x)_z \neq 1$. By Lemma 6.11 and double-transitivity, $|(M_j^x)_z| = |(M_i^y)_x|$. Since H^i fixes z and normalizes M_j^x, H^i normalizes $(M_j^x)_z$. By Lemma 6.5, if $f \in H^i$, and $f \neq 1$, the centralizer of f on $(M_j^x)_z$ is precisely $(M_j^x)_{zy}$. Therefore, we have a $(H^i, (M_j^x)_z, (M_j^x)_{zy})$ configuration with $|H^i| = |(M_j^x)_z|$. Applying Lemma 4.2 of part I to each Sylow subgroup of H^i and $(M_j^x)_z$, it follows that $(M_j^x)_z = (M_j^x)_{zy}$. Since $|(M_j^x)_z| = |(M_j^x)_y|$, $(M_j^x)_z = (M_j^x)_y$. Therefore, $z \in F_{L^j}$.

Thus, $F_{H^i} \subseteq F_{L^j}$. Similarly, $F_{L^j} \subseteq F_{H^i}$.

We can now bring §6 to a close. Set $Y = F_{H^1}$. By Lemma 6.13, $Y = F_{L^1} = F_{L^2}$. Now L^2 centralizes M_1^x and therefore M_1^x fixes the set Y. But the kernel of the restriction map $M_1^x \longrightarrow M_1^x|Y$ is not 1 as $L^1 \neq 1$ and L^1 fixes all points of Y. Since M_1^x is simple, M_1^x fixes y, a contradiction.

7. In this section we complete the proof of Theorem A. Here, G is a doubly-transitive group on X, $N^x \lhd G_x$, and $N^x = M_1^x \times M_2^x \times \dots \times M_k^x$, with M_1^x isomorphic simple groups. By the last section we may assume $M_i^x \cap M_j^y \neq 1$ for some $x, y \in X$, $x \neq y$. For definiteness, we assume $M_1^x \cap M_1^y \neq 1$.

We derive a contradiction if $k \geqslant 2$.

Crucial to many of our arguments will be the counting methods of §1.

We take $s_1, \dots, s_k$ to be the lengths of the orbits of y under $M_1^x, \dots, M_k^x$ and $t_1, \dots, t_k$ to be the lengths of the orbits of x under $M_1^y, \dots, M_k^y$, respectively.

We define a graph $\mathcal{G}_x$ on $X - x$ preserved by G_x as follows: $\mathcal{G}_x = \{\{a, b\} \mid f(a) = b$ for some $f \in M_i^x$ and some $i\}$. Thus, $\{a, b\} \in \mathcal{G}_x$ if $\{a, b\}$ lies in some orbit of some M_i^x, $1 \leqslant i \leqslant k$. We let v be the valence of $\mathcal{G}_x$.

LEMMA 7.1. $v \leqslant (s_1 - 1) + (s_2 - 1) + \dots + (s_k - 1)$.

PROOF. Fix $a \in X - x$ and let $\Delta_1, \dots, \Delta_k$ be the orbits of a under $M_1^x, \dots, M_k^x$ respectively. Then $\{a, b\} \in \mathcal{G}_x$ if and only if $b \in \bigcup_{i=1}^k (\Delta_i - a)$. Since the latter set has size at most $(s_1 - 1) + \dots + (s_k - 1)$, the lemma follows.

LEMMA 7.2. *If $f \in G$, $f \neq 1$, set*

$$T(f) = \{z \mid f \in M_i^z \text{ for some } i,\ 1 \leqslant i \leqslant k\}.$$

Then $|T(f)| \leqslant v$.

PROOF. Take $a, b \in X$, $a \neq b$, such that $f(a) = b$. Set $\tau(a, b) = \{z \mid \{a, b\} \in G_z\}$. By Lemma 1.1 of §1, $|\tau(a, b)| = v$. Now if $z \in T(f)$, $f \in M_i^z$, and $f(a) = b$. Thus, $\{a, b\} \in G_z$. So $T(f) \subseteq \tau(a, b)$, and $|T(f)| \leqslant v$.

Our first step will be to show that $k = 2$. So, we first assume $k \geqslant 3$.

LEMMA 7.3. *Either* (i) $s_1 > s_2, \ldots, s_1 > s_k$, *or* (ii) $k = 3$ *and* $(M_2^x)_y = (M_3^x)_y = 1$.

PROOF. If (i) fails, we may take $s_2 \geqslant s_1$ and $s_2 \geqslant s_3, \ldots, s_2 \geqslant s_k$. Set $Y = \{z \mid M_1^x \cap M_1^y \subseteq M_i^z$ for some $i\}$. By Lemmas 7.1 and 7.2, $|Y| \leqslant v \leqslant (s_1 - 1) + \ldots + (s_k - 1)$. Clearly, $\{x, y\} \subseteq Y$. Moreover, $C_G(M_1^x \cap M_1^y)$ fixes Y.

Now take $\overline{M}_1^x = M_2^x \times \ldots \times M_k^x$. Since $[\overline{M}_1^x, M_1^x] = 1$, $\overline{M}_1^x$ fixes Y. Thus, Y contains the orbit of y under $\overline{M}_1^x$. The length of this orbit is $|\overline{M}_1^x : (\overline{M}_1^x)_y|$. Thus, $|\overline{M}_1^x : (\overline{M}_1^x)_y| < s_1 + \ldots + s_k$.

Let π be the projection of $\overline{M}_1^x$ onto $M_3^x \times \ldots \times M_k^x$. Let $T = \pi((\overline{M}_1^x)_y)$. Then, $|(\overline{M}_1^x)_y| = |T|\,|(M_2^x)_y|$. So $|\overline{M}_1^x : (\overline{M}_1^x)_y| = |M_2^x : (M_2^x)_y|\,|M_3^x \times \ldots \times M_k^x : T|$. Therefore, $s_1 + \ldots + s_k > s_2\,|M_3^x \times \ldots \times M_k^x : T|$. Since $s_2 \geqslant s_i$ for all i, $k > |M_3^x \times \ldots \times M_k^x : T|$.

If T contains no normal subgroup $\neq 1$ of $M_3^x \times \ldots \times M_k^x$, by Lemma 3.5, $k > 5^{k-2}$. Hence, $k \leqslant 2$.

So we may assume T contains M_3^x. Let L be the largest normal subgroup of $M_3^x \times \ldots \times M_k^x$ contained in T. Then, there is a subgroup H of $(\overline{M}_1^x)_y$ such that $(M_2^x)_y \subseteq H$ and $\pi(H) = L$. Let π_2 be the projection of $\overline{M}_1^x$ onto M_2^x. Let $\overline{H} = \ker(\pi_2 | H)$. Then, $\overline{H} \subseteq M_3^x \times \ldots \times M_k^x$ and $\pi(\overline{H}) = \overline{H}$. Since $\overline{H} \lhd H$ and $\pi(H) = L \lhd M_3^x \times \ldots \times M_k^x$, $\overline{H} \lhd M_3^x \times \ldots \times M_k^x$. Since also $\overline{H}$ fixes y, if $\overline{H} \neq 1$, we have a contradiction by the proposition of §4. Hence, $\overline{H} = 1$. Therefore, H is isomorphic to a subgroup of M_2^x. Since also H has homomorphic image L and L contains M_3^x, $|H| = |M_2^x|$, and H is a diagonal of M_2^x and M_3^x. Since H fixes y, it follows that $(M_2^x)_y = (M_3^x)_y = 1$.

Since T/M_3^x contains no normal subgroup of $M_3^x \times \ldots \times M_k^x/M_3^x$, by Lemma 3.5, $k > 5^{k-3}$. Hence, $k \leqslant 3$. Therefore, $k = 3$ and $(M_2^x)_y = (M_3^x)_y = 1$.

LEMMA 7.4. *Either* (i) $s_1 > s_2, \ldots, s_1 > s_k$ *and* $t_1 > t_2, \ldots, t_1 > t_k$, *or*

(ii) $k = 3$ *and* $(M_2^x)_y = (M_3^x)_y = (M_2^y)_x = (M_3^y)_x = 1$.

PROOF. Since Lemma 7.3 also applies to N^y, it suffices to show that if $k = 3$ and $(M_2^x)_y = (M_3^x)_y = 1$, then $(M_2^y)_x = (M_3^y)_x = 1$. But if $k = 3$ and

$(M_2^x)_y = (M_3^x)_y = 1$, M_2^x and M_3^x are both regular on their orbits which contain y. By double-transitivity, two of the factors of N^y are regular on their orbits which contain x. Since M_1^y is not regular on its x-orbit, as $1 \neq M_1^x \cap M_1^y \subseteq (M_1^y)_x$, M_2^y and M_3^y are regular on their x-orbits. Hence, $(M_2^y)_x = (M_3^y)_x = 1$.

LEMMA 7.5. *$M_i^x \cap M_j^y = 1$ if $(i, j) \neq (1, 1)$.*

PROOF. We use Lemma 7.4. In case (ii) of Lemma 7.4, Lemma 7.5 is clear, as $(M_i^x)_y = 1$ if $i \neq 1$ and $(M_j^y)_x = 1$ if $j \neq 1$. Hence, $s_1 > s_i$ if $i \neq 1$ and $t_1 > t_j$ if $j \neq 1$.

Now if $M_i^x \cap M_j^y \neq 1$ for $(i, j) \neq (1, 1)$, we may suppose, say $i \neq 1$. But the proof of Lemma 7.4 is based only on the fact that $M_1^x \cap M_1^y \neq 1$. Hence, if $M_i^x \cap M_j^y \neq 1$ for $i \neq 1$, by Lemma 7.4, $s_i > s_1$, a contradiction.

It follows that for each $a, b \in X$, $a \neq b$, there is a unique (i, j) such that $M_i^a \cap M_j^b \neq 1$. We set

$$B(a, b) = \{c \mid 1 \neq M_i^a \cap M_j^b \subseteq M_k^c \text{ for some } k\}.$$

LEMMA 7.6. *$\{B(a, b) \mid a, b \in X, a \neq b\}$ form a block design (with $\lambda = 1$) on X preserved by G.*

PROOF. By double-transitivity, if $M_i^a \cap M_j^b \neq 1$, then $|M_i^a \cap M_j^b| = |M_1^x \cap M_1^y|$.

To prove Lemma 7.6, it suffices to show that if $\{c, d\} \subseteq B(a, b)$, then $B(a, b) = B(c, d)$. But if $\{c, d\} \subseteq B(a, b)$, there are subscripts k and l such that $1 \neq M_i^a \cap M_j^b \subseteq M_k^c$ and $1 \neq M_i^a \cap M_j^b \subseteq M_l^d$. Hence, $1 \neq M_i^a \cap M_j^b \subseteq M_k^c \cap M_l^d$. By the previous paragraph, $M_i^a \cap M_j^b = M_k^c \cap M_l^d$. Thus, $B(a, b) = B(c, d)$.

LEMMA 7.7. *$k = 2$.*

PROOF. Suppose $k \geq 3$ and let $\mathcal{B}$ be the block design of Lemma 7.7. Let $B = B(x, y)$. Then $G_B^* \mid B$ is doubly-transitive. Let $\overline{M}_1^x = M_2^x \times \ldots \times M_k^x$. Since $\overline{M}_1^x$ centralizes $M_1^x \cap M_1^y$, $\overline{M}_1^x$ fixes the set B. Since $M_1^x \cap M_1^y$ fixes all points of B and M_1^x is simple, M_1^x does not fix the set B. Therefore, $\overline{M}_1^x$ is the largest normal subgroup of N^x which fixes B. Therefore, $\overline{M}_1^x \lhd (G_B^* \mid B)_x$. Since $k \geq 3$, $\overline{M}_1^x$ has at least two simple factors. By induction, we obtain a contradiction.

Before proceeding to the case $k = 2$, we prove some general lemmas which we shall use later on.

LEMMA 7.8. *Let G be a doubly-transitive group on a set X and K^x a normal subgroup of G_x of prime index p in G_x. Suppose also*

(i) *$|X| \equiv 1 \pmod{p}$, and*

(ii) *if $x, y \in X$, $x \neq y$, $K_y^x = K_x^y = K^x \cap K^y$. Then, there is a normal subgroup $\overline{G}$ of G such that $\overline{G}_x = K^x$.*

PROOF. Take Q a Sylow p-subgroup of G_x and set $P = Q \cap K^x$. By (i), Q is a Sylow p-subgroup of G. We use Grun's theorem [7], to show that G has a normal subgroup $\overline{G}$ of index p such that $\overline{G} \cap Q = P$.

Thus, it suffices to show that $[N_G(Q), Q] \subseteq P$ and if Q' is another Sylow p-subgroup of G, $[Q', Q'] \cap Q \subseteq P$.

Since Q fixes only the point x, $N_G(Q) = N_{G_x}(Q)$. Since $K^x \lhd G_x$ and $K^x \cap Q = P$, $[N_{G_x}(Q), Q] \subseteq P$. Likewise, if Q' is a Sylow p-subgroup of G and $Q' \subseteq G_x$, $[Q', Q'] \cap Q \subseteq P$.

So we may suppose Q' is a Sylow p-subgroup of G and $Q' \subseteq G_y$ for some $y \neq x$. Then, $[Q', Q'] \subseteq K^y$. Thus, $[Q', Q'] \cap Q \subseteq K^y_x = K^x_y$. Since $K^x \cap Q = P$, $[Q', Q'] \cap Q \subseteq P$.

From this it follows:

LEMMA 7.9. *Let G be a doubly-transitive group on X, $x \in X$, K^x a normal subgroup of G_x of prime index p in G_x such that K^x is intransitive on $X - x$.*

Then there is a normal subgroup $\overline{G}$ of G such that $\overline{G}_x = K^x$.

PROOF. Since $|G_x : K^x| = p$ and K^x is intransitive on $X - x$, if $f \in G_x$ fixes $y \in X - x$, $f \in K^x$. Thus, $K^y_x \subseteq K^x$ and so $K^x_y = K^y_x$. Since K^x has p orbits of equal length of $X - x$, $|X - x| \equiv 0 \pmod p$.

Now we begin the analysis with $k = 2$. We proceed in two cases according as G_{xy} normalizes M^x_1 or G_{xy} does not normalize M^x_1. First we treat the case in which G_{xy} normalizes M^x_1.

LEMMA 7.10. *G_{xy} normalizes M^x_1, M^x_2, M^y_1, M^y_2.*

Now set $L^x = N_{G_x}(M^x_1) = N_{G_x}(M^x_2)$. By Lemma 2.3, $L^x \subset G_x$. Thus, L^x is of index 2 in G_x. Since G_{xy} is contained in L^x, L^x has two orbits of length, say l, on $X - x$. By Lemma 7.9, there is a normal subgroup $\overline{G}$ of G such that $\overline{G}_x = L^x$.

Then, $\overline{G}$ is a rank 3 permutation group having subdegrees $1, l, l$. Since $M^x_1 \subseteq \overline{G}$, $|\overline{G}|$ is even. Therefore, l is even and both orbits of $\overline{G}_x$ are self-paired. Since $G_{xy} \subseteq \overline{G}$, for all $x, y \in X$, $x \neq y$, and since $\overline{G}$ contains elements interchanging x and y, $G^*_{\{x,y\}} \subseteq \overline{G}$.

Now M^x_1 and M^x_2 are not conjugate in $\overline{G}$. Thus, we may index M^y_1 and M^y_2 for other $y \in X - x$ so that M^x_1 and M^y_1 are conjugate in $\overline{G}$. If j is an element of $\overline{G}$ interchanging x and y, then $j(M^x_1)j^{-1} = M^y_1$. It follows that the y-orbit of M^x_1 has the same length as the x-orbit of M^y_1, which we denote by s_1. s_2 is defined similarly.

Because of our reindexing we need no longer have $M^x_1 \cap M^y_1 \neq 1$. However, still for some i, j, $M^x_i \cap M^y_j \neq 1$.

LEMMA 7.11. $M^x_1 \cap M^y_2 = M^x_2 \cap M^y_1 = 1$.

PROOF. Suppose $M_1^x \cap M_2^y \neq 1$ and set $Y = \{z \mid M_1^x \cap M_2^y \subseteq M_m^z$ for some $m\}$. By Lemma 7.2, $|Y| \leqslant (s_1 - 1) + (s_2 - 1)$. Also, $\{x, y\} \subseteq Y$.

Suppose for definiteness that $s_2 \geqslant s_1$. Since $[M_2^x, M_1^x \cap M_2^y] = 1, M_2^x$ fixes the set Y. We study the orbits of M_2^x on $Y - x$. The orbit of $y \in Y - x$ under M_2^x is of length s_2. Any other orbit M_2^x on $Y - x$ is of length $\geqslant s_1$. But as $s_1 + s_2 > (s_1 - 1) + (s_2 - 1), M_2^x$ has only one orbit on $Y - x$.

Since $[M_1^y, M_1^x \cap M_2^y] = 1, M_1^y$ fixes Y. Since M_1^y does not fix x, M_1^y moves x into $Y - x$. Thus, $\langle M_2^x, M_1^y \rangle$ is doubly-transitive on Y. Also, $\langle M_2^x, M_1^y \rangle \subseteq \overline{G}$. Thus, there is a $j \in \langle M_2^x, M_1^y \rangle$ such that j interchanges x and y, and it follows that $j(M_1^y)j^{-1} = M_1^x$. Therefore, M_1^x fixes Y, a contradiction, as $M_1^x \cap M_2^y$ fixes all points of Y and M_1^x is simple.

Now by hypothesis $M_i^x \cap M_j^y \neq 1$ for some i, j. If both $M_1^x \cap M_1^y \neq 1$ and $M_2^x \cap M_2^y \neq 1$, if necessary, renumber M_1^x and M_2^x so that $s_2 \geqslant s_1$. Otherwise, choose the numbering so that $M_1^x \cap M_1^y \neq 1$ and $M_2^x \cap M_2^y = 1$.

Now set $B = \{z \mid M_1^x \cap M_1^y \subseteq M_i^z$ for some $i\}$.

LEMMA 7.12. *$\{g(B) \mid g \in G\}$ form a block design on X (with $\lambda = 1$) preserved by G. We call this block design $\mathcal{B}$.*

PROOF. If $M_2^x \cap M_2^y = 1$, the proof of Lemma 7.6 is valid. So we may suppose that $M_2^x \cap M_2^y \neq 1$. By our choice of numbering, $s_2 \geqslant s_1$.

Since $[M_2^x, M_1^x \cap M_1^y] = 1, M_2^x$ fixes B. By Lemmas 7.1 and 7.2, $|B| \leqslant (s_1 - 1) + (s_2 - 1)$. Also, $\{x, y\} \subseteq B$. Studying the orbits of M_2^x on $B - x$, it follows, as in Lemma 7.11, that M_2^y is transitive on $B - x$, and that $\langle M_2^x, M_2^y \rangle$ is doubly-transitive on B.

Since G_{xy} normalizes M_1^x and M_1^y, $M_1^x \cap M_1^y \lhd G_{xy}$. Therefore, G_{xy} fixes B.

Now since $G_B^* \mid B$ is doubly-transitive and G_{xy} fixes B, it is readily verified that the translates of B under the action of G form a block design preserved by G.

Take $\Delta(x, y)$ the orbit of y under L^x. If $z \in x - \Delta(x, y)$, $\Delta(x, z)$ is the orbit of z under L^x.

LEMMA 7.13. *If $C \in \mathcal{B}$ and $x \in C$, either* (i) *$C \subseteq \Delta(x, y)$ and M_2^x fixes C, or* (ii) *$C \subseteq \Delta(x, z)$ and M_1^x fixes C.*

PROOF. First take $C = B$, the block containing x and y. It is clear from the definition of B that M_2^x fixes B. Moreover, M_1^x does not fix B as M_1^x is simple and $M_1^x \cap M_1^y$ fixes all points of B. Therefore, M_2^x is the largest normal subgroup of N^x which fixes B. Therefore, $M_2^x \lhd (G_B^*)_x$. Therefore, $(G_B^*)_x \subseteq L^x$. Since also $(G_B^*)_x$ is transitive on $B - x$, $B \subseteq \Delta(x, y)$.

Now if C is any block of $\mathcal{B}$ containing x, $f(B) = C$ for some $f \in G_x$. If

$f \in L^x$, $C \subseteq \Delta(x, y)$ and M_2^x fixes C. If $f \in G_x - L^x$, $C \subseteq \Delta(x, z)$ and M_1^x fixes C.

We are now in a position to obtain a final contradiction to our assumption that G_{xy} normalizes M_1^x. Recall $|\Delta(x, y)| = 1 + l$ and $|X| = 1 + 2l$. Set $|B| = 1 + r$.

Take $B = \{x_1, \ldots, x_{r+1}\}$. Then there is a $y_i \in X - x_i$ such that $B \subseteq \Delta(x_i, y_i)$, by Lemma 7.13. Take $z_i \in X - \Delta(x_i, y_i)$.

Now for each $x_i \in B$, there is an index t such that $M_1^x \cap M_1^y \subseteq M_t^{x_i}$. By Lemma 7.13, $M_t^{x_i}$ fixes either all the blocks of $\mathcal{B}$ containing x_i in $\Delta(x_i, y_i)$ or those in $\Delta(x_i, z_i)$. Since $B \subseteq \Delta(x_i, y_i)$ and $M_1^x \cap M_1^y$ fixes all points of B, $M_t^{x_i}$ does not fix B. Therefore, $M_t^{x_i}$ fixes all blocks contained in $\Delta(x_i, z_i)$. Therefore, $M_1^x \cap M_1^y$ fixes all the blocks of $\mathcal{B}$ which contain x_i and lie in $\Delta(x_i, z_i)$. Moreover, if C is such a block, as $M_t^{x_i}$ fixes C, $M_1^x \cap M_1^y$ has a nontrivial orbit on C.

Since each $\Delta(x_i, z_i)$ contain l/r blocks which contain x_i and since there are $r + 1$ choices for $x_i \in B$, we obtain a family $\mathcal{C}$ of $(1 + r)l/r$ distinct blocks such that $M_1^x \cap M_1^y$ fixes each $C \in \mathcal{C}$ and $M_1^x \cap M_1^y$ has a nontrivial orbit on C.

Since no two blocks of $\mathcal{B}$ have in common a pair of points, no two blocks of $\mathcal{C}$ have in common a nontrivial orbit of $M_1^x \cap M_1^y$.

Then, if Y is the union of the nontrivial orbits of $M_1^x \cap M_1^y$, $|Y| \geq 2(1 + r)l/r$.

Since $M_1^x \cap M_1^y$ also fixes the $1 + r$ points of B, it follows that $(1 + r) + 2(1 + r)l/r \leq |X| = 1 + 2l$. Therefore, $1 + r + 2l \leq 1 + 2l$, a contradiction.

Thus, in the remainder of this section we may assume that G_{xy} does not normalize M_1^x.

Again, we set $L^x = N_{G_x}(M_1^x) = N_{G_x}(M_2^x)$. Since $G_x = L^x \cdot G_{xy}$, it follows that L^x is transitive on $X - x$.

We suppose that $M_1^x \cap M_1^y \neq 1$. We take $\Delta_1(x, y)$ to be the union of x and the orbit of y under M_1^x and $\Delta_1(y, x)$ to be the union of y and the orbit of x under M_1^y. We define $\Delta_2(x, y)$ and $\Delta_2(y, x)$ analogously.

Since L^x is transitive on $X - x$, all orbits of M_1^x on $X - x$ are of the same length s. Since M_1^x and M_2^x are conjugate in G_x, the same applies to M_2^x. Thus, $|\Delta_1(x, y)| = |\Delta_2(x, y)| = |\Delta_1(y, x)| = |\Delta_2(y, x)| = s + 1$.

LEMMA 7.14. *Let $f \in M_i^x \cap M_j^y$, $f \neq 1$. Set $T(f) = \{z \mid f \in M_k^z$ for some $k\}$. Writing $N^x = M_i^x \times M_{i'}^x$ and $N^y = M_j^y \times M_{j'}^y$, we have $T(f) = \Delta_{i'}(x, y) = \Delta_{j'}(y, x)$.*

PROOF. By Lemmas 7.1 and 7.2, $|T(f)| \leq 2s - 2$. Also, $\{x, y\} \subseteq T(f)$, and $M_{i'}^x$ fixes $T(f)$, as $M_{i'}^x$ centralizes $M_i^x \cap M_j^y$. Thus, $\Delta_{i'}(x, y) \subseteq T(f)$. By the forgoing inequality on $|T(f)|$ and since each orbit of $M_{i'}^y$ on $T(f) - x$ is of length s, it follows that $\Delta_{i'}(x, y) = T(f)$. Similarly, $\Delta_{j'}(y, x) = T(f)$.

LEMMA 7.15. (i) $|M_1^x \cap M_1^y| = |M_2^x \cap M_2^y|$; (ii) $M_1^x \cap M_2^y = M_2^x \cap M_1^y = 1$.

PROOF. Since $M_1^x \cap M_1^y \neq 1$, by Lemma 7.14, $\Delta_2(x, y) = \Delta_2(y, x)$. If also $M_1^x \cap M_2^y \neq 1$, then $\Delta_2(x, y) = \Delta_1(y, x)$, again by Lemma 7.14. Thus, $\Delta_1(y, x) = \Delta_2(y, x)$. Since M_2^y is transitive on $\Delta_2(y, x)$ and M_1^y fixes $\Delta_2(y, x)$, while centralizing M_2^y, M_1^y is semiregular on $\Delta_2(y, x)$, a contradiction as $1 \neq M_1^y \cap M_1^x \subseteq (M_1^y)_x$.

Thus $M_1^x \cap M_2^y = M_2^x \cap M_1^y = 1$.

Now take $t \in G_{xy}$ such that $t(M_1^x)t^{-1} = M_2^x$. Then, $t(M_1^x \cap M_1^y)t^{-1} = M_2^x \cap t(M_1^y)t^{-1}$. Since $M_2^x \cap M_1^y = 1$, $tM_1^yt^{-1} = M_2^y$. Thus, $|M_1^x \cap M_1^y| = |M_2^y \cap M_2^y|$.

LEMMA 7.16. (i) $L_y^x = L_x^y = L^x \cap L^y$; (ii) $|X|$ *is even and* s *is odd.*

PROOF. (i) Since L_y^x normalizes M_1^x, $M_1^x \cap M_1^y \neq 1$, and $M_1^x \cap M_2^y = 1$, L_y^x normalizes M_1^y. Thus, $L_y^x \subseteq L^y$. So $L_y^x = L_x^y = L^x \cap L^y$.

(ii) Now if $|X|$ is odd, by Lemma 7.8, there is a normal subgroup $\overline{G}$ of G such that $\overline{G}_x = L^x$. Since L^x is transitive on $X - x$, $\overline{G}$ is doubly-transitive on X. On the other hand, $M_1^x \lhd \overline{G}_x$, a contradiction by Lemma 2.3.

Thus, $|X|$ is even. As $s \mid |X - x|$, s is odd.

LEMMA 7.17. $(M_1^x)_y = M_1^x \cap M_1^y = (M_1^y)_x$.

PROOF. Set $B = \Delta_2(x, y) = \Delta_2(y, x)$ (using Lemma 7.14). Thus, $\langle M_2^x, M_2^y\rangle$ is doubly-transitive on B. Now $B = \{z \mid M_1^x \cap M_1^y \subseteq M_i^z\}$, by Lemma 7.14. By Lemma 7.16(i), $(M_1^x)_y$ normalizes M_1^y. Therefore, $M_1^x \cap M_1^y \lhd (M_1^x)_y$. Thus, also $(M_1^x)_y$ fixes B.

Since M_2^x is transitive on $B - x$, and $(M_1^x)_y$ fixes $y \in B - x$ and centralizes M_2^x, $(M_1^x)_y$ fixes all points of B.

Now $(M_1^x)_y$ normalizes M_2^y (as it normalizes M_1^y). Therefore, $(M_2^y)\cdot((M_1^x)_y)$ is a group which fixes the set B. The subgroup of $M_2^y \cdot (M_1^x)_y$ fixing all points of B is precisely $(M_1^x)_y$. Therefore, M_2^y normalizes $(M_1^x)_y$. Since M_2^y does not fix x, $(M_1^x)_y \subseteq M_1^x \cap M_1^{x'}$ for some $x' \neq x$ and some i. But by Lemma 7.15, $|M_1^x \cap M_1^y| = |M_1^x \cap M_i^{x'}|$. Therefore, $|(M_1^x)_y| \leqslant |M_1^x \cap M_1^y|$. Since also $M_1^x \cap M_1^y \subseteq (M_1^x)_y$, $M_1^x \cap M_1^y = (M_1^x)_y$.

LEMMA 7.18. *Let* B *be the fixed point set of* $(M_1^x)_y$. *Then* (i) $B = \Delta_2(x, y) = \Delta_2(y, x)$, *and* (ii) $(M_1^x)_y$ *is semiregular on* $X - B$.

PROOF. Suppose $f \in (M_1^x)_y$, $f \neq 1$, and suppose f fixes z. We show $z \in \Delta_2(x, y)$, proving both (i) and (ii).

By Lemma 7.15 and double-transitivity, either $M_1^x \cap M_1^z \neq 1$ and $M_1^x \cap M_2^z = 1$ or $M_1^x \cap M_1^z = 1$ and $M_1^x \cap M_2^z \neq 1$. So, for definiteness, say $M_1^x \cap M_1^z \neq 1$. By Lemma 7.17 and double-transitivity, $(M_1^x)_z = (M_1^z)_x$. Therefore, $f \in M_1^z$. Since

$f \in M_1^x \cap M_1^y$, it follows from Lemma 7.14, that $z \in \Delta_2(x, y) = \Delta_2(y, x)$.

LEMMA 7.19. (i) *M_1^x is $SL(2, q)$, $Sz(q)$, $U_3(q)$, with $q = 2^\alpha$.*

(ii) *$(M_1^x)_y$ is a Sylow 2-subgroup of M_1^x.*

PROOF. Now $|M_1^x : (M_1^x)_y| = s$ is odd, by Lemma 7.16 and $(M_1^x)_y$ is semi-regular on the complement of its fixed point set in $\Delta_1(x, y)$.

Set $H = N_{M_1^x}((M_1^x)_y)$. If T is a subgroup of $(M_1^x)_y$ and $T \neq 1$, then $(M_1^x)_y$ and T have the same fixed point set. Therefore, $N_{M_1^x}(T) \subseteq H$.

In particular, as s is odd, $(M_1^x)_y$ contains a Sylow 2-subgroup P of M_1^x, and $N_{M_1^x}(P) \subseteq H$. Also, if j is an involution of H, $j \in (M_1^x)_y$ (as $(M_1^x)_y \lhd H$ and $|H : (M_1^x)_y|$ is odd), and $C_{M_1^x}(j) \subseteq H$. Thus, H is a strongly embedded subgroup of M_1^x.

By Bender's Theorem [1], and since M_1^x is simple, M_1^x is $SL(2, q)$, $Sz(q)$, $U_3(q)$, with $q = 2^\alpha > 2$. Moreover, H is a Sylow 2-normalizer in M_1^x.

Next we claim: $P = (M_1^x)_y$.

By the structure of $SL(2, q)$, $Sz(q)$, and $U_3(q)$, it follows that if $a \in N_{M_1^x}(P)$, $a \neq 1$, and $|a|$ is odd, then $M_1^x = \langle P, N_{M_1^x}(\langle a \rangle)\rangle$. Thus, if $(M_1^x)_y$ contains an element $a \neq 1$ of odd order, P and $N_{M_1^x}(\langle a \rangle)$ are contained in H and $H = M_1^x$, a contradiction.

LEMMA 7.20. (i) *If $M_1^x = SL(2, q)$, $|\Delta(x, y) - x| = (1 + q)(q - 1)$ and $|X - x| = (1 + q)^2(q - 1)$.*

(ii) *If $M_1^x = Sz(q)$, $|\Delta(x, y) - x| = (1 + q^2)(q - 1)$ and $|X - x| = (1 + q^2)^2(q - 1)$.*

(iii) *If $M_1^x = U_3(q)$, $|\Delta(x, y) - x| = (1 + q^3)(q^2 - 1)$ and $|X - x| = (1 + q^3)^2(q^2 - 1)$.*

PROOF. Now all orbits of M_1^x on $X - x$ are of length s. Thus, each point $y \in X - x$ is fixed by a unique Sylow 2-subgroup of M_1^x.

By Lemma 7.18, $\Delta_2(x, y) - x$ is the fixed point set of a Sylow 2-subgroup of M_1^x. By Lemma 7.19, the size of $\Delta(x, y) - x$ is as given. Thus, $|X - x|$ is $|\Delta_1(x, y) - x|$ times the number of Sylow 2-subgroups of M_1^x.

LEMMA 7.21. *The family of sets $\{g(\Delta_1(x, y)) \mid g \in G\}$ form a (b, v, r, k, λ)-design on X with $\lambda = 2$.*

PROOF. We must show that each 2 element subset of X belongs to exactly two of the sets $g(\Delta_1(x, y))$. It suffices to show that if $\{x, y\} \subseteq g(\Delta_1(x, y))$, then $g(\Delta_1(x, y)) = \Delta_1(x, y)$ or $\Delta_2(x, y)$.

Then, $g(a) = x$ and $g(b) = y$ for $\{a, b\} \subseteq \Delta_1(x, y)$. Since $\langle M_2^x, M_2^y\rangle$ is doubly-transitive on $\Delta_1(x, y)$, there is an h which fixes $\Delta_1(x, y)$ such that $h(x) = a$ and $h(y) = b$.

Then $(g \cdot h)(\Delta_1(x, y)) = g(\Delta_1(x, y))$ and $g \cdot h \in G_{xy}$.

Now $|G_{xy} : L_y^x| = 2$ and L_y^x fixes $\Delta_1(x, y)$. Thus, if $gh \in L_y^x$, $g(\Delta_1(x, y)) = \Delta_1(x, y)$ and if $g \cdot h \in G_{xy} - L_y^x$, $g(\Delta_1(x, y)) = \Delta_2(x, y)$.

We now obtain a final contradiction, and complete the proof of Theorem A. $v = |X|$ and $k = |\Delta(x, y)|$. Then b is the number of sets $\{g(\Delta_i(x, y))\}$ and satisfies $b = 2v(v - 1)/k(k - 1)$.

Therefore, $2v(v - 1) \equiv 0 \pmod{k(k - 1)}$.

Since $k - 1$ divides $v - 1$, $2v((v - 1)/(k - 1)) \equiv 0 \pmod{k}$. Now $(v - 1)/(k - 1)$ is the number of Sylow 2-subgroups of M_1^x and $k - 1$ is the number of cosets of a Sylow 2-subgroup of M_1^x, so $(v - 1)/(k - 1)$ is relatively prime to k, and so $2v \equiv 0 \pmod{k}$.

In case (i), $v = 1 + (1 + q)^2(q - 1)$ and $k = 1 + (1 + q)(q - 1)$. Thus, $2(1 + (1 + q)^2(q - 1)) \equiv 0 \pmod{q^2}$. Thus, $2(q^3 + q^2 - q) \equiv 0 \pmod{q^2}$. So $q/2$, in contradiction to the simplicity of M_1^x.

In case (ii), $2(1 + (1 + q^2)^2(q - 1)) \equiv 0 \pmod{((1 + q^2)(q - 1) + 1)}$, and as $(1 + q^2)(q - 1) + 1 = q^3 - q^2 + q$, $2(1 + (1 + q^2)^2(q - 1)) \equiv 0 \pmod{(q^2 - q + 1)}$. Since the modulus is odd and $1 + q^2 \equiv q \pmod{(q^2 - q + 1)}$, $1 + q^2(q - 1) \equiv 0 \pmod{(q^2 - q + 1)}$. Then, $q^2 - 2q + 2 \equiv 0 \pmod{(q^2 - q + 1)}$. So $q - 1 \equiv 0 \pmod{(q^2 - q + 1)}$. It follows that $q^2 - q + 1 \leqslant q - 1$, or $q^2 \leqslant 2(q - 1)$, a contradiction.

In case (iii), a similar procedure works, completing the proof of Theorem A.

8. In this section we prove Theorem B. First, however, we must bound the size of the fixed point sets of elements of a doubly-transitive group under certain conditions.

PROPOSITION 8. *Let G be a doubly-transitive group on X and suppose G_x admits a system of imprimitivity on $X - x$ having imprimitive block Δ. Suppose $|X| = n$ and $|\Delta| = s$. Suppose no nonidentity element of G fixes all points of $x \cup \Delta$.*

Then, if $g \in G$, $g \neq 1$, g fixes at most $s\sqrt{n/2}$ points of X.

PROOF. Let Y be the fixed point set of g, and suppose $t = |Y|$. We take $\Delta(x, y) = x \cup \Delta$ and obtain a predesign function for G. As usual, we take $\mathcal{G}_x$ to be a graph on $X - x$ with $\{a, b\} \in \mathcal{G}_x$ if $\Delta(x, a) = \Delta(x, b)$. Also, $\tau(a, b) = \{x \mid \{a, b\} \subseteq \Delta(x, y) - x$ for some $y \in X - x\}$. Then, $\mathcal{G}_x$ is of valence $s - 1$ and $|\tau(a, b)| = s - 1$ by Lemma 1.1.

We take Ω to be the set of ordered pairs $(x, \{a, b\})$ where

(i) g fixes x,

(ii) there is some $y \in X - x$ such that $\{a, b\} \subseteq \Delta(x, y) - x$,

(iii) $a \neq b$ and a and b lie in the same orbit of g.

Note that (ii) is equivalent to the statement that $x \in \tau(a, b)$. Also, it follows from these hypotheses that $g(\Delta(x, y)) = \Delta(x, y)$.

We bound $|Y|$ by estimating in two different ways the size of Ω. Clearly, it suffices to do this when g is of prime order p.

First we obtain a lower bound for $|\Omega|$. Since no element of G fixes all points of $\Delta(x, y)$, the number of fixed points of g on $\Delta(x, y) - x$ is at most $s - p$. Thus, for a fixed $x \in Y$, the number of sets $\Delta(x, z)$ on which g fixes a point of $\Delta(x, z) - x$ is greater than or equal to $(t - 1)/(s - p)$. Thus, the number of possibilities for $\Delta(x, y)$ in the definition of Ω is at least $t(t - 1)/(s - p)$. On each such set $\Delta(x, y)$, g has at least one orbit of length p, and so for this $\Delta(x, y)$, the number of possibilities for $\{a, b\}$ is at least $p(p - 1)/2$. Therefore,

$$(t(t-1)/(s-p))(p(p-1)/2) \leqslant |\Omega|.$$

Next we obtain an upper bound for $|\Omega|$. On $X - Y$, g has $(n - t)/p$ orbits of length p. Thus, there are at most $(n - t)/p \cdot p(p - 1)/2$ possibilities for $\{a, b\}$ in the definition of Ω. As $x \in \tau(a, b)$ if $(x, \{a, b\}) \in \Omega$, for a given $\{a, b\}$, there are at most $s - 1$ choices for x. Therefore, $|\Omega| \leqslant ((n - t)/p)(p(p - 1)/2)(s - 1)$. It follows that $t(t - 1) \leqslant (s - 1)(s - p)(n - t)/p$. Since $p \geqslant 2$, we obtain $t(t - 1) \leqslant (s - 1)(s - 2)(n - t)/2$.

It follows from this that $t^2 \leqslant s^2 n/2$. For if not, $s^2 n < 2t + (s-1)(s-2) \cdot (n - t)$. Then, $(3s - 2)n < (2 - (s - 1)(s - 2))t$. If $s \geqslant 3$, we have the contradiction, $(3s - 2)n \leqslant 0$. If $s = 1$ or 2, it follows from the fact that no element $\neq 1$ of G fixes all points of $x \in \Delta$ that $G_{xy} = 1$ and so $t \leqslant 1$. Thus, $t \leqslant s\sqrt{n/2}$.

REMARK. Later we shall use the inequality $t(t - 1) \leqslant (s - 1)(s - 2)(n - t)/2$.

Next we prove:

THEOREM B. *Let G be a doubly-transitive group on X, $x \in X$, and suppose N^x is a normal subgroup of G_x. Suppose $|X| = n$ and that the orbits of N^x on $X - x$ are of length s. Then, one of the following holds:*

(i) *N^x is semiregular on $X - x$, or*

(ii) *G is a normal extension of $L_n(q)$, or*

(iii) *$n < 2(s - 1)^2$.*

PROOF. We take $\Delta(x, y) = x \cup \{f(y) \mid f \in N^x\}$. $\Delta(x, y)$ is the predesign function associated with the orbits of N^x on $X - x$. By hypothesis, $|\Delta(x, y)| = 1 + s$.

We define a graph G_x on $X - x$ by connecting $a, b \in X - x$ if a and b belong to the same orbit of N^x. Equivalently, a and b are connected in $\{a, b\} \subseteq \Delta(x, y) - x$ for some $y \in X - x$. G_x is a graph of valence $s - 1$. By Lemma 1.2, if $x \neq y$, $|G_x \cap G_y| = (s - 1)(s - 2)/2$.

Now if N^x is semiregular on $X - x$, (i) holds. If N^x is not semiregular on

$X - x$, but $N^x \cap N^y = 1$ if $x \neq y$, by Theorem A of part I, (ii) holds. Thus we may take $f \in N^x \cap N^y$, f of prime order p.

If N^x does not restrict faithfully to $\Delta(x, y)$, by Theorem A, (ii) holds. Thus, we may suppose N^x restricts faithfully to $\Delta(x, y)$.

We claim that no element of G fixes all points of $\Delta(x, y)$. Indeed, let $W = G_{\Delta(x,y)} \neq 1$. Then, as N^x fixes $\Delta(x, y)$, $[N^x, W] \subseteq W$. As W fixes x, $[N^x, W] = N^x$. By the last paragraph, $N^x \cap W = 1$. Thus, $[N^x, W] = 1$. Therefore, $C_{G_x}(N^x)$ is not semiregular on $X - x$. By Corollary B.3 of [8], there is a block design $\mathcal{B}$, such that N^x fixes all blocks of $\mathcal{B}$ which contain x. By Lemma 2.8 of [8], it follows that $N^x \cap N^y = 1$ if $x \neq y$, contrary to hypothesis. Thus, the claim follows.

Let t be the number of fixed points of f. By Proposition 8 of this section, $t(t - 1) \leqslant (s - 1)(s - 2)(n - t)/2$.

Since $f \in N^x \cap N^y$, all pairs of points in orbits of f lie in $\mathcal{G}_x \cap \mathcal{G}_y$. Since f has $(n - t)/p$ orbits in the complement of its fixed point set and since each orbit contains $p(p - 1)/2$ pairs of points, it follows that

$$\frac{(n-t)}{p}\frac{p(p-1)}{2} \leqslant \frac{(s-1)(s-2)}{2}.$$

Thus, $n - t \leqslant (s - 1)(s - 2)/(p - 1)$. Since $p \geqslant 2$, $n - t \leqslant (s - 1)(s - 2)$.

By the previous paragraph, $t(t - 1) \leqslant (s - 1)^2(s - 2)^2/2$. Thus, $t \leqslant (s - 1)^2/\sqrt{2}$. Thus, $n \leqslant (s - 1)^2/\sqrt{2} + (s - 1)(s - 2) < 2(s - 1)^2$.

REMARK. In case (i) and (ii) of Theorem C, no bound of n in terms of s is possible. Indeed, if $s = 2$, n can be arbitrarily large.

There are certain circumstances under which the bound of (iii) can be sharpened. We say the normal subgroup N^x of G_x is balanced if $N^x_y = N^y_x = N^x \cap N^y$. In other words, N^x is balanced if N^x is a strongly closed subgroup of G_x in G.

In all known doubly-transitive groups, if N^x is intransitive on $X - x$, either N^x is balanced or $N^x \cap N^y = 1$ if $x \neq y$.

Under the condition of balance, we can obtain stronger bounds for the degree of G.

THEOREM C. *Let G be a doubly-transitive group on X, $x \in X$, and N^x, a normal subgroup of G_x.*

Suppose that the orbits of N^x on $X - x$ are of length s and that N^x is a permutation group of rank r on each of these orbits.

Suppose that N^x is balanced and $N^x_y \neq 1$ if $y \in X - x$. Then (i) *$n \leqslant (s - 1)^2$, and* (ii) *the number of orbits of N^x on $X - x$ is less than or equal to $r(s - 1)(s - 2)/s(s - r)$.*

Before proceeding with the proof of this result, we remark that if $r < s/2$ (which is certainly true if N^x is primitive on its orbits in $X - x$), then the number of orbits of N^x on $X - x$ is less than $2r$.

If $r = 2$, then N^x is transitive on $X - x$. If $r = 3$, N^x has at most 3 orbits on $X - x$.

The proof of (i) follows the lines of Theorem B.

LEMMA 8.1. *Suppose G is doubly-transitive on X, $x \in X$, and N^x is a balanced normal subgroup of G_x. Suppose the orbits of N^x on $X - x$ are of length s.*

Let $f \in N^x$, $f \neq 1$. Then f fixes at most $s - 1$ points of X.

PROOF. Let Y be the fixed point set of f. Then if $y \in Y$, $f \in N_y^x = N_x^y$. Thus, $f \in N^y$. In the notation of Lemma 1.3, it follows that $Y \leqslant S(f)$. By Lemma 1.3, $|S(f)| \leqslant s - 1$, and $|Y| \leqslant s - 1$.

To complete the proof of (i) of Theorem C, let t be the number of fixed points $N^x \cap N^y$, if $x, y \in X$, $x \neq y$. By the proof of Theorem B, it follows that $n - t \leqslant (s - 1)(s - 2)$. Since also by the previous lemma, $t \leqslant s - 1$, it follows that $n \leqslant (s - 1)^2$.

Next we prove (ii) of Theorem C with $\Delta(x, y)$ and $\mathcal{G}_x$ defined as in the proof of Theorem B.

LEMMA 8.2. *Let $a_1, \ldots, a_r$ be positive real numbers and suppose $a_1 + \ldots + a_r = s$. Then $a_1^2 + \ldots + a_r^2 \geqslant s^2/r$.*

PROOF. Apply the Cauchy-Schwarz inequality to the vectors $a = (a_1, a_2, \ldots, a_r)$ and $b = (1, 1, \ldots, 1)$. Then $|\Sigma a_1| = s \leqslant (a_1^2 + \ldots + a_r^2)^{1/2} r^{1/2}$. Thus, $a_1^2 + \ldots + a_r^2 \geqslant s^2/r$.

LEMMA 8.3. *Let G be a group having two permutation representations of rank r with respective point stabilizers H_1 and H_2.*

Then H_2 has no more than r orbits on cosets of H_1.

PROOF. Let χ_1 and χ_2 be the respective permutation characters. Since G is of rank r, $(\chi_1, \chi_1) = (\chi_2, \chi_2) = r$. Then the number of orbits of H_2 on cosets of H_1 is

$$(\chi_1 \chi_2, 1) = (\chi_1, \chi_2) \leqslant (\chi_1, \chi_1)^{1/2} (\chi_2, \chi_2)^{1/2} = r.$$

We now prove (ii). Since $N_y^x = N_x^y$, all orbits of N_y^x lie in the sets $\Delta(y, z)$, for some $z \in X - y$. Thus, any two element subset of an orbit of N_y^x lies in $\mathcal{G}_x \cap \mathcal{G}_y$.

We estimate the number of two element subsets of X contained in the orbits of N_y^x contained in some $\Delta(x, z)$, $z \in X - x$. Let the lengths of the orbits of N_y^x on $\Delta(x, z) - x$ be $a_1, \ldots, a_t$. Then, $a_1 + \ldots + a_t = s$. By

Lemma 4.2, $a_1^2 + \ldots + a_t^2 \geq s^2/t$. By Lemma 4.3, $t \leq r$. Thus, $a_1^2 + \ldots + a_t^2 \geq s^2/r$. Thus, $a_1(a_1 - 1)/2 + \ldots + a_t(a_t - 1)/2 \geq 1/2(s^2/r - s)$. Thus, the number of two element subsets of $\Delta(x, z)$ contained in some orbit of N_y^x is greater than or equal to $1/2(s^2/r - s)$.

Since the number of choices for $\Delta(x, z)$ is m, with m the number of orbits of N^x on $X - x$, it follows that $|G_x \cap G_y| \geq m/2(s^2/r - s)$.

But by Lemma 3.2, $|G_x \cap G_y| = (s-1)(s-2)/2$. Thus, $m \leq r(s-1)(s-2)/s(s-r)$, proving (ii).

REMARK. The bounds of Theorem C do not seem to be susceptible to much improvement. Indeed, if $q = 2^m$, there is a doubly-transitive group of degree q^2 and order $q^2(q^2 - 1) \cdot 2$, such that G_x has a normal subgroup N^x of order $(q + 1) \cdot 2$. The nontrivial orbits of N^x are of length $(q + 1)$ and of rank $q/2 + 1$. Thus, in this case, $n = (s - 1)^2$ and the number of orbits of N^x is the greatest integer less than $r(s-1)(s-2)/s(s-r)$.

The crucial condition in the proof of Theorem C is that N^x be balanced. We note some conditions under which this holds.

If N_y^x is simple, then, as $N^x \cap N^y \lhd N_y^x$, either $N^x \cap N^y = 1$ or $N_y^x = N_x^y = N^x \cap N^y$. By Theorem A of part I, we cannot have $N^x \cap N^y = 1$ and N_y^x simple. Thus, $N_y^x = N_x^y$, and N^y is balanced.

If we assume N^x is the largest normal subgroup of G_x which fixes the orbits of N^x, there are other conditions under which N^x is balanced.

If, for example, N_y^x is an intravariant subgroup of N^x, then N_y^x fixes some point in each orbit of N^x on $X - x$. By Theorem B of [8], N_x^y fixes some point on each orbit of N^x on $X - x$. Thus, N_x^y fixes all orbits of N^x on $X - x$. Thus, $N_x^y \subseteq N^x$. It follows that N^x is balanced.

Also, if N_y^x is a Hall subgroup of N^x, N^x is balanced. Indeed, under these conditions, if p is a prime dividing N_y^x, a Sylow p-subgroup p of N_y^x fixes some point in each orbit of N^x. By Lemma 2.6 of [8], some Sylow p-subgroup of N_x^y fixes a point in each orbit of N^x. Since p was any prime divisor of $|N_y^x|$, it follows that N_x^y fixes all orbits of N^x. Thus, N^x is balanced.

We have proved

LEMMA 8.4. *Let G be a doubly-transitive group on X, $x, y \in X$, $x \neq y$. Suppose N^x is a normal subgroup of G_x. Suppose also that N^x is the largest normal subgroup of G_x fixing all orbits of N^x. Then N^x is balanced if*

(i) *N_y^x is an intravariant subgroup of N^x, or*

(ii) *N_y^x is a Hall subgroup of N^x.*

9. In this section we prove:

THEOREM D. *Let G be a doubly-transitive group on a set X. Suppose $x \in X$ and $N^x \lhd G_x$. Suppose $N^x \neq 1$ is doubly-transitive on each of its orbits on*

$X - x$. *Then either*

(i) *G is triply-transitive on X, or*

(ii) *G is a normal extension of $L_n(q)$, or*

(iii) $|N^x| = 2$ *and G has a regular normal subgroup.*

As usual, we let $\Delta(x, y)$ be the predesign function associated with the orbits of N^x. We suppose $|\Delta(x, y)| = 1 + s$.

LEMMA 9.1. *Let r^k be the highest power of the prime r dividing $s - 1$. Let R be a Sylow r-subgroup of N_y^x. Then all orbits of R are of length* 1 *or of length divisible by r^k.*

PROOF. Since N^x is doubly-transitive of degree s, R is a Sylow r-subgroup of N^x. Thus, on any orbit $\Delta(x, z) - x$ of N^x on $X - x$, R fixes some point z' of $\Delta(x, z) - x$. So R is a Sylow r-subgroup of $N_{z'}^x$. Since $N_{z'}^x$ is transitive on $\Delta(x, z) - \{x, z'\}$, all other orbits of R or $\Delta(x, z) - x$ are of length divisible by r^k.

LEMMA 9.2. *N_x^y is transitive on $\Delta(x, y) - \{x, y\}$.*

PROOF. Take R' a Sylow r-subgroup of N_x^y for some prime r dividing $s - 1$. Let r^k be the highest power of r dividing $s - 1$. We claim: all orbits of R' on $\Delta(x, y) - \{x, y\}$ are of length divisible by r^k. If this is false, by Lemma 9.1, R' fixes some point $z \in \Delta(x, y) - \{x, y\}$. By Lemma 2.6 of [8], some Sylow r-subgroup R of N_y^x also fixes z. But all orbits of R on $\Delta(x, y) - \{x, y\}$ are of length divisible by r^k, as N_y^x is transitive on $\Delta(x, y) - \{x, y\}$. This contradiction proves the claim.

We now prove Theorem D.

Since all orbits of N_x^y on $X - y$ are contained in $\Delta(y, z)$ for some $z \in X - y$, it follows that $\Delta(x, y) - \{x, y\} \subseteq \Delta(y, z)$. Thus, $|(\Delta(x, y) - x) \cap (\Delta(y, z) - y)| = s - 1$.

Applying Lemma 9.2 to N_y^x instead of N_x^y, it follows that N_y^x is transitive on $\Delta(y, x) - \{x, y\}$. Thus there is a point $z' \in X - x$ such that $|(\Delta(y, x) - y) \cap (\Delta(x, z') - x)| = s - 1$.

As in §1, we define $\mathcal{G}_x$ so that $\{a, b\} \in \mathcal{G}_x$ if a and b belong to the same orbit of N^x. Then $\mathcal{G}_x$ is of valence $s - 1$, and by Lemma 1.2, if $x \neq y$, $|\mathcal{G}_x \cap \mathcal{G}_y| = (s - 1)(s - 2)/2$.

Now all pairs of points in $(\Delta(x, y) - x) \cap (\Delta(y, z) - y)$ and $(\Delta(y, x) - y) \cap (\Delta(x, z') - x)$ belong to $\mathcal{G}_x \cap \mathcal{G}_y$. Thus, if either $\Delta(x, y) \neq \Delta(x, z')$ or $\Delta(y, x) \neq \Delta(y, z)$, we have

$$(\Delta(x, y) - x) \cap (\Delta(y, z) - y) \cap (\Delta(y, x) - y) \cap (\Delta(x, z') - x) = \varnothing$$

and so $|G_x \cap G_y| \geq (s-1)(s-2)$, a contradiction if $s > 2$.

Thus, if $s \neq 2$,

$$(\Delta(x, y) - x) \cap (\Delta(y, z) - y) = (\Delta(y, x) - y) \cap (\Delta(x, z') - x).$$

Therefore, $\Delta(x, y) = \Delta(y, x)$. It follows (by Lemma 1.4 of [8]) that $\{\Delta(x, y) \mid x, y \in X, x \neq y\}$ form a block design preserved by G and that N^x fixes all blocks which contain x. Moreover, as $s > 2$, $N_y^x \neq 1$. By Theorem B of part I, if $\Delta(x, y) \subset X$, G is a normal extension of $L_n(q)$. If $\Delta(x, y) = X$, of course, G is triply-transitive.

Thus, we need only consider the case $s = 2$. Then, N^x is an elementary abelian 2-group. If $|N^x| > 2$, by Theorem A of [8], G is a normal extension of $L_n(2)$. If $|N^x| = 2$, by Glauberman's Z^*-Theorem [4], $O(G) \neq 1$, and G has a regular normal subgroup.

BIBLIOGRAPHY

1. H. Bender, *Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt,* J. Algebra **17** (1971), 527–554. MR **44** #5370.

2. R. Brauer and M. Suzuki, *On finite groups of even order whose 2-Sylow group is a quaternion group,* Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1757–1759. MR **22** #731.

3. W. Feit and J. G, Thompson, *Solvability of groups of odd order,* Pacific J. Math. **13** (1963), 755–1029. MR **29** #3538.

4. G. Glauberman, *Central elements in core-free groups,* J. Algebra **4** (1966), 403–420. MR **34** #2681.

5. ———, *On the automorphism group of a finite group having no non-identity normal subgroups of odd order,* Math. Z. **93** (1966), 154–160. MR **33** #2713.

6. D. Gorenstein and J. H. Walter, *On finite groups with dihedral Sylow 2-subgroups,* Illinois J. Math. **6** (1962), 553–593. MR **26** #188.

7. O. Grun, *Beitrage zur Gruppen theorie.* I, J. Reine Angew. Math. **174** (1935), 1–14.

8. M. E. O'Nan, *A characterization of $L_n(q)$ as a permutation group,* Math. Z. **127** (1972), 301–314. MR **47** #310.

9. ———, *Normal structure of the one-point stabilizer of a doubly-transitive permutation group.* I. Trans. Amer. Math. Soc. **214** (1975), 43–74

10. B. Shult, *On the fusion of an involution in its centralizer* (unplublished).

11. M. Aschbacher, *F-sets and permutation groups,* J. Algebra **30** (1974), 400–416.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637