

CELL-LIKE CLOSED-0-DIMENSIONAL DECOMPOSITIONS OF R^3 ARE R^4 FACTORS

BY

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ABSTRACT. It is proved that the product of a cell-like closed-0-dimensional upper semicontinuous decomposition of R^3 with a line is R^4 . This establishes at once this feature for all the various dogbone-inspired decompositions of R^3 . The proof makes use of an observation of L. Rubin that the universal cover of a wedge of circles admits a 1-1 immersion into the wedge crossed with R^1 .

Introduction. This paper is intended to rectify the proof in [R] (see Erratum) of the following result (definitions in §1).

THEOREM.⁽³⁾ *Suppose G is an upper semicontinuous decomposition of R^3 such that each nondegenerate element $G \in G$ is cell-like, and the closure of the image in the decomposition space of the nondegenerate elements of G is 0-dimensional. Then $R^3/G \times R^1 \approx R^3 \times R^1$; that is, the product of the decomposition space with the real line is homeomorphic to R^4 .*

ADDENDUM. *In the usual fashion, the decomposition $H = \{G \times t | G \in G, t \in R^1\}$ of R^4 is realizable by pseudoisotopy of R^4 . That is, there is a proper homotopy $h_s: R^4 \rightarrow R^4$, $s \in [0, 1]$, with $h_0 = \text{identity}$ and h_s a homeomorphism for $s < 1$, such that each path $\{h_s(x) | s \in [0, 1]\}$, $x \in R^4$, lies in some member of an arbitrarily small preassigned open saturated cover of R^4 and such that the point inverses of h_1 are precisely the elements of H .*

REMARK. R^3 can be replaced by any manifold-without-boundary M^3 , provided one assumes that each nondegenerate element of G has an irreducible manifold neighborhood in M^3 , thus circumventing the Poincaré conjecture. (Some conditions under which such neighborhoods exist are given in [M] and [M-R, Theorem 4].) Clearly R^1 can be replaced by any manifold N , provided $\dim \partial N \neq 0$.

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⁽³⁾This Theorem has also been obtained by W. T. Eaton and C. Pixley, in: L. C. Glaser and T. B. Rushing, eds. Geometric Topology Conference, Park City, Utah, 1974, Lecture Notes in Mathematics #438, Springer-Verlag 1974, pp. 166–194.

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This Theorem generalizes a series of results which originated with Bing's proof that his dogbone decomposition has the property stated in the Theorem [B]. Some subsequent papers are referenced in [R]. A good introduction to this decomposition problem, and to decompositions of manifolds in general, is contained in the survey series of Armentrout [A₁]–[A₃].

Our proof of the Theorem introduces a construction which we call *window building* (Lemmas 1 and 2).⁽⁴⁾ This construction provides an alternative to the familiar staircase construction introduced by Bing in the article mentioned above (cf. remark in §1 below).

Our interest in this question was aroused because it can loosely be regarded as a low dimensional version of the double suspension problem (which is equivalent to the question: Do noncombinatorial triangulations of topological manifolds exist? [G]). That is, the Theorem above implies the $n = 3$ case of the conjecture below; the $n \geq 4$ case, for X a codimension 2 contractible subpolyhedron of R^n , is strictly equivalent to the double suspension question.

CONJECTURE (cf. [A₃]). Suppose X is a cell-like compactum in R^n . Then $R^n/X \times R^1 \approx R^{n+1}$.

We recall that in general, if G is a cell-like 0-dimensional upper semicontinuous decomposition of R^n which has a defining neighborhood sequence (see below) with spines of dimension $\leq n - 3$, then $R^n/G \approx R^n$. This was observed in [A–S]. As a consequence, Theorem 2 (with $k \geq 4$) of [R] is somewhat vacuous.

1. Definitions; statement of the Proposition. A *decomposition* G of R^n is a partition of R^n into disjoint compact subsets whose union is all of R^n . The decomposition space, abusively but commonly denoted R^n/G , is given the quotient topology. The decomposition G is *upper semicontinuous* provided the quotient map $R^n \rightarrow R^n/G$ is closed.

A compact subset $X \subset R^n$ is *cell-like* if it lies cellularly in some $R^{n+k} \supset R^n$ ($k = 1$ suffices). Recall that cell-like is an intrinsic property of X not dependent on the embedding $X \hookrightarrow R^n$ [L].

A *cube-with-handles* is a compact orientable 3-manifold obtained by attaching to a 3-cell a finite number of 1-handles (this is made more explicit below). Theorem 1 of [S–A] (which observes that Theorem 1 of [L–S] holds with *cell-like* in place of *point-like*) provides a useful characterization of the decomposition G of our Theorem. It says that G can be described in the following manner: there is a *defining neighborhood sequence* of closed subsets M_1, M_2, \dots of R^3 ,

⁽⁴⁾This has no relation to W. Thurston's window building construction in *The theory of foliations of codimension greater than one*, Comm. Math. Helv. 49 (1974), 214–231. Our choice of notation was coincidental.

where, for each i , M_i is the union of a disjoint locally finite collection of cubes-with-handles such that $M_{i+1} \subset \text{int } M_i$, and furthermore for each component $M_{i,j}$ of each M_i , the inclusion map $M_{i+1} \cap M_{i,j} \hookrightarrow M_{i,j}$ is null-homotopic, and the nondegenerate components of $\bigcap_{i=1}^{\infty} M_i$ coincide with the nondegenerate elements of G .

The following Proposition is the basic result of this paper. In its application to the Theorem, first i is chosen large and then U is taken to be any one of the components $\text{int } M_{i,j}$ of $\text{int } M_i$; the cube-with-handles T is the union of the components of $M_{i+1} \cap M_{i,j}$ strung together with a thickened arc, and X is $M_{i+2} \cap M_{i,j}$.

PROPOSITION. *Suppose $X \subset \text{int } T \subset T \subset U \subset R^3$, where U is open, T is a cube-with-handles, X is compact and the inclusion maps $X \hookrightarrow T$ and $T \hookrightarrow U$ are null homotopic. Then given any $\epsilon > 0$, there exists a uniformly continuous homeomorphism $h: R^3 \times R^1 \rightarrow R^3 \times R^1$ such that*

- (1) $h = \text{identity off of } U \times R^1$, and for each $t \in R^1$, $h(U \times t) \subset U \times [t - \epsilon, t + \epsilon]$, and
- (2) for each $t \in R^1$, $\text{diam } h(X \times t) < \epsilon$.

Using the substitutions mentioned above, the Theorem follows from the Proposition as a consequence of the well-known shrinking criterion of Bing ([B, Theorem 3] and subsequently several other places). As usual, the shrinking homeomorphism h in the Proposition is actually isotopic to the identity in a well-controlled manner, hence the Addendum to the Theorem can be achieved. Since this process is widely understood, we suppress further mention.

Note of clarification. Theorem 5 of [E–G] shows that the Bing shrinking criterion works even for upper semicontinuous decompositions into compacta of complete metric spaces (previous proofs were for locally compact metric spaces). In shortening the proof for publication, the authors deleted the nontrivial verification of surjectivity of the limit shrinking map. These details have been recaptured in [M–V] with a proof simpler than the original; in particular, they show that $\lambda = 0$ suffices in condition (a) of the proof.

REMARK. One feature of this Proposition is that it minimizes the number of successive neighborhoods in the defining neighborhood sequence that one must use in order to shrink the sets $\{X \times t \mid t \in R^1\}$ small. For example, in the Bing staircase construction, which was used for the dogbone space [B, §5] and subsequently for arcs [A–C], the smaller one wanted to shrink the arc-components, the deeper one had to go into the defining neighborhood sequence. The above Proposition shows that no matter how small one wishes to shrink the sets $\{X \times t \mid t \in R^1\}$ keeping the support inside a given neighborhood $M_i \times R^1$ of the

sequence, one need only make use of the next two neighborhoods in the sequence.

In this paper \approx denotes homeomorphism, $N(A, \epsilon)$ denotes the open ϵ -neighborhood in R^3 of a subset A of R^3 , ∂ denotes manifold boundary and $\text{fr}_B A$ denotes the topological frontier of a subset A in a space B . For concreteness, a 3-dimensional cube-with-handles $T = B \cup \bigcup_{i=1}^n F_i$ consists of a 3-cell B together with n 3-cells F_1, \dots, F_n , all disjoint, each attached to ∂B as a 1-handle (Figure 1). Thus for each i , $F_i \approx [-1, 1] \times D^2$ such that $F_i \cap B = F_i \cap \partial B$ corresponds to $\{-1, 1\} \times D^2$. It is useful to extend this parametrization of F_i to a neighborhood $F_i^+ \approx [-2, 2] \times D^2$ of F_i in T , so that $F_i^+ \cap \text{cl}(T - F_i^+)$ corresponds to $\{-2, 2\} \times D^2$. We assume the F_i^+ 's are disjoint. Each $a \times D^2$ is called a *transverse slice* of F_i or F_i^+ . The subset $[a - \delta, a + \delta] \times D^2 \subset F_i^+$ has the obvious meaning whenever $-2 \leq a - \delta < a + \delta \leq 2$, and if these bounds are exceeded, we interpret $[a - \delta, a + \delta] \times D^2$ to mean $([a - \delta, a + \delta] \cap [-2, 2]) \times D^2$.

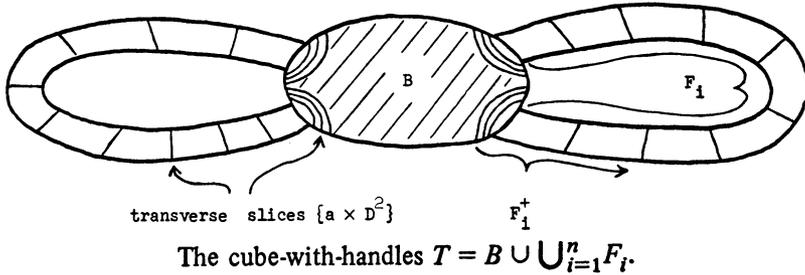
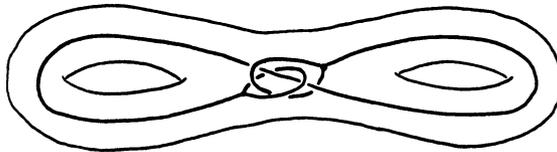


FIGURE 1

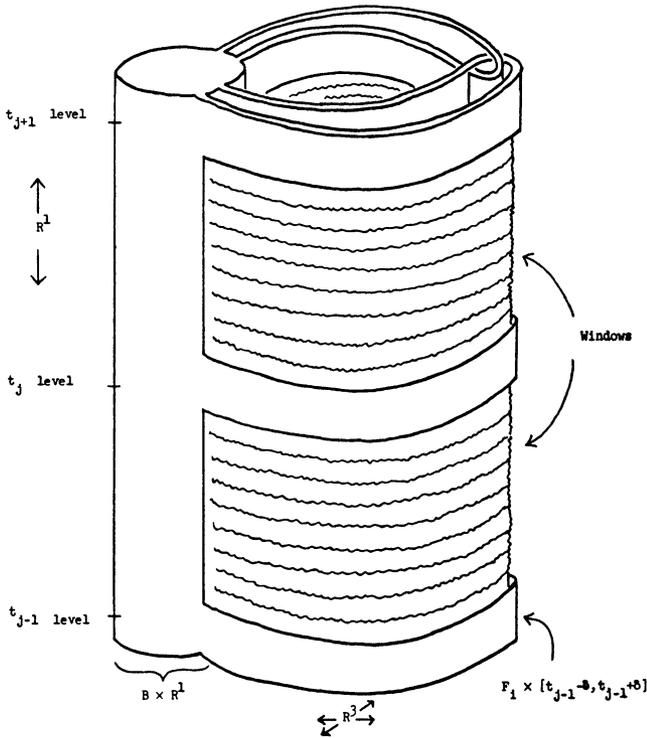


This figure depicts a cell-like set worthwhile to consider while reading the proof. Let $X \subset R^3$ be an intersection of solid figure 8's, each embedded in the previous one as shown above. Let G be the decomposition of R^3 having X as its only nondegenerate element.

FIGURE 2

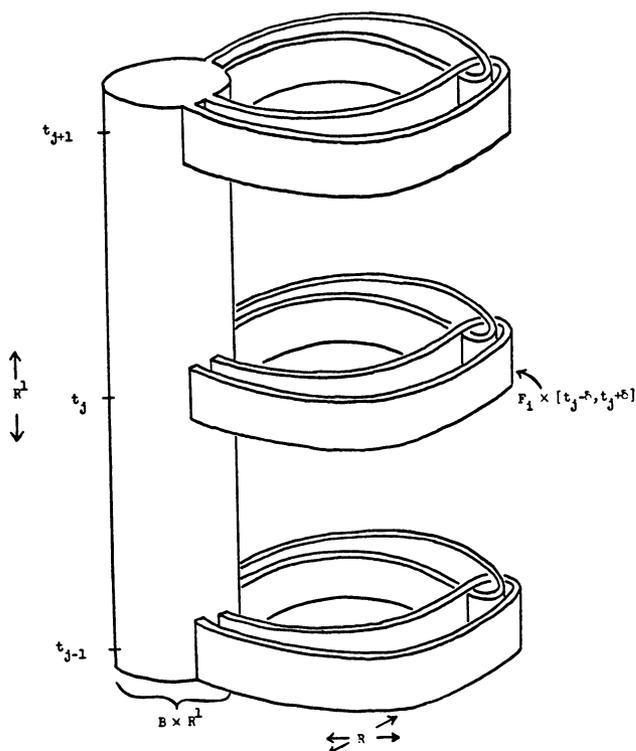
2. Window building. In the Proposition above, clearly one cannot hope to simultaneously shrink every level $T \times t$ of $T \times R^1$ small in $U \times R^1$. Nevertheless one can shrink any individual level $T \times t_0$ small since the handles of $T \times t_0$, which may link in $U \times t_0$, can be separated in $U \times R^1$ by pushing them up or down slightly. Then one can pull in the handles of $T \times t_0$, at the expense of stretching the handles of some other levels $T \times t_i, i = 1, 2, \dots, n$.

Window building can loosely be described as follows. An individual window in $T \times R^1$ is a handle of T crossed with a vertical interval. Clearly one can choose a countable number of windows in $T \times R^1$ as suggested in Figure 3, so that if the material in the windows is disregarded, then the remainder of $T \times R^1$ (call it P ; see Figure 4) can be pulled in close to $B \times R^1$, by first separating the handles as described above and then pulling them in. Of course in pulling them in they may stretch some of the material in the windows. The basic idea of window building is to first isotope $X \times R^1$ inside of $T \times R^1$, so that when done all of the images of the $X \times t$'s which intersect the windows are extremely small. (These images resemble chain mail filling up the window; see Figure 3.) Then these images will stay relatively small when the above shrinking of P is performed.



The windows in $T \times R^1$. The chain-mail-like material in the windows represents the images of the $X \times t$'s after they have been shrunk small by the homeomorphism f of Lemma 1.

FIGURE 3



The set $P \equiv T \times R^1$ minus the windows. The handles can be separated and pulled in close to $B \times R^1$.

FIGURE 4

In Lemma 1, the windows are the sets $\{F_i \times [t_j + \delta, t_{j+1} - \delta]\}$ appearing in condition 2(b).

LEMMA 1 (WINDOW BUILDING). *Suppose $T = B \cup \bigcup_{i=1}^n F_i$ is a cube-with-handles decomposed as described above, and suppose $X \subset \text{int } T$ is a compact subset such that the inclusion map is null homotopic. Then given any $\delta > 0$, there exists a uniformly continuous homeomorphism $f: R^3 \times R^1 \rightarrow R^3 \times R^1$ such that (letting $t_j = 2j - 1$ for all $j \in Z$):*

(1) *$f = \text{identity off of } T \times R^1$, and for each $j \in Z$ and each $t \in [t_j, t_{j+1}]$, $f(T \times t) \subset T \times [t_j, t_{j+1}]$, and*

(2) *for each $t \in R^1$, either*

(a) *$f(X \times t) \subset [a - \delta, a + \delta] \times D^2 \times s$ for some transverse slice $a \times D^2$ of some F_i , and some $s \in R^1$, or*

(b) *$f(X \times t) \subset T \times [t_j, t_{j+1}] - \bigcup_{i=1}^n F_i \times [t_j + \delta, t_{j+1} - \delta]$ for some $j \in Z$.*

The model form of this lemma is

LEMMA 2 (BUILDING A SINGLE WINDOW). *Suppose the data of Lemma 1. Then given any $\delta > 0$, there exists a homeomorphism $f: R^3 \times R^1 \rightarrow R^3 \times R^1$ such that*

- (1) $f = \text{identity off of } T \times [-1, 1]$, and
- (2) for each $t \in [-1, 1]$, either
 - (a) $f(X \times t) \subset [a - \delta, a + \delta] \times D^2 \times s$ for some transverse slice $a \times D^2 \subset F_1$, and some $s \in [-1, 1]$, or
 - (b) $f(X \times t) \subset T \times [-1, 1] - F_1 \times [-1 + \delta, 1 - \delta]$.

PROOF OF LEMMA 2. Without loss X is connected; if it is not one can either modify the proof below, or else can initially replace X with a small polyhedral neighborhood which can then be connected by running an arc through it.

Let $T^\# = T \cup \partial T \times [0, 1]$ denote T with an exterior collar (in $R^3 - \text{int } T$) attached. We explicitly prove the following *a priori* weaker conclusion for Lemma 2: Given $\delta > 0$ and an arbitrarily small collar neighborhood $T^\#$ of T , there exists $\lambda > 0$ and a homeomorphism $f: R^3 \times R^1 \rightarrow R^3 \times R^1$ such that

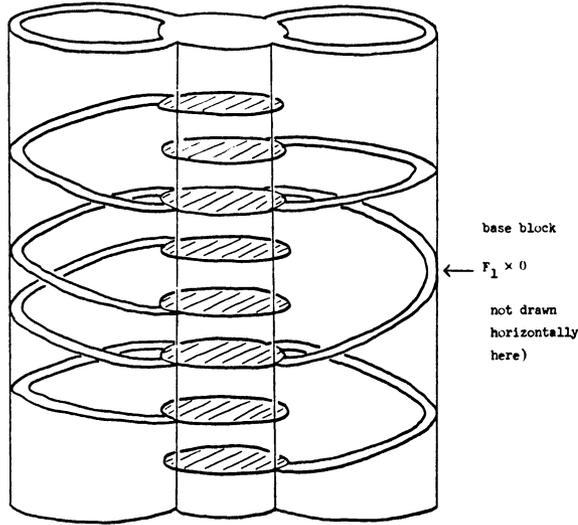
- (1') $f = \text{identity off of } T^\# \times [-1, 1]$, and
- (2') for each $t \in [-1, 1]$, either
 - (a) $f(X \times t) \subset [a - \delta, a + \delta] \times D^2 \times s$ for some transverse slice $a \times D^2 \subset F_1$ and some $s \in [-1, 1]$, or
 - (b) $f(X \times t) \subset T \times [-1, 1] - F_1 \times [-\lambda, \lambda]$.

Given any f satisfying (1') and (2'), one obtains an f satisfying (1) and (2) by vertically expanding $F_1 \times [-\lambda, \lambda]$ to coincide with $F_1 \times [-1 + \delta, 1 - \delta]$ (thus arranging (1') and (2)), and then appropriately conjugating $T^\#$ to T .

The proof of Lemma 2 is based on the key observation of Rubín in [R, §4] that there is a 1-1 immersion $\omega: \tilde{T} \rightarrow T \times (-1, 1)$ of the universal cover \tilde{T} of T into $T \times (-1, 1)$, such that $\text{proj}_T \circ \omega = \text{covering projection: } \tilde{T} \rightarrow T$. We have included a sketch of the proof in the Appendix. The image of \tilde{T} in $T \times (-1, 1)$ suggests a multitiered parking structure with its spiraling ramps (see Figure 5).

To simplify notation we abusively identify \tilde{T} with its image $\omega(\tilde{T})$ in $T \times (-1, 1)$, and we call a subset of $\tilde{T} \subset T \times (-1, 1)$ compact if its preimage in the genuine \tilde{T} is compact. Likewise $\text{fr}_{\tilde{T}} W$ of a subset $W \subset \tilde{T}$ means the frontier of W in the topology of the genuine \tilde{T} .

Assuming $X \cap F_1 \neq \emptyset$, choose a basepoint $* \in X \cap F_1$. Then $\tilde{T} \cap * \times R^1$ consists of a countable number of points, each of which determines a lifted copy $X_\nu \subset \tilde{T} \cap X \times R^1$ of X which projects homeomorphically onto X . Note that the lifts $\{X_\nu\}$ are totally ordered by their comparative heights (measured in the



The ramp $W \subset \tilde{T} \subset T \times (-1, 1)$ (not drawn with $F_1^+ \times 0$ horizontal).

FIGURE 5

R^1 coordinate), that is, for any two disjoint lifts X_ν and X_μ , either all the points of X_ν lie over the corresponding points of X_μ , or vice versa. Let $\tilde{X} \equiv \bigcup\{X_\nu\} = \tilde{T} \cap X \times R^1$. For convenience we assume that \tilde{T} is horizontal along $F_1^+ \times 0$, that is, $F_1^+ \times 0 \subset \tilde{T}$.

The cells $\{B, F_1, \dots, F_n\}$ of T lift to cells in \tilde{T} which we call *blocks*. Each X_ν intersects the same number of blocks; call this number m .

Suppose $\delta > 0$ is given. The homeomorphism f is the composition $f = \beta\alpha$ of two homeomorphisms, defined below in the reverse order of their composition.

DEFINITION OF β . Briefly, β pulls a large compact subset of blocks of \tilde{T} along the inclines of \tilde{T} towards $F_1^+ \times 0$, to arrange that the intersection of the image of any block with $F_1^+ \times 0$ is small.

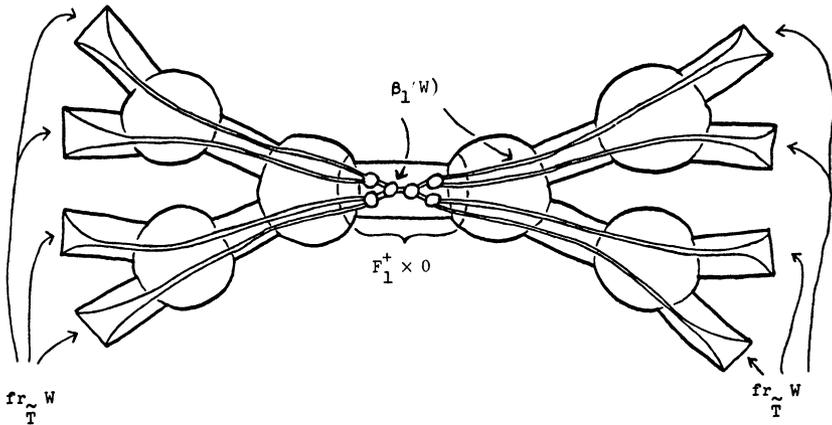
Let $p > 2m/\delta$ be an integer, and let W be the connected compact subset of \tilde{T} consisting of all blocks which can be joined to the base block $F_1^+ \times 0 \subset \tilde{T}$ by a chain of $2p + 1$ or fewer blocks. Then W resembles a many branched tree, with $\text{fr}_{\tilde{T}} W$ being the ends of the branches. We define a certain embedding $\beta_1: W \rightarrow W$. It is the tail end of an isotopic deformation of W into itself (not onto), always fixed on $\text{fr}_{\tilde{T}} W$, which pulls most of the blocks of W into $F_1^+ \times 0$, at the same time making them small. The only blocks not made small are those which intersect $\text{fr}_{\tilde{T}} W$; instead they are stretched very long to take up the slack. Figure 6 suggests the construction of β_1 . The reader can provide the details. Its required properties are

- (i) $\beta_1 = \text{identity}$ on $\text{fr}_{\tilde{T}} W$, and β_1 is (nonambiently) isotopic to $\beta_0 = \text{id}(W)$ through embeddings $\beta_r: W \rightarrow W, r \in [0, 1]$, and

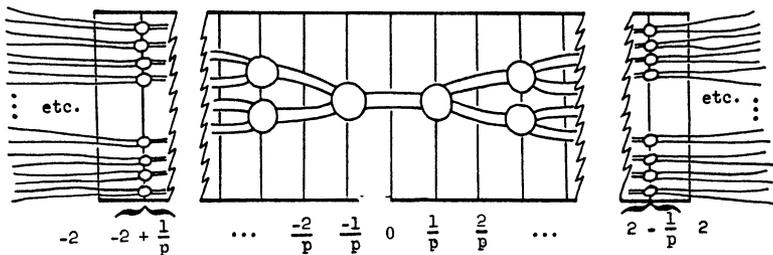
(ii) if F is any block of W , then $\beta_1(F) \cap F_1^+ \times 0 \subset [a - 1/p, a + 1/p] \times D^2 \times 0$ for some transverse slice $a \times D^2$ of F_1^+ .

Regarding W as an inclined ramp in $T \times (-1, 1)$, thicken it vertically by a small amount, to a subset denoted by (and homeomorphic to) $W \times [-2\lambda, 2\lambda]$, where $\lambda > 0$ is small and each interval $(z, t) \times [-2\lambda, 2\lambda] \subset W \times [-2\lambda, 2\lambda]$ corresponds linearly to $z \times [t - 2\lambda, t + 2\lambda] \subset T \times (-1, 1)$. Extend β_1 to an embedding $\beta_2: W \times [-2\lambda, 2\lambda] \rightarrow W \times [-2\lambda, 2\lambda]$ by applying β_1 to each of the individual levels in $W \times [-\lambda, \lambda]$, and damping to the identity using β_r in the remaining levels. Specifically, $\beta_2(W \times u) \subset W \times u$ for each $u \in [-2\lambda, 2\lambda]$, and

$$\beta_2|_{W \times u} = \begin{cases} \beta_1 & \text{if } |u| \leq \lambda, \\ \beta_{2-|u|/\lambda} & \text{if } \lambda \leq |u| \leq 2\lambda. \end{cases}$$



W (unwound) and the image $\beta_1(W)$ in W . Here $2p + 1 = 5$. In reality, each of the blocks of W , except the end blocks, intersects an even number of other blocks.



$\beta_1(W) \cap F_1^+ \times 0$, magnified (with large p value).
The definition of $\beta_1: W \rightarrow W$

FIGURE 6

Observe $\beta_2 = \text{identity}$ on

$$\text{fr}_{\tilde{T}} W \times [-2\lambda, 2\lambda] \cup W \times \{-2\lambda, 2\lambda\} = \text{fr}_{T \times (-1,1)}(W \times [-2\lambda, 2\lambda])$$

(which is *not* $\text{fr}_{T \times (-1,1)}\beta_2(W \times [-2\lambda, 2\lambda])$). Define the desired β by extending β_2 via the identity over the rest of $T \times (-1, 1)$ and over $R^3 \times R^1 - T^\# \times (-1, 1)$, and then by extending to a homeomorphism of the remaining regions $T^\# \times (-1, 1) - T \times (-1, 1) \cong T^\# \times (-1, 1) - \beta_2(T \times (-1, 1))$ using an appropriate isotopy extension construction.

DEFINITION OF α . The homeomorphism α will move points only vertically, to arrange that certain levels $X \times t$ in $X \times R^1$ go into levels $W \times u$ in the inclined $W \times [-\lambda, \lambda] \subset T \times (-1, 1)$. A certain finite number of $X \times t$ levels will go into each $W \times u$ level.

Let $\{X_k \mid -q \leq k \leq q\}$ be the finite collection consisting of all X_ν in $\{X_\nu\}$ such that $X_\nu \subset \beta_1^{-1}(F_1^+ \times 0) \subset W$. (It is natural to regard them as ordered by increasing height so that $X_0 \cap F_1 \times 0 \neq \emptyset$.) Assume $\delta < 1$. Thus

- (*) for any $X_\nu \in \{X_\nu\}$ such that $X_\nu \cap \beta_1^{-1}(F_1 \times 0) \neq \emptyset$,
 then $X_\nu = X_k$ for some k , and hence $\beta_1(X_\nu) \subset$
 $[a - \delta, a + \delta] \times D^2 \times 0$ for some transverse slice
 $a \times D^2 \times 0$ of $F_1 \times 0$.

Define $s_k \in R^1$ by $(*, s_k) = X_k \cap * \times R^1$. Let $L = \bigcup_{k=-q}^q [s_k - \lambda, s_k + \lambda] \subset (-1, 1)$. Define an embedding $\alpha_0: X \times L \rightarrow X \times (-1, 1)$ by letting

$$\alpha_0(x \times [s_k - \lambda, s_k + \lambda]) = (x, t) \times [-\lambda, \lambda] \subset W \times [-\lambda, \lambda]$$

linearly, where $x \in X$ and t is defined (as a function of x and k) by the statement $(x, t) \in X_k$. Extend α_0 over $X \times R^1$ first via the identity on $X \times ((-\infty, -1] \cup [1, \infty))$, and then linearly on the remaining intervals. Next extend via the identity over $R^3 \times R^1 - T \times (-1, 1)$, and finally extend over $(T - X) \times (-1, 1)$ by means of vertical damping in the usual fashion, thus producing α .

The homeomorphism $f = \beta\alpha$ has properties (1') and (2') listed above. We verify (2') in detail. It is equivalent to showing that for any $t \in R^1$, if $X \times t \cap f^{-1}(F_1 \times [-\lambda, \lambda]) \neq \emptyset$, then $f(X \times t) \subset [a - \delta, a + \delta] \times D^2 \times s$ for some transverse slice $a \times D^2$ of F_1 and some $s \in R^1$. By (*) above, $X \times R^1 \cap \beta_1^{-1}(F_1 \times 0) \subset \bigcup_{k=-q}^q X_k$. By definition of β from β_1 , $X \times R^1 \cap \beta^{-1}(F_1 \times [-\lambda, \lambda]) \subset \bigcup_{k=-q}^q X_k \times [-\lambda, \lambda]$, where $X_k \times [-\lambda, \lambda] \subset W \times [-\lambda, \lambda] \subset T \times (-1, 1)$ has the natural meaning. Now $\alpha(X \times L) = \bigcup_{k=-q}^q X_k \times [-\lambda, \lambda]$, so applying α^{-1} to the above inclusion yields $X \times R^1 \cap f^{-1}(F_1 \times [-\lambda, \lambda]) \subset X \times L$. So it suffices to show that each slice $X \times t \subset X \times L$ satisfies $f(X \times t) \subset [a - \delta, a + \delta] \times D^2 \times s$ for some transverse slice $a \times D^2$ of F_1 and some $s \in R^1$. But this is clear by (*) and the definitions of α and β . This completes Lemma 2.

PROOF OF LEMMA 1 FROM LEMMA 2. We give the details for the definition of $f|T \times [-1, 1]$, arranging that $f| = \text{identity on } \partial(T \times [-1, 1])$. From this, the desired f can be defined by stacking vertical translates.

The idea in constructing $f|T \times [-1, 1]$ is to first select some nonoverlapping vertical translates of the desired window blocks $\{F_i \times [-1 + \delta, 1 - \delta]\}$, and to apply Lemma 2 separately to make a window in each of these completely disjoint blocks. These windows are then translated to the original positions, and a preliminary vertical expansion is applied, to produce the desired $f|T \times [-1, 1]$.

In detail. Let $h: T \times [-1, 1] \rightarrow T \times [-1, t_n]$ (where t_n is from Lemma 1) be a homeomorphism which moves points only vertically, with the property that for each $1 \leq i \leq n$, $h|F_i^+ \times [-1 + \delta, 1 - \delta]$ is vertical translation by distance $(2i - 2)$, so that $h(F_i^+ \times [-1 + \delta, 1 - \delta]) = F_i^+ \times [t_{i-1} + \delta, t_i - \delta]$. For each separate interval $[t_{i-1}, t_i]$, let $f_i: T \times [t_{i-1}, t_i] \rightarrow T \times [t_{i-1}, t_i]$, with $f_i = \text{identity on } \partial(T \times [t_{i-1}, t_i])$, be a homeomorphism which builds a window in $F_i \times [t_{i-1} + \delta, t_i - \delta]$, as provided by Lemma 2. Let $g: T \times [-1, 1] \rightarrow T \times [-1, t_n]$ be a homeomorphism which moves points only vertically, such that $g|\partial T \times [-1, 1] = h|\partial T \times [-1, 1]$, and such that for each $t \in [-1, 1]$, there exists an $s = s(t) \in [-1, t_n]$ for which $g(X \times t) = X \times s$. Define $f|T \times [-1, 1] = h^{-1} \circ (+f_i) \circ g$ where $+f_i$ denotes the disjoint union of the f_i , $1 \leq i \leq n$. This completes Lemma 1.

3. Proof of Proposition from Lemma 1. Let $T = B \cup \bigcup_{i=1}^n F_i$ as above. By squeezing B small, we can assume that $\text{diam } B < \epsilon/4$.

The homeomorphism $h: R^3 \times R^1 \rightarrow R^3 \times R^1$ will be defined as a composition $h = g \circ f$ of three homeomorphisms. To simplify epsilomics, we will define h to satisfy properties (1_h) and (2_h) below, which imply (1) and (2) of the Proposition merely by reparametrizing vertically (e.g., replace the h below by $\tau^{-1} h \tau$, where $\tau(z, t) = (z, 24t/\epsilon)$).

(1_h) $h = \text{identity off of } U \times R^1$, and for each $t \in R^1$, $h(U \times t) \subset U \times [t - 3, t + 3]$, and

(2_h) for each $t \in R^1$, either

(a) $h(X \times t) \subset N(B, \epsilon/4) \times [t - 3, t + 3]$, or

(b) $\text{diam } h(X \times t) < \epsilon$.

DEFINITION OF g . The complete properties of the homeomorphism $g: R^3 \times R^1 \rightarrow R^3 \times R^1$ are that g is uniformly continuous and (still letting $t_j = 2j - 1$ for $j \in \mathbb{Z}$):

(1_g) $g = \text{identity off of } U \times \bigcup_{j \in \mathbb{Z}} [t_j - \frac{1}{2}, t_j + \frac{1}{2}]$, and $g = \text{identity on } B \times R^1$, and

(2_g) for each t_j , $g(T \times t_j) \subset N(B, \epsilon/8) \times R^1$.

This is routine unknotting in R^4 . Consider for example $T \times 1 \hookrightarrow U \times 1$. The problem in isotoping $T \times 1$ into $N(B, \epsilon/8) \times 1 \text{ rel } B \times 1$ always keeping the

image in $U \times 1$, is that the handles of $T \times 1$ may be geometrically linked, either with each other or self-linked. But in $U \times [1/2, 3/2]$ this linking problem vanishes, as the handles $\{F_i \times 1\}$ have 1-dimensional spines.

DEFINITIONS OF f AND r . The homeomorphism f is exactly as provided by Lemma 1, for the sets $X \subset \text{int } T$, with δ value as specified below.

Let $r_0: T \rightarrow T$ be an embedding which squeezes T close to its spine (with closeness specified below), such that $r_0(B) \subset B$ and $r_0(a \times D^2) \subset a \times D^2$ for each transverse slice $a \times D^2$ in each F_i^+ . Extend r_0 to a homeomorphism $r_0^\#$ of R^3 with support in U , and let $r = r_0^\# \times \text{id}(R^1)$.

The smallness of δ and r_0 are determined by the following requirements (t_j as above):

(1 $_\delta$) for each t_j , $g(T \times [t_j - \delta, t_j + \delta]) \subset N(B, \epsilon/4) \times R^1$, and

(2 $_\delta$) for any transverse slice $a \times D^2$ of any F_i^+ , and for any $s \in R^1$, $\text{diam } gr([a - \delta, a + \delta] \times D^2 \times s) < \epsilon$.

Then $h = grf$ satisfies properties (1 $_h$) and (2 $_h$) above. This completes the proof of the Proposition.

APPENDIX. Here we sketch the proof of Rubin's observation about covering spaces, which we used in Lemma 2.

PROPOSITION. Suppose $S = \bigvee_{i \in N} S_i^1$ is a finite or countably infinite wedge (= one point union) of circles. Then there is a 1-1 immersion $\mu: \tilde{S} \rightarrow S \times R^1$ of the universal cover \tilde{S} of S , such that $\pi_S \mu = p_S$, where $p_S =$ covering projection: $\tilde{S} \rightarrow S$ and $\pi_S =$ projection: $S \times R^1 \rightarrow S$.

This leads to the exceedingly general

COROLLARY. Suppose T is a space having the homotopy type of a connected CW complex, whose fundamental group lies in an exact sequence of groups $1 \rightarrow H \rightarrow \pi_1(T, *) \rightarrow F \rightarrow 1$, where F is free and countably generated. Then there is a 1-1 immersion $\omega: \tilde{T} \rightarrow T \times R^1$ such that $\pi_T \omega = p_T$, where $p_T: \tilde{T} \rightarrow T$ is the covering of T corresponding to the subgroup $H \subset \pi_1(T, *)$.

PROOF OF PROPOSITION. One is tempted to look for an explicit analytical definition of μ , but we have not found any simple one. At any rate, the recursive construction of a suitable μ is quite trivial. Calling the closures of the components of $p_S^{-1}(\bigvee_{i \in N} S_i^1 - *)$ in \tilde{S} intervals, let $J_1 \subset J_2 \subset \dots$ be an increasing sequence of subsets of \tilde{S} such that $\tilde{S} = \bigcup_{i=1}^\infty J_i$ and each J_i is a connected union of i intervals in \tilde{S} . Inductively assuming the existence of any embedding $\mu_{i-1}: J_{i-1} \rightarrow S \times R^1$ such that $\pi_S \mu_{i-1} = p_S|_{J_{i-1}}$, it is routine point set topology to extend this to μ_i .

PROOF OF COROLLARY. Let S be a wedge of circles such that $\pi_1(S) = F$, and by elementary construction let $r: (T, *) \rightarrow (S, *)$ be a map such that $r_\#:$

$\pi_1(T, *) \rightarrow F$ is the hypothesized homomorphism. Retaining the notation of the Proposition, let \tilde{T} and p_T be defined by the pullback diagram

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{r}} & \tilde{S} \\ \downarrow p_T & & \downarrow p_S \\ T & \xrightarrow{r} & S \end{array}$$

That is, $\tilde{T} = \{(x, y) \in T \times \tilde{S} \mid r(x) = p_S(y)\}$. Now define $\omega = p_T \times \pi_{R^1} \mu \tilde{r}$, where $\mu: \tilde{S} \rightarrow S \times R^1$ is from the Proposition.

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