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AN ASYMPTOTIC FORMULA FOR AN INTEGRAL IN STARLIKE FUNCTION THEORY

BY

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ABSTRACT. The paper is concerned with the integral

$$H = \int_0^{2\pi} |f|^\sigma |F|^\tau (\operatorname{Re} F)^\kappa \, d\theta$$

in which f is a function regular and starlike in the unit disc, $F = zf'/f$, and the parameters σ, τ, κ are real. A study of H is of interest since various well-known integrals in the theory, such as the length of $f(|z| = r)$, the area of $f(|z| < r)$, and the integral means of f, are essentially obtained from it by suitably choosing the parameters. An asymptotic formula, valid as $r \longrightarrow 1$, is obtained for H when f is a starlike function of positive order α, and the parameters satisfy $\alpha\sigma + \tau + \kappa > 1$, $\tau + \kappa \geq 0$, $\kappa \geq 0$, $\sigma > 0$. Several easy applications of this result are made; some to obtaining old results, two others in proving conjectures of Holland and Thomas.

1. **Introduction.** Let a function f be regular in the open unit disc D, and such that $f(0) = 0, f'(0) = 1$. Suppose a function F exists, regular in D, and of positive real part, for which

$$F(z) = zf'(z)/f(z) \quad (0 < |z| < 1), \qquad F(0) = 1. \tag{1.1}$$

Then f is called starlike. It is well known that a starlike function is univalent, and maps D onto a set starshaped with respect to the origin.

Suppose now that f is starlike, and let (1.1) define F. We consider for $0 < r < 1$ and real σ, τ, and κ, the integral

$$H(r, \sigma, \tau, \kappa) = \int_0^{2\pi} |f(re^{i\theta})|^\sigma |F(re^{i\theta})|^\tau (\operatorname{Re} F(re^{i\theta}))^\kappa \, d\theta.$$

With various choices of σ, τ, and κ, this integral is well known in starlike (and univalent) function theory. For example,

$$H(r, 1, 1, 0) = \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})| \, d\theta = \int_0^{2\pi} r|f'(re^{i\theta})| \, d\theta$$

is the length of $f(\{z: |z| = r\})$, and

$$\begin{aligned} H(r, 2, 0, 1) &= \int_0^{2\pi} |f(re^{i\theta})|^2 \operatorname{Re} F(re^{i\theta}) \, d\theta \\ &= 2\int_0^r \int_0^{2\pi} \rho |f'(\rho e^{i\theta})|^2 \, d\theta d\rho = 2A(r, f), \end{aligned} \tag{1.2}$$

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where $A(r, f)$ is the area of $f(\{z: |z| \leq r\})$. Also, for $\lambda > 0$,

$$H(r, \lambda, 0, 0) = \int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta = 2\pi I(r, \lambda, f), \tag{1.3}$$

and

$$\begin{aligned} H(r, \lambda, \lambda, 0) &= \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda \, d\theta \\ &= \int_0^{2\pi} r^\lambda |f'(re^{i\theta})|^\lambda \, d\theta = 2\pi r^\lambda J(r, \lambda, f), \end{aligned} \tag{1.4}$$

where $(I(r, \lambda, f))^{1/\lambda}$ and $(J(r, \lambda, f))^{1/\lambda}$ are integral means of f and f' respectively. In the present paper we suppose f to have a certain minimal growth, and find, for suitable σ, τ, and κ, as r tends to one, an asymptotic formula for $H(r, \sigma, \tau, \kappa)$.

A few remarks and definitions precede the statement of this result. From a classical theorem on regular functions of positive real part, and the relationship (1.1), we find for a starlike function f the well-known representation

$$f(z) = z \exp\left(-\int_0^{2\pi} \log(1 - ze^{-it}) \, d\mu(t)\right) \qquad (z \in D), \tag{1.5}$$

where μ is nondecreasing on $[0, 2\pi]$ and $\int_0^{2\pi} d\mu(t) = 2$. Any such function μ satisfying (1.5) is continuous apart from jumps of height at most $\int_0^{2\pi} d\mu(t)$. Pommerenke has shown (implicitly) in [4] that

$$\Delta(\varphi, f) = \lim_{r \to 1} \frac{\log |f(re^{i\theta})|}{-\log(1 - r)}$$

is the jump of μ at φ if $0 < \varphi < 2\pi$, and the sum of the jumps at 0 and 2π when $\varphi = 0$ or 2π. For $\alpha(f) = \sup_\varphi \Delta(\varphi, f)$ and $M(r) = \max_{|z|=r} |f(z)|$ $(0 < r < 1)$ he has shown

$$\alpha(f) = \lim_{r \to 1} \frac{\log M(r, f)}{-\log(1 - r)}. \tag{1.6}$$

We call $\alpha(f)$ the order of f, and $\Delta(\varphi, f)$ the radial order of f on $\{re^{i\varphi}\}$.

Starlike functions of positive order are the main concern of the present paper. We shall, in the following, make implicit use of the fact that for such a function f, $\{\varphi: \Delta(\varphi, f) > 0\}$ is countable; and for $0 < c \leq \alpha$, $\{\varphi: \Delta(\varphi, f) \geq c\}$ is finite nonempty.

Our result is as follows:

THEOREM 1. *Let f be a starlike function of positive order α, and denote by $\varphi_1, \ldots, \varphi_N$ the values of φ in $[0, 2\pi)$ for which α is the radial order of f on $\{re^{i\varphi}\}$. Then if $\sigma > 0$, $\kappa \geq 0$, $\tau + \kappa \geq 0$, and $\alpha\sigma + \tau + \kappa > 1$,*

$$H(r, \sigma, \tau, \kappa) \sim \alpha^{\tau+\kappa} C(\alpha\sigma + \tau + 2\kappa)(1 - r)^{1-\tau-\kappa} \sum_{\nu=1}^{N} |f(re^{i\varphi_\nu})|^\sigma,$$

as $r \to 1$, *where, for* $x > 1$, $C(x) = \int_{-\infty}^{\infty} dt/(1 + t^2)^{\frac{1}{2}x} = \Gamma(\frac{1}{2}x - \frac{1}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}x)$.

Theorem 1 has some interesting applications. Let f be starlike of order α. Recalling (1.2) to (1.4), and using, with Theorem 1, the relationship

$$M(r, f) \sim \max(|f(re^{i\varphi_1})|, \ldots, |f(re^{i\varphi_N})|) \qquad (\alpha > 0) \tag{1.7}$$

(proved at the end of §2) we obtain for $\alpha > 0$:

$$\liminf_{r\to 1} \frac{A(r, f)}{M^2(r, f)} \geq \frac{1}{2}\alpha C(2\alpha + 2), \qquad \limsup_{r\to 1} \frac{A(r, f)}{M^2(r, f)} \leq \frac{1}{2} N\alpha C(2\alpha + 2),$$

for $\alpha > 0$, $\alpha\lambda > 1$:

$$\liminf_{r\to 1} \frac{I(r, \lambda, f)}{(1 - r)M^\lambda(r, f)} \geq C(\alpha\lambda), \qquad \limsup_{r\to 1} \frac{I(r, \lambda, f)}{(1 - r)M^\lambda(r, f)} \leq N C(\alpha\lambda),$$

and for $\alpha > 0$, $(1 + \alpha)\lambda > 1$:

$$\liminf_{r\to 1} \frac{J(r, \lambda, f)}{(1 - r)^{1-\lambda}M^\lambda(r, f)} \geq \alpha^\lambda C(\alpha\lambda + \lambda),$$

$$\limsup_{r\to 1} \frac{J(r, \lambda, f)}{(1 - r)^{1-\lambda}M^\lambda(r, f)} \leq N\alpha^\lambda C(\alpha\lambda + \lambda).$$

These are all results of Sheil-Small [5].

From (1.2), $A(r, f) = \frac{1}{2}H(r, 2, 0, 1)$ and $A'(r, f) = H(r, 2, 2, 0)/r$; so for $\alpha > 0$ Theorem 1 also yields

$$\lim_{r\to 1} \frac{(1 - r)A'(r, f)}{A(r, f)} = 2\alpha. \tag{1.8}$$

A proof of this result, and a simple proof of (1.8) in the case $\alpha = 0$, are to be found in [3]. We are also able to prove, using Theorem 1, that, for $\alpha\lambda > 1$,

$$\lim_{r\to 1} \frac{(1 - r)I'(r, \lambda, f)}{I(r, \lambda, f)} = \alpha\lambda - 1, \tag{1.9}$$

a result conjectured in [2]. In fact, once we have noticed, from (1.3), that, for any $\lambda > 0$, $I(r, \lambda, f) = H(r, \lambda, 0, 0)/2\pi$, and

$$I'(r, \lambda, f) = \frac{\lambda}{2\pi r} \int_0^{2\pi} |f(re^{i\theta})|^\lambda \operatorname{Re} F(re^{i\theta})\, d\theta = \frac{\lambda}{2\pi r} H(r, \lambda, 0, 1),$$

the proof via Theorem 1 follows simply on noting

$$\alpha\lambda C(\alpha\lambda + 2) = (\alpha\lambda - 1) C(\alpha\lambda), \quad \text{for } \alpha\lambda > 1.$$

Another conjecture in [2] is that, for $(1 + \alpha)\lambda > 1$,

$$\lim_{r\to 1} \frac{(1 - r)J'(r, \lambda, f)}{J(r, \lambda, f)} = (1 + \alpha)\lambda - 1. \tag{1.10}$$

A corollary of Theorem 1 yields a proof of (1.10) when $(1 + \alpha)\lambda > 1$, $\alpha > 0$, as we shall see in §4.

Our proof of Theorem 1 begins in §2, where some preliminary results are obtained, and is completed in §3. In all that follows we assume that $0 < r < 1$, and that θ is a real number. Also that the o, 0, and $\sim$ notations refer to behaviour as r tends to one. The term r near one means all values of r in $(\eta, 1)$, for some η in $(0, 1)$.

2. **Preliminaries.** In this section we prove a number of results on what are essentially powers of starlike functions. A function g will be called star-powered whenever

$$g(z) = z \exp\left(-\int_0^{2\pi} \log(1 - e^{-it}z)d\nu(t)\right) \qquad (z \in D), \tag{2.1}$$

for some nondecreasing function ν on $[0, 2\pi]$. For such a g, we define as for a starlike function g the terms *order of* g and *radial order of* g on $\{re^{i\varphi}\}$.

The results we now prove are directed towards finding for a star-powered function g, and the function

$$G\colon G(z) = zg'(z)/g(z) \quad (z \in D\backslash\{0\}), \qquad G(0) = 1 \tag{2.2}$$

information about behaviour on various subsets of D.

LEMMA 2.1. *Let g be a star-powered function of positive order β, and in the above notation, put $K = \int_0^{2\pi} d\nu(t)$. Then*

(i) *for $\epsilon > 0$, and r near one, $M(r, g) < (1 - r)^{-\beta-\epsilon}$;*

(ii) *with G defined by* (2.2), $M(r, G) \leqslant 1 + Kr(1 - r)^{-1}$.

This is a well-known result when g is starlike [4], [3] and the extension to star-powered functions is simple enough to omit.

LEMMA 2.2. *Let g be a star-powered function of positive order β, and denote by $\psi_1, \ldots, \psi_p$ the values of ψ in $[0, 2\pi)$ for which β is the radial order of f on $\{re^{i\varphi}\}$. Put*

$$T(r) = [0, 2\pi]\backslash \bigcup_{k=1}^{p} \{\theta \bmod 2\pi\colon |\theta - \psi_k| < l(r)\},$$

where $l(r) = (-\log 1 - r)^{-1}$, and let γ be the largest radial order of g less than β. Then for any positive ϵ, and r near one,

$$\sup_{\theta \in T(r)} |g(re^{i\theta})| = O(1)(1 - r)^{-\gamma-\epsilon}.$$

PROOF. The function h,

$$h(z) = g(z) \prod_{k=1}^{p} (1 - ze^{-i\psi_k})^{\beta},$$

is star-powered and has order γ, so by Lemma 2.1, for $\epsilon > 0$, and r near one, $M(r, h) < (1 - r)^{-\gamma-\frac{1}{2}\epsilon}$. Also, for r near one, we have uniformly when $\theta \in T(r)$

$$\prod_{k=1}^{p} |1 - re^{i(\theta-\psi_k)}|^{-\beta} \leqslant |1 - re^{il(r)}|^{-\beta p} \leqslant (\tfrac{1}{2}l^2(r))^{-\frac{1}{2}\beta p}.$$

Hence

$$\sup_{\theta \in T(r)} |g(re^{i\theta})| = O(1)(1-r)^{-\gamma-\frac{1}{2}\epsilon}(-\log(1-r))^{\beta p},$$

from which the stated result follows.

LEMMA 2.3. *Let g be a star-powered function of positive radial order β on $\{re^{i\varphi}\}$ $(0 \leqslant \varphi < 2\pi)$. Suppose that λ is a positive number, and δ a positive function on $(0, 1)$ for which $\lambda(1-r) < \delta(r) = o(1)$. Then*

$$|g(re^{i\theta})| < (1 + \lambda^2 + o(1))^{-\frac{1}{2}\beta} |g(re^{i\varphi})|$$

uniformly when $\lambda(1-r) < |\theta - \varphi| < \delta(r)$.

PROOF. Let

$$g(z) = zh(z)(1 - ze^{-i\varphi})^{-\beta} \qquad (z \in D) \tag{2.3}$$

then, by an elementary argument,

$$|g(re^{i\theta})| < (1 + o(1))r|h(re^{i\theta})|(1-r)^{-\beta}\left(1 + \left(\frac{\theta - \varphi}{1-r}\right)^2\right)^{-\frac{1}{2}\beta} \tag{2.4}$$

uniformly when $|\theta - \varphi| < \delta(r)$. For h we shall prove

$$|h(re^{i\theta})| < |h(re^{i\varphi})| \exp\left(o(1)\int_0^{|\theta-\varphi|/2(1-r)} (1 + t^2)^{-\frac{1}{2}} dt\right) \tag{2.5}$$

uniformly when $\lambda(1-r) < |\theta - \varphi| < \delta(r)$. The lemma then follows by combining (2.3) and (2.4) to form

$$|g(re^{i\theta})| < (1 + o(1))|g(re^{i\varphi})| \left(1 + \left(\frac{\theta - \varphi}{1-r}\right)^2\right)^{-\frac{1}{2}\beta}$$

$$\cdot \exp\left(o(1)\int_0^{|\theta-\varphi|/2(1-r)} (1 + t^2)^{-\frac{1}{2}} dt\right),$$

valid uniformly for $\lambda(1-r) < |\theta - \varphi| < \delta(r)$, and by noting that for any suitably small positive ϵ

$$y_\epsilon : y_\epsilon(x) = (1 + x^2)^{-\frac{1}{2}\beta} \exp\left(\epsilon \int_0^{\frac{1}{2}x} (1 + t^2)^{-\frac{1}{2}} dt\right) \qquad (x > \lambda)$$

is a decreasing function.

For the proof of (2.5), let η be a positive function on $(0, 1)$ for which $\eta(r) = o(1)$, yet $\delta(r) = o(1)|1 - re^{i\eta(r)}|$, and put

$$E(r) = \{\theta \bmod 2\pi : |\theta - \varphi| < \eta(r)\}, \qquad E'(r) = [0, 2\pi] \backslash E(r).$$

Write

$$g(z) = z \exp\left(-\int_0^{2\pi} \log(1 - e^{-it}z)\, d\nu(t)\right) \qquad (z \in D),$$

then from (2.3) we deduce

$$h(z) = \exp\left(-\int_0^{2\pi} \log(1 - e^{-it}z)\, d\tau(t)\right) \qquad (z \in D) \tag{2.6}$$

where

$$\tau(t) = \nu(t) - \begin{cases} \beta, & \varphi < t \leqslant 2\pi, \\ 0, & 0 \leqslant t \leqslant \varphi. \end{cases} \tag{2.7}$$

We have, for any real t,

$$\log|1 - re^{i(\varphi-t)}| - \log|1 - re^{i(\theta-t)}| = \operatorname{Re}\int_{\varphi-t}^{\theta-t} \frac{\partial}{\partial u}\left(\log\frac{1}{1-re^{iu}}\right) du$$

$$\leqslant \max_{0\leqslant t\leqslant 2\pi} \int_t^{t+|\theta-\varphi|} \frac{du}{|1-re^{iu}|} = 2\int_0^{\frac{1}{2}|\theta-\varphi|} \frac{du}{|1-re^{iu}|}.$$

So with the aid of (2.7) we obtain

$$\int_{E(r)} (\log|1 - re^{i(\varphi-t)}| - \log|1 - re^{i(\theta-t)}|)\, d\tau(t)$$

$$\leqslant 2\left(\int_{E(r)} d\tau(t)\right)\left(\int_0^{\frac{1}{2}|\theta-\varphi|} \frac{du}{|1-re^{iu}|}\right) \tag{2.8}$$

$$= o(1)\int_0^{\frac{1}{2}|\theta-\varphi|} \frac{du}{|1-re^{iu}|} = o(1)\int_0^{|\theta-\varphi|/2(1-r)} (1+u^2)^{-\frac{1}{2}}\, du,$$

uniformly when $|\theta - \varphi| < \delta(r)$. Now observe that, by the choice of η,

$$\left|\frac{1-re^{i(\theta-t)}}{1-re^{i(\varphi-t)}} - 1\right| \leqslant \left|\frac{e^{i(\theta-\varphi)}-1}{1-re^{i\eta(r)}}\right| = o(1)$$

uniformly for $t \in E'$ and $|\theta - \varphi| < \delta(r)$. Hence

$$\int_{E'(r)} (\log|1 - re^{i(\varphi-t)}| - \log|1 - re^{i(\theta-t)}|)\, d\tau(t) < o(1)\int_0^{2\pi} d\tau(t) = o(1) \tag{2.9}$$

uniformly when $|\theta - \varphi| < \delta(r)$. Combining (2.8) and (2.9), and using (2.6) we obtain (2.5) uniformly for $\lambda(1-r) < |\theta - \varphi| < \delta(r)$.

LEMMA 2.4. *Let g be a star-powered function of positive radial order β on $\{re^{i\varphi}\}$ $(0 \leqslant \varphi < 2\pi)$, and define G by* (2.2). *Then for $0 < c < C$*

(i)
$$|G(re^{i\theta})| \sim \beta(1-r)^{-1}\left(1 + \left(\frac{\theta-\varphi}{1-r}\right)^2\right)^{-\frac{1}{2}},$$

and

$$\operatorname{Re} G(re^{i\theta}) \sim \beta(1-r)^{-1}\left(1 + \left(\frac{\theta-\varphi}{1-r}\right)^2\right)^{-1},$$

uniformly when $|\theta - \varphi| < C(1-r)$.

(ii)
$$\operatorname{Im} G(re^{i\theta}) \sim \beta(\theta-\varphi)(1-r)^{-2}\left(1 + \left(\frac{\theta-\varphi}{1-r}\right)^2\right)^{-1}$$

uniformly when $c(1-r) < |\theta - \varphi| < C(1-r)$.

(iii) $$|g(re^{i\theta})| \sim |g(re^{i\varphi})| \left(1 + \left(\frac{\theta - \varphi}{1-r}\right)^2\right)^{-\frac{1}{2}\beta}$$

uniformly when $|\theta - \varphi| < C(1-r)$.

PROOF. Parts (i) and (ii) are easily proved if $G(z) = \beta z/(1-z)$, $\varphi = 0$, so in deriving (i) and (ii) we shall prove no more than

$$\left| G(re^{i\theta}) - \frac{\beta re^{i(\theta-\varphi)}}{1 - re^{i(\theta-\varphi)}} \right| = \frac{o(1)}{1-r} \tag{2.10}$$

uniformly when $|\theta - \varphi| < \delta(r)$, where δ is any positive $o(1)$ function on $(0, 1)$. We have, from (2.1) and (2.2), for some nondecreasing function ν,

$$G(z) = 1 + \int_0^{2\pi} \frac{e^{-it}z}{1 - e^{-it}z}\, d\nu(t) \qquad (z \in D)$$

and this we rewrite in terms of

$$\tau(t) = \nu(t) - \begin{cases} \beta, & \varphi < t \leqslant 2\pi, \\ 0, & 0 \leqslant t \leqslant \varphi, \end{cases} \tag{2.11}$$

as

$$G(re^{i\theta}) - \frac{\beta re^{i(\theta-\varphi)}}{1 - re^{i(\theta-\varphi)}} = 1 + \int_0^{2\pi} \frac{re^{i(\theta-t)}}{1 - re^{i(\theta-t)}}\, d\tau(t). \tag{2.12}$$

Let η and ϵ be positive functions on $(0, 1)$ for which $\epsilon(r) = \delta(r) + \eta(r) = o(1)$, and $1 - r = o(1)|1 - re^{i\eta(r)}|$. Put

$$P(r) = \{\theta \bmod 2\pi \colon |\theta - \varphi| < \epsilon(r)\}, \qquad Q(r) = [0, 2\pi] \backslash P(r),$$

and consider now only values of r for which $Q(r)$ is nonempty. Then from (2.11) we see that

$$\left| \int_{P(r)} \frac{re^{i(\theta-t)}}{1 - re^{i(\theta-t)}}\, d\tau(t) \right| \leqslant \frac{1}{1-r} \int_{P(r)} d\tau(t) = \frac{o(1)}{1-r} \tag{2.13}$$

uniformly for real θ. Moreover, for $|\theta - \varphi| < \delta(r)$ and $t \in Q(r)$, $\eta(r) \leqslant |\theta - t| \leqslant 2\pi - \eta(r)$, so we also have, uniformly for $|\theta - \varphi| < \delta(r)$,

$$\left| \int_{Q(r)} \frac{re^{i(\theta-t)}}{1 - re^{i(\theta-t)}}\, d\tau(t) \right| \leqslant \left| \frac{1}{1 - re^{i\eta(r)}} \right| \int_{Q(r)} d\tau(t) = \frac{o(1)}{1-r}, \tag{2.14}$$

by the choice of η. Now combining (2.12), (2.13) and (2.14), the required estimate (2.10) is easily obtained.

We next derive (iii). Using the identity

$$\frac{\partial}{\partial\theta} \log |g(re^{i\theta})| = -\operatorname{Im} G(re^{i\theta}) \tag{2.15}$$

and (ii) of this lemma, we have, uniformly for $c(1-r) < \theta - \varphi < C(1-r)$,

$$\log\left|\frac{g(re^{i\theta})}{g(re^{i(\varphi+c(1-r))})}\right| \sim -\beta\int_{\varphi+c(1-r)}^{\theta} \frac{t-\varphi}{(1-r)^2+(t-\varphi)^2}\,dt$$

$$= -\beta\int_{c(1-r)}^{\theta-\varphi} \frac{u}{(1-r)^2+u^2}\,du = -\beta\int_{c}^{(\theta-\varphi)/1-r} \frac{t}{1+t^2}\,dt$$

$$= -\frac{\beta}{2}\log\left(\frac{1+((\theta-\varphi)/(1-r))^2}{1+c^2}\right).$$

From this, and a similar argument, we deduce

$$\left|\frac{g(re^{i\theta})}{g(re^{i(\varphi+c(1-r))})}\right| \sim \left(\frac{1+c^2}{1+((\theta-\varphi)/(1-r))^2}\right)^{\frac{1}{2}\beta},$$

and

$$\left|\frac{g(re^{i\theta})}{g(re^{i(\varphi-c(1-r))})}\right| \sim \left(\frac{1+c^2}{1+((\theta-\varphi)/(1-r))^2}\right)^{\frac{1}{2}\beta}$$

valid uniformly for $c(1-r) < \theta - \varphi < C(1-r)$, and $-C(1-r) < \theta - \varphi < -c(1-r)$ respectively. When $|\theta - \varphi| \leqslant c(1-r)$ we have, using (2.15) again and Lemma 2.1(ii),

$$\left|\log\left|\frac{g(re^{i\theta})}{g(re^{i\varphi})}\right|\right| = \left|\int_{\varphi}^{\theta} \operatorname{Im} G(re^{it})\,dt\right| \leqslant c(1-r)\left(1+\frac{Kr}{1-r}\right),$$

from which

$$e^{-c(1-r+Kr)} \leqslant \left|\frac{g(re^{i\theta})}{g(re^{i\varphi})}\right| \leqslant e^{c(1-r+Kr)},$$

and this modifies trivially to

$$e^{-c(1-r+Kr)}\left(\frac{1}{1+((\theta-\varphi)/(1-r))^2}\right)^{\frac{1}{2}\beta} \leqslant \left|\frac{g(re^{i\theta})}{g(re^{i\varphi})}\right|$$

$$\leqslant e^{c(1-r+Kr)}\left(\frac{1+c^2}{1+((\theta-\varphi)/(1-r))^2}\right)^{\frac{1}{2}\beta}.$$

Since c is an arbitrary positive number, the last two results imply (iii).

To conclude this sequence of lemmas we shall prove a result assumed in §1.

LEMMA 2.5. *Let g be a star-powered function of positive order β, and denote by $\psi_1, \ldots, \psi_p$ the values of ψ in $[0, 2\pi)$ for which β is the radial order of f on $\{re^{i\varphi}\}$. Then*

$$M(r, g) \sim \max(|g(re^{i\psi_1})|, \ldots, |g(re^{i\psi_p})|.$$

PROOF. Let

$$T(r) = [0, 2\pi] \setminus \bigcup_{k=1}^{p} \{\theta \bmod 2\pi\colon |\theta - \psi_k| < l(r)\},$$

then from Lemma 2.2, and the inequalities

$$|g(z)| \geq r(1 + r)^{\nu(2\pi)-\nu(0)} |1 - ze^{-i\psi_k}|^{-\beta} \qquad (|z| = r, k = 1, \ldots, p),$$

which follows easily from (2.1), we have

$$\sup_{\theta \in T(r)} |g(re^{i\theta})| = o(1) M(r, g). \tag{2.16}$$

Now using Lemma 2.3, we see via (2.16) that if $\epsilon > 0$ and

$$W(r, \epsilon) = [0, 2\pi] \setminus \bigcup_{k=1}^{p} \{\theta \bmod 2\pi\colon |\theta - \psi_k| < \epsilon(1 - r)\},$$

then, for r near one,

$$\sup_{\theta \in W(r,\epsilon)} |g(re^{i\theta})| < (1 + \tfrac{1}{2}\epsilon^2)^{-\frac{1}{2}\beta} M(r, g).$$

So $|g(re^{i\eta(r)})| = M(r, g)$ where, for the same ϵ and r, $|\eta(r) - \psi_k| \leq \epsilon(1 - r)$ and $k = k(r) \in \{1, \ldots, N\}$. Since ϵ is an arbitrary positive number we deduce from Lemma 2.4(iii) that

$$M(r, g) < (1 + o(1)) |g(re^{i\psi_k})|$$

where $k = k(r)$. Hence

$$M(r, g) < (1 + o(1)) \max(|g(re^{i\psi_1})|, \ldots, |g(re^{i\psi_p})|),$$

and obviously this completes the proof.

3. **Proof of Theorem 1.** Let f be a starlike function of positive order α, and denote by $\varphi_1, \ldots, \varphi_N$ the values of φ in $[0, 2\pi)$ for which α is the radial order of f on $\{re^{i\varphi}\}$. Let $l(r) = (-\log(1 - r))^{-1}$, and put

$$U_k(r) = \{\theta\colon |\theta - \varphi_k| < l(r)\} \qquad (k = 1 \text{ to } N),$$

$$T(r) = [0, 2\pi] \setminus \bigcup_{k=1}^{N} \{\theta \bmod 2\pi\colon \theta \in U_k(r)\}.$$

Then, for real σ, τ and κ, and r near one, $H(r, \sigma, \tau, \kappa) = \Sigma_{k=1}^{N} X_k + Y$, where

$$X_k = \int_{U_k(r)} |f|^\sigma |F|^\tau (\operatorname{Re} F)^\kappa \, d\theta, \qquad Y = \int_{T(r)} |f|^\sigma |F|^\tau (\operatorname{Re} F)^\kappa \, d\theta.$$

When $\alpha\sigma + \tau + \kappa > 1, \tau + \kappa \geq 0, \kappa \geq 0, \sigma > 0$, we shall find for each X_k the asymptotic formula

$$X_k \sim \alpha^{\tau+\kappa} C(\alpha\sigma + \tau + 2\kappa)(1 - r)^{1-\tau-\kappa} |f(re^{i\varphi_k})|^\sigma, \tag{3.1}$$

and for Y the estimate

$$Y = o(1)(1 - r)^{1-\alpha\sigma-\tau-\kappa}. \tag{3.2}$$

Since, by the representation (1.5), $|f(re^{i\varphi_k})| \geqslant r(1+r)^{-2}(1-r)^{-\alpha}$, we then have $Y = o(1)\Sigma_{k=1}^N X_k$; and consequently $H(r, \sigma, \tau, \kappa) \sim \Sigma_{k=1}^N X_k$. Our proof of (3.1) is in §3.1, and that of (3.2) in §3.2.

3.1. Denote by φ any one of the φ_k, and by $U(r)$ the corresponding $U_k(r)$. We have to prove

$$\begin{aligned} &\int_{U(r)} |f(re^{i\theta})|^\sigma |F(re^{i\theta})|^\tau (\operatorname{Re} F(re^{i\theta}))^\kappa\, d\theta \\ &\qquad \sim \alpha^{\tau+\kappa} C(\alpha\sigma+\tau+2\kappa)(1-r)^{1-\tau-\kappa}|f(re^{i\varphi})|^\sigma \end{aligned} \tag{3.3}$$

for $\alpha\sigma + \tau + \kappa > 1$, $\tau + \kappa \geqslant 0$, $\kappa \geqslant 0$, $\sigma > 0$. Let

$$V(r, x) = \{\theta \colon |\theta - \varphi| < x(1-r)\} \qquad (x > 0)$$

and write

$$\int_{U(r)} = \int_{V(r,x)} + \int_{U(r)\setminus V(r,x)} = I_1 + I_2, \tag{3.4}$$

say, where the missing integrand is that in (3.3). For I_1 we have, by Lemma 2.4, for real σ, τ, and κ, and for $x > 0$,

$$\begin{aligned} I_1 &\sim \alpha^{\tau+\kappa}(1-r)^{-\tau-\kappa}|f(re^{i\varphi})|^\sigma \int_{V(r,x)} \left(1 + \left(\frac{\theta-\varphi}{1-r}\right)^2\right)^{-\frac12(\alpha\sigma+\tau+2\kappa)} d\theta \\ &= \alpha^{\tau+\kappa}(1-r)^{1-\tau-\kappa}|f(re^{i\varphi})|^\sigma \int_{-x}^{x} (1+t^2)^{-\frac12(\alpha\sigma+\tau+2\kappa)}\, dt. \end{aligned} \tag{3.5}$$

For I_2 consider first the case $\tau + \kappa \neq 0$. Let p and q be chosen so that $\alpha\sigma p > 1$, $(\tau+\kappa)q > 1$, $p^{-1} + q^{-1} = 1$, $p > 1$; it is easy to verify that this is possible. Then, using the inequality $\kappa \geqslant 0$, and Hölder's inequality, we obtain

$$I_2 \leqslant \int_{U\setminus V} |f|^\sigma |F|^{\tau+\kappa} d\theta \leqslant \left(\int_{U\setminus V} |f|^{\sigma p}\right)^{1/p} \left(\int_0^{2\pi} |F|^{(\tau+\kappa)q}\right)^{1/q},$$

where $U \equiv U(r)$, $V \equiv V(r, x)$. With $I(\lambda) = \int_{U\setminus V} |f|^\lambda$, and use of Hayman's well-known estimate [1], applicable here since $(\tau+\kappa)q > 1$, this becomes

$$I_2 = O(1)(1-r)^{(1/q)-\tau-\kappa}(I(\sigma p))^{1/p}. \tag{3.6}$$

To deal with I we put

$$g(z) = z(1 - ze^{-i\varphi})^{\frac12(\alpha\lambda+1)}(f(z)/z)^\lambda \qquad (z \in D\setminus\{0\})$$

where $\alpha\lambda > 1$, and write

$$\begin{aligned} r^{1-\lambda} I(\lambda) &= \int_{U\setminus V} |g(re^{i\theta})||1 - re^{i(\theta-\varphi)}|^{-\frac12(\alpha\lambda+1)} d\theta \\ &= \int_{\varphi-l(r)}^{\varphi-x(1-r)} + \int_{\varphi+x(1-r)}^{\varphi+l(r)} = J_1 + J_2. \end{aligned}$$

On $\{re^{i\varphi}\}$ g has radial order $\frac{1}{2}(\alpha\lambda - 1) > 0$, so the lemmas of §2 apply. From Lemma 2.4(ii) and (2.15), we see that, for r near one, $|g(re^{i\theta})|$ increases throughout the interval $(\varphi - x'(1-r), \varphi - x(1-r))$. So on applying Lemma 2.3 to $|g(re^{i\theta})|$ for θ in $(\varphi - l(r), \varphi - x'(1-r))$ we have for $r > r_0(x)$ that in $(\varphi - l(r), \varphi - x(1-r))$, $|g(re^{i\theta})| < |g(re^{i(\varphi - x(1-r))})|$. This result applied to J_1, and a similar one applied to J_2, give for $r > r_0(x)$

$$r^{1-\lambda} I(\lambda) \leqslant \max|g(re^{i(\varphi \pm x(1-r))})| \int_{U\setminus V} |1 - re^{i(\theta-\varphi)}|^{-\frac{1}{2}(\alpha\lambda+1)} d\theta.$$

Now using Lemma 2.4(iii) we obtain, for $r > r_0(x)$,

$$I(\lambda) < |g(re^{i\varphi})| \int_{U\setminus V} |1 - re^{i(\theta-\varphi)}|^{-\frac{1}{2}(\alpha\lambda+1)} d\theta.$$

For $\epsilon > 0$, x suitably large, and r near one, it is easy to prove that

$$\int_{U\setminus V} |1 - re^{i(\theta-\varphi)}|^{-\frac{1}{2}(\alpha\lambda+1)} d\theta < \epsilon \int_V |1 - re^{i(\theta-\varphi)}|^{-\frac{1}{2}(\alpha\lambda+1)} d\theta$$

by making the substitution $\theta - \varphi = t(1-r)$ and using $\alpha\lambda > 1$. Moreover, for any given positive x, and r near one,

$$\int_V |1 - re^{i(\theta-\varphi)}|^{-\frac{1}{2}(\alpha\lambda+1)} d\theta \leqslant \int_0^{2\pi} |1 - re^{i(\theta-\varphi)}|^{-\frac{1}{2}(\alpha\lambda+1)} d\theta$$

$$< A(1-r)^{-\frac{1}{2}(\alpha\lambda-1)}$$

where A denotes an absolute constant. So when $x > x_0(\epsilon)$ and $r > r_0(\epsilon)$ we have

$$I(\lambda) < A\epsilon |g(re^{i\varphi})|(1-r)^{-\frac{1}{2}(\alpha\lambda-1)} < A\epsilon(1-r)|f(re^{i\varphi})|^{\lambda} \tag{3.7}$$

for $\alpha\lambda > 1$, and since $\alpha\sigma p > 1$ we may substitute (3.7) into (3.6) with $\lambda = \sigma p$ and obtain, for any $\epsilon > 0$,

$$I_2 < A\epsilon(1-r)^{1-\tau-\kappa} |f(re^{i\varphi})|^{\sigma}. \qquad (x > x_0(\epsilon), r > r_0(\epsilon)). \tag{3.8}$$

To prove (3.8) when $\tau + \kappa = 0$ we first deduce $I_2 \leqslant I(\sigma)$ using $\kappa \geqslant 0$, and then note that (3.7) applies with $\lambda = \sigma$ since $\alpha\sigma = \alpha\sigma + \tau + \kappa > 1$. Now using (3.4), (3.5) and (3.8) we easily obtain (3.3).

3.2. We have to prove that for $\alpha\sigma + \tau + \kappa > 1$, $\tau + \kappa \geqslant 0$, $\kappa \geqslant 0$, $\sigma > 0$,

$$\int_{T(r)} |f(re^{i\theta})|^{\sigma} |F(re^{i\theta})|^{\tau} (\operatorname{Re} F(re^{i\theta}))^{\kappa} d\theta = o(1)(1-r)^{1-\alpha\sigma-\tau-\kappa}. \tag{3.9}$$

Assume $\tau + \kappa > 0$ so that p and q exist satisfying $\alpha\sigma p > 1$, $(\tau+\kappa)q > 1$, $p^{-1} + q^{-1} = 1$, $p > 1$. Then, since $\kappa \geqslant 0$, we have by Hölder's inequality

$$\begin{aligned} \int_{T(r)} |f|^{\sigma} |F|^{\tau} (\operatorname{Re} F)^{\kappa} &\leqslant \left(\int_{T(r)} |f|^{\sigma p}\right)^{1/p} \left(\int_0^{2\pi} |F|^{(\tau+\kappa)q}\right)^{1/q} \\ &= O(1)(1-r)^{1/q-\tau-\kappa} \left(\int_{T(r)} |f|^{\sigma p}\right)^{1/p}, \end{aligned} \tag{3.10}$$

where we have used Hayman's result [1], applicable since $(\tau + \kappa)q > 1$.

We now let $\lambda = \sigma p$, so that $\alpha\lambda > 1$, and consider $\int_{T(r)} |f|^\lambda d\theta$. Denote by $\varphi_1, \dots, \varphi_N, \dots$ the sequence of φ for which on $\{re^{i\varphi}\}$ the radial order of f is positive, and by α_k the radial order of f on $\{re^{i\varphi_k}\}$. Then, since $\alpha\lambda > 1$, we define a star-powered function g by

$$g(z) = z\left(\frac{f(z)}{z}\right)^\lambda \prod_{0<\gamma_k<1} (1 - ze^{-i\varphi_k})^{\gamma_k + \xi} \qquad (z \in D\backslash\{0\})$$

where $\gamma_k = \alpha_k\lambda - \alpha\lambda + 1$, $\xi = \min(\alpha\lambda - 1, 1 - \gamma_k)$.

Any radial order of g less than the order is also less than $\alpha\lambda - 1$. Thus, by Lemma 2.2, for some positive δ,

$$\sup_{\theta \in T(r)} |g(re^{i\theta})| = O(1)(1 - r)^{1-\alpha\lambda+\delta}.$$

Hence

$$\int_{T(r)} |f|^\lambda d\theta = O(1)(1 - r)^{1-\alpha\lambda+\delta} \int_0^{2\pi} \prod_{0<\gamma_k<1} |1 - re^{i(\theta-\varphi_k)}|^{-\gamma_k-\xi} d\theta$$

$$= O(1)(1 - r)^{1-\alpha\lambda+\delta} \int_0^{2\pi} |1 - re^{i\theta}|^{-1} d\theta,$$

since $\gamma_k + \xi \leqslant 1$, and the points φ_k for which $0 \leqslant \gamma_k < 1$ are distinct and finite in number. The last integral is $O(\log 1/(1 - r))$, so

$$\int_{T(r)} |f|^\lambda d\theta = o(1)(1 - r)^{1-\alpha\lambda} \tag{3.11}$$

and (3.9) follows easily from this (with $\lambda = \sigma p$) and (3.10). To prove (3.9) when $\tau + \kappa = 0$, we first use $\kappa \geqslant 0$ to deduce

$$\int_{T(r)} |f|^\sigma |F|^\tau (\operatorname{Re} F)^\kappa d\theta \leqslant \int_{T(r)} |f|^\sigma d\theta,$$

and then note that (3.11) applies with $\lambda = \sigma$, since $\alpha\sigma = \alpha\sigma + \tau + \kappa > 1$.

4. Proof of a conjecture by Holland and Thomas. With the aid of a corollary to Theorem 1, we can outline a proof of the conjecture (1.10) in the case $\alpha > 0$, that is (in the notation of (1.10))

$$\lim_{r\to 1} \frac{(1 - r)J'(r, \lambda)}{J(r, \lambda)} = (1 + \alpha)\lambda - 1, \quad \text{for } \alpha > 0, (1 + \alpha)\lambda > 1 \tag{4.1}$$

where

$$J(r, \lambda) = \frac{1}{2\pi r^\lambda} \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda d\theta. \tag{4.2}$$

In proving a similar result for I (also conjectured by Holland and Thomas), we used Theorem 1 to represent both I and I'. The same approach will suffice for J but not for J' since we have

$$J'(r, \lambda) = \frac{\lambda}{2\pi r^{\lambda+1}} \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda (\operatorname{Re} F(re^{i\theta}) + \operatorname{Re} \mathcal{F}(re^{i\theta}) - 1)\, d\theta \tag{4.3}$$

where

$$\mathcal{F}(z) = zF'(z)/F(z) \qquad (z \in D).$$

However, when $(1 + \alpha)\lambda > 1$, a representation of J' can be squeezed out from the proof of Theorem 1.

The following results on $\mathcal{F}$ are needed:

(i) If f has positive radial order on $\{re^{i\varphi}\}$

$$\operatorname{Re} \mathcal{F}(re^{i\theta}) \sim \left(1 + \left(\frac{\theta - \varphi}{1 - r}\right)^2\right)^{-1/2} (1 - r)^{-1}$$

uniformly when $|\theta - \varphi| < O(1)(1 - r)$.

(ii) $$|\mathcal{F}(z)| \leqslant 2r/(1 - r^2), \qquad (|z| = r).$$

(i) follows from an argument similar to that in Lemma 2.4(i); for (ii) see [3]. Next we have the

COROLLARY TO THEOREM 1. *Let f and $\varphi_1, \ldots, \varphi_N$ be given as in Theorem 1, and suppose that $\sigma > 0$, $\tau \geqslant 0$, $\alpha\sigma + \tau > 1$. If Φ is any real function on D for which*

(i) $$\Phi(re^{i\theta}) \sim \left(1 + \left(\frac{\theta - \varphi_k}{1 - r}\right)^2\right)^{-1/2} (1 - r)^{-1}$$

uniformly when $|\theta - \varphi_k| < O(1)(1 - r)$, and

(ii) $$|\Phi(z)| = O(1)(1 - r)^{-1} \qquad (|z| = r),$$

then

$$\int_0^{2\pi} |f(re^{i\theta})|^\sigma |F(re^{i\theta})|^\tau \Phi(re^{i\theta})\, d\theta \sim \alpha^\tau C(\alpha\sigma + \tau + 1)(1 - r)^{-\tau} \sum_{k=1}^N |f(re^{i\varphi_k})|^\sigma.$$

Clearly we may take $\Phi = \operatorname{Re} \mathcal{F}$ in the corollary. In this way we find a representation for

$$\int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda \operatorname{Re} \mathcal{F}(re^{i\theta})\, d\theta$$

when $\alpha > 0$, $(1 + \alpha)\lambda > 1$. Theorem 1 supplies representations for

$$\int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda \operatorname{Re} F(re^{i\theta})\, d\theta$$

when $\alpha > 0$, $\lambda > 0$, and for $J(r, \lambda)$ when $\alpha > 0$, $(1 + \alpha)\lambda > 1$. With these results it is not difficult to obtain (4.1) via (4.2) and (4.3).

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