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# SOME $C^*$-ALGEBRAS WITH A SINGLE GENERATOR[1]

BY

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ABSTRACT. This paper grew out of the following question: If $X$ is a compact subset of $C^n$, is $C(X) \otimes M_n$ (the $C^*$-algebra of $n \times n$ matrices with entries from $C(X)$) singly generated? It is shown that the answer is affirmative; in fact, $A \otimes M_n$ is singly generated whenever $A$ is a $C^*$-algebra with identity, generated by a set of $n(n+1)/2$ elements of which $n(n-1)/2$ are selfadjoint. If $A$ is a separable $C^*$-algebra with identity, then $A \otimes K$ and $A \otimes U$ are shown to be singly generated, where $K$ is the algebra of compact operators in a separable, infinite-dimensional Hilbert space, and $U$ is any UHF algebra. In all these cases, the generator is explicitly constructed.

**1. Introduction.** This paper grew out of a question raised by Claude Schochet and communicated to us by J. A. Deddens: If $X$ is a compact subset of $\mathbf{C}^n$, is $C(X) \otimes M_n$ (the $C^*$-algebra of $n \times n$ matrices with entries from $C(X)$) singly generated? We show that the answer is affirmative; in fact, $A \otimes M_n$ is singly generated whenever $A$ is a $C^*$-algebra with identity, generated by a set of $n(n+1)/2$ elements of which $n(n-1)/2$ are selfadjoint. Working towards a converse, we show that $A \otimes M_2$ need not be singly generated if $A$ is generated by a set consisting of four elements. If $A$ lacks an identity, our results are weaker, and we obtain them only in the commutative case.

Informally, one might say that there are enough degrees of freedom in $M_n$ to allow a small generating set for $A$ to be combined into a single generator for $A \otimes M_n$. For countably generated $A$ we prove two natural infinite analogs: If $A$ is any separable $C^*$-algebra with identity, then $A \otimes K$ and $A \otimes U$ are singly generated, where $K$ is the algebra of compact operators on a separable, infinite-dimensional Hilbert space and where $U$ is any UHF algebra. In all these cases, we explicitly construct a generator.

Single generators for $C^*$-algebras and for von Neumann algebras have been studied by R. G. Douglas, C. Pearcy, T. Saitô, N. Suzuki, D. Topping, W. Wogen

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and others (see [5], [8]–[10], [12]–[16]). Our results generalize some of this work.

Throughout, we write $\mathbf{M}_n$ for the $C^*$-algebra of $n \times n$ scalar matrices. If $\mathcal{A}$ is any $C^*$-algebra then the elements of $\mathcal{A} \otimes \mathbf{M}_n$ may be viewed as $n \times n$ matrices with entries from $\mathcal{A}$. If $B_1, \ldots, B_k$ are elements of the $C^*$-algebra $\mathcal{B}$, then by $C^*(B_1, \ldots, B_k)$ we mean the $C^*$-subalgebra generated by $B_1, \ldots, B_k$. We write $\sigma(B)$ for the spectrum of $B \in \mathcal{B}$. For general facts about $C^*$-algebras, the reader is referred to [4] or [11].

## 2. Matrix algebras.

THEOREM 1. *Let $\mathcal{A}$ be a $C^*$-algebra with identity which is generated by a set of $n(n+1)/2$ elements, of which $n(n-1)/2$ are selfadjoint. Then $\mathcal{A} \otimes \mathbf{M}_n$ has a single generator.*

PROOF. Suppose that $\{a_1, a_2, \ldots, a_n, b_1, \ldots, b_{n(n-1)/2}\}$ is a generating set for $\mathcal{A}$, where each $b_j$ is selfadjoint. By translating with scalar multiples of $1_{\mathcal{A}}$, we can assume that each $b_j$ is positive, that all the generators are invertible, and that their spectra are contained in disjoint discs in $\mathbf{C}$. Our generator for $\mathcal{A} \otimes \mathbf{M}_n$ will be the upper triangular matrix:

$$T = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & a_2 & b_n & \cdots & b_{2n-3} \\ 0 & 0 & a_3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & b_{n(n-1)/2} \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

It suffices to show that $C^*(T)$ contains all the matrices of the form:

$$A_i = \begin{pmatrix} a_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} b_j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

(for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n(n-1)/2$) and all the elementary matrices $E_{km}$, where $E_{km}$ has $1_{\mathcal{A}}$ in the $k$th row and $m$th column and zeros elsewhere (for $1 \leqslant k \leqslant m$, $1 \leqslant m \leqslant n$).

Observe first that $\sigma(T) \subset \bigcup \sigma(a_i)$. Since the spectra of the $a_i$ are in disjoint discs in $\mathbf{C}$, we can find [5, p. 22] a sequence $\{p_s\}$ of complex polynomials which converge uniformly on some neighborhood of $\sigma(T)$ to a function $f$ which is 1 near $\sigma(a_1)$ and 0 near $\sigma(a_i)$ for $i \neq 1$. Using the analytic functional calculus, we conclude that $\{p_s(T)\}$ converges uniformly to an element $f(T) \in C^*(T)$. Also,

$\{p_s(a_i)\}$ converges in $A$ to $f(a_i)$, where $f(a_1) = 1_A$, $f(a_i) = 0$ for $i \neq 1$. Now, for each $s$, $p_s(T)$ has the form

$$p_s(T) = \begin{pmatrix} p_s(a_1) & & & & * \\ & p_s(a_2) & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & p_m(a_n) \end{pmatrix},$$

with some elements of $A$ above the diagonal. Thus

$$f(T) = \begin{pmatrix} f(a_1) & & & & * \\ & f(a_2) & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & f(a_n) \end{pmatrix} = \begin{pmatrix} 1_A & & & & * \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 0 \end{pmatrix}$$

A direct calculation shows that $[f(T)]^n[f(T)^*]^n$ has a nonzero entry only in the upper left-hand corner, and that entry is of the form $1_A + p$ where $p$ is a positive element of $A$. Since $1_A + p$ is invertible in $A$, it follows that $1_A \in C^*(1_A + p) \subset A$. Hence $E_{11} \in C^*(T)$. This means that

$$T - E_{11}T = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & a_2 & b_n & \dots & b_{2n-3} \\ 0 & 0 & a_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & b_{n(n-1)/2} \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}$$

is in $C^*(T)$. Now we can imitate the preceding argument to get $E_{22} \in C^*(T)$. Then $T - E_{11}T - E_{22}T \in C^*(T)$, and continuing in this fashion, we get $E_{ii} \in C^*(T)$, $1 \leqslant i \leqslant n$.

From these, we obtain the following elements of $C^*(T)$:

$$E_{11}TE_{22} = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$(E_{11}TE_{22})(E_{11}TE_{22})^* = \begin{pmatrix} b_1^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

and

$$[(E_{11}TE_{22})(E_{11}TE_{22})^*]^{1/2} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = B_1.$$

Since $b_1$ is invertible in $\mathsf{A}$, $b_1^{-1} \in C^*(b_1)$; by multiplying

$$\begin{pmatrix} b_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = E_{12},$$

we conclude that $E_{12} \in C^*(T)$.

Similar arguments show that each $A_i$, $B_j$, and $E_{km}$ is in $C^*(T)$. We conclude that $C^*(T) = \mathsf{A} \otimes \mathrm{M}_n$, as desired.

As an immediate corollary, we have an affirmative answer to the question raised by Schochet.

COROLLARY 2. *If $X$ is a compact subset of $\mathbf{C}^n$, then $C(X) \otimes \mathrm{M}_n$ is singly generated.*

PROOF. This follows immediately from Theorem 1. We note that, if $X$ is appropriately translated, then a simple generator for $C(X) \otimes \mathrm{M}_n$ is

$$T = \begin{pmatrix} z_1 & 1 & 0 & 0 & \dots & 0 \\ 0 & z_2 & 1 & 0 & \dots & 0 \\ 0 & 0 & z_3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 1 & 0 \\ \vdots & \vdots & \vdots & & z_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 0 & z_n \end{pmatrix}$$

REMARK 3. The proof of Theorem 1 works equally well when $\mathsf{A}$ is a von Neumann algebra and we interpret generation in the sense of von Neumann algebras. This result for von Neumann algebras and the following corollary generalize theorems of W. Wogen [16].

COROLLARY 4. *If $\mathsf{A}$ is a finitely generated $C^*$-algebra with identity (resp. von Neumann algebra) and $\mathsf{A} \simeq \mathsf{A} \otimes \mathrm{M}_n$ for some $n$, then $\mathsf{A}$ has a single generator.*

We do not know whether these are the best possible results. This issue is settled for $C(X) \otimes \mathrm{M}_2$ by the following theorem, first proved by Donald Hadwin.

THEOREM 5. *If $C(X) \otimes \mathrm{M}_2$ is singly generated, then $X$ is homeomorphic to a subset of $\mathbf{C}^2 \times \mathbf{R}$, and $C(X)$ is generated by a set of 3 elements of which one is selfadjoint.*

PROOF. We will view $C(X) \otimes \mathrm{M}_2$ as the algebra of continuous functions from $X$ into $\mathrm{M}_2$ (with the usual topology). Let $T$ be a generator for $C(X) \otimes \mathrm{M}_2$. If $x$ and $y$ are distinct points of $X$ then there is a polynomial in $T$ and $T^*$ whose value at $x$ is close to the zero matrix and whose value at $y$ is close to the identity matrix; it follows that the matrices $T(x)$ and $T(y)$ are not unitarily equivalent. Moreover, since $T$ generates $C(X) \otimes \mathrm{M}_2$ it follows that $T(z)$ generates $\mathrm{M}_2$ for each $z \in X$; i.e., $T(z)$ is an irreducible matrix. It is well known [1] that each irreducible $2 \times 2$ complex matrix is unitarily equivalent to a matrix of the form

$$A = \begin{pmatrix} \alpha & p \\ 0 & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbf{C}$, $p > 0$, and that this representation is unique up to the interchange of $\alpha$ and $\beta$. Let $\operatorname{tr}(A)$ denote the trace of $A$. Note that $\alpha + \beta = \operatorname{tr}(A)$, $\alpha^2 + \beta^2 = \operatorname{tr}(A^2)$, $|\alpha|^2 + |\beta|^2 + p^2 = \operatorname{tr}(A^*A)$. Since the trace is invariant under unitary equivalence, it follows that the map $\varphi\colon X \longrightarrow \mathbf{C}^2 \times \mathbf{R}$ defined by

$$\varphi(x) = (\operatorname{tr}(T(x)), \operatorname{tr}(T(x)^2), \operatorname{tr}(T(x)^*T(x)))$$

is a continuous one-one mapping and thus a homeomorphism. Since the coordinate functions generate $C(\varphi(X))$, which is isomorphic to $C(X)$, the proof is complete.

For a $C^*$-algebra without identity, the situation is more complicated and we obtain results only in the commutative case. If $\mathsf{A}$ is a commutative $C^*$-algebra, we identify $\mathsf{A}$ with $C_0(Y)$ for some locally compact Hausdorff space $Y$. Suppose that $\mathsf{A} = C_0(Y)$ is generated by the set $\{a_1, \ldots, a_k\}$ of $k$ functions. Since these functions must separate the points of $Y$ from $0$, $f = \Sigma |a_i|$ is a strictly positive function in $C_0(Y)$. Set $b_i = \operatorname{Re}(a_i) + \|a_i\| + 1$, $c_i = \operatorname{Im}(a_i) + \|a_i\| + 1$ for each $i$; each $b_i$ and $c_i$ is then a strictly positive, continucus function on $Y$ (although not in $C_0(Y)$ of course). The Stone-Weierstrass theorem now shows that the set $\{f, fb_1, fc_1, \ldots, fb_k, fc_k\}$ of $2k + 1$ strictly positive functions generates $C_0(Y)$. (In general it will not be possible to obtain a set of $2k$ strictly positive generators:

consider the closed unit disc minus the origin.) In view of these observations we formulate our next result as follows.

THEOREM 6. *If $Y$ is a locally compact Hausdorff space and $C_0(Y)$ is generated by a set of $2 + n(n-1)/2$ strictly positive functions, then $C_0(Y) \otimes M_n$ is singly generated.*

PROOF. Write $2 + n(n-1)/2 = m$ and let $\{a_1, a_2, \ldots, a_m\}$ be a set of strictly positive functions which generates $C_0(Y)$. Set $b = a_1 + ia_2$. We will show that the matrix

$$T = \begin{pmatrix} b & a_3 & a_4 & \cdots & a_{n+1} \\ 0 & 0 & a_{n+2} & \cdots & a_{2n-1} \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & a_m \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

generates $C_0(Y) \otimes M_n$.

To begin, we remark that, if $f$ is any strictly positive function in $C_0(Y)$, then the closed subalgebra generated by $f$ is a $C^*$-subalgebra and contains a sequence $\{h_k\}$ of strictly positive functions such that $\{h_k f\}$ is an approximate identity for $C_0(Y)$. Now,

$$T^n(T^n)^* = \begin{pmatrix} g & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

where $g$ is a strictly positive function. The above remark implies that $C^*(T)$ contains matrices

$$U_k = \begin{pmatrix} u_k & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

where $\{u_k\}$ is an approximate identity for $C_0(Y)$. Hence, $U_k T$ converges to an element $S$ of $C^*(T)$ where

$$S = \begin{pmatrix} b & a_3 & a_4 & \cdots & a_{n+1} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then

$$(T-S)^{n-2} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & h \\ 0 & \cdots & 0 & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$[(T-S)^{n-2}(T^*-S^*)^{n-2}]^{1/2} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & h & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $h$ is some strictly positive function in $C_0(Y)$. Arguing as above, we see that $C^*(T)$ contains matrices

$$V_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & v_k & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $\{v_k\}$ is an approximate identity for $C_0(Y)$. Then, $\{SV_k\}$ converges to $R \in C^*(T)$ where

$$R = \begin{pmatrix} 0 & a_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and thus $C^*(T)$ contains

$$A_3 = (RR^*)^{1/2} = \begin{pmatrix} a_3 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Using our initial remark, we conclude that $C^*(T)$ contains matrices

$$H_k = \begin{pmatrix} h_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where $\{h_k a_3\}$ is an approximate identity for $C_0(Y)$. Then

$$H_k A_3 = \begin{pmatrix} 0 & h_k a_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is also in $C^*(T)$. We continue this process and complete the proof as for Theorem 1, with a sequence of matrices which have the elements of an approximate identity for $\mathcal{A}$ in the $i$th row and $j$th column (such as $\{V_k\}$ and $\{H_k A_3\}$) playing the role of the elementary matrices used in Theorem 1.

COROLLARY 7. *If $Y$ is a locally compact subset of $\mathbf{C}^n$, then $C_0(Y) \otimes \mathbf{M}_n$ has a single generator whenever $n \geqslant 5$.*

PROOF. For a locally compact $Y \subset \mathbf{C}^n$, $C_0(Y)$ can always be generated by a set of $2n + 1$ strictly positive functions: let $f$ be a strictly positive function in $C_0(Y)$, and translate $Y$ so that the real coordinate functions $x_1, \dots, x_{2n}$ are strictly positive. Then $\{f, fx_1, \dots, fx_{2n}\}$ generates $C_0(Y)$. If $n \geqslant 5$, then $2n + 1 \leqslant 2 + n(n - 1)/2$.

3. **Infinite-dimensional algebras.** In this section we establish two infinite analogs of the results of §2. Let $\mathcal{K}$ be the $C^*$-algebra of compact operators on a separable, infinite-dimensional Hilbert space $\mathcal{H}$.

THEOREM 8. *If $\mathcal{A}$ is a separable $C^*$-algebra with identity, then the $C^*$ tensor product $\mathcal{A} \otimes \mathcal{K}$ has a single generator.*

PROOF. The algebra $\mathcal{A}$ is generated by a countable set $\{a_n\}$ of elements, and we can assume these are selfadjoint. Using scalar multiplication and translating by scalar multiples of $1_{\mathcal{A}}$, we assume without loss of generality that $\sigma(a_n) \subset [2^{-2n-1}, 2^{-2n}]$, for each $n$. In particular, each $a_n$ is positive and invertible.

We first describe a generating set for $\mathcal{A} \otimes \mathcal{K}$ consisting of two elements $A$ and $B$. Let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Let $E_{ij} \in \mathcal{K}$ be defined by $E_{ij}(e_j) = e_i$, and $E_{ij}(e_k) = 0$ if $k \neq j$. Now, $\|a_n \otimes E_{nn}\| = \|a_n\| \, \|E_{nn}\| \leqslant 2^{-2n}$, so the series $\Sigma_{n=1}^{\infty} a_n \otimes E_{nn}$ converges in norm to a positive element $A \in \mathcal{A} \otimes$

$\mathcal{K}$. Let $S \in \mathcal{K}$ be the weighted backward shift defined by $Se_n = n^{-1}e^{n-1}$, for $n \geq 2$, and $Se_1 = 0$. Set $B = 1_{\mathcal{A}} \otimes S$. We claim that $C^*(A, B) = \mathcal{A} \otimes \mathcal{K}$.

Observe first that since $S$ is irreducible, $C^*(S) = \mathcal{K}$. Thus $1_{\mathcal{A}} \otimes K \in C^*(A, B)$ for each $K \in \mathcal{K}$. Since

$$a_m \otimes E_{11} = (1_{\mathcal{A}} \otimes E_{1m})(1_{\mathcal{A}} \otimes E_{mm})A(1_{\mathcal{A}} \otimes E_{m1}),$$

we conclude that $a_m \otimes E_{11} \in C^*(A, B)$, each $m$, and therefore $a \otimes E_{11} \in C^*(A, B)$ for each $a \in \mathcal{A}$. But then, for any $a \in \mathcal{A}$,

$$a \otimes E_{ij} = (1 \otimes E_{i1})(a \otimes E_{11})(a \otimes E_{1j})$$

is in $C^*(A, B)$, and linear combinations of such elements are dense in $\mathcal{A} \otimes \mathcal{K}$. Thus our claim is proved.

We now note the following isometric $*$-isomorphisms of $C^*$ tensor products:

$$(\mathcal{A} \otimes \mathcal{K}) \otimes M_2 \simeq \mathcal{A} \otimes (\mathcal{K} \otimes M_2) \simeq \mathcal{A} \otimes \mathcal{K}.$$

Thus it suffices to exhibit a single generator for this first algebra, which we will identify with the $2 \times 2$ matrices whose entries lie in $\mathcal{A} \otimes \mathcal{K}$. Our generator is

$$T = \begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix}.$$

Now

$$TT^* = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$AA^* + BB^* = \sum a_n^2 \otimes E_{nn} + 1_{\mathcal{A}} \otimes \sum n^{-2}E_{nn}$$

$$= \sum [(a_n^2 + n^{-2}1_{\mathcal{A}}) \otimes E_{nn}].$$

Notice that

$$\sigma(a_n^2 + n^{-2}1_{\mathcal{A}}) \subset [2^{-4n-2} + n^{-2}, 2^{-4n} + n^{-2}]$$

and that these are disjoint intervals. Furthermore,

$$\sigma(AA^* + BB^*) = \bigcup \sigma(a_n^2 + n^{-2}1_{\mathcal{A}}) \cup \{0\}.$$

Thus the characteristic function of $\sigma(a_n^2 + n^{-2}1_{\mathcal{A}})$ is continuous on $\sigma(AA^* + BB^*)$; so using the functional calculus, we obtain $1_{\mathcal{A}} \otimes E_{nn} \in C^*(AA^* + BB^*)$, for each $n$.

In particular, this means that $U_k = 1_{\mathcal{A}} \otimes \Sigma_{n=1}^k E_{nn}$ is in $C^*(AA^* + BB^*)$. Furthermore, $\{U_k\}$ forms a positive approximate identity for $\mathcal{A} \otimes \mathcal{K}$; so we conclude that

$$\lim \begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

is in $C^*(T)$. Thus $C^*(T)$ contains $\left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} AA^* & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ since $A$ is positive. Hence, if $X_k = \Sigma_{n=1}^k a_n^{-1} \otimes E_{nn}$ then

$$\begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U_k \\ 0 & 0 \end{pmatrix}$$

is in $C^*(T)$ for each $k$. Finally, as in previous proofs, it is clear that the $2 \times 2$ matrices over $\mathcal{A} \otimes \mathcal{K}$ are generated by the set

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & U_k \\ 0 & 0 \end{pmatrix}, k = 1, 2, \ldots \right\}$$

and the proof is complete.

Recall that a uniformly hyperfinite (UHF) algebra of type $\{p_n\}$ is a $C^*$-algebra $\mathcal{U}$, which is the closure of the union of an increasing sequence $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ of $C^*$-subalgebras containing $1_{\mathcal{U}}$, where each $\mathcal{M}_n$ is isomorphic to $\mathbf{M}_{p_n}$. For each $n$, $p_n$ divides $p_{n+1}$ [7].

THEOREM 9. *Let $\mathcal{A}$ be a separable $C^*$-algebra with identity and let $\mathcal{U}$ be a UHF algebra. Then $\mathcal{A} \otimes \mathcal{U}$ is singly generated.*

PROOF. We will give a detailed proof only when $\mathcal{U}$ is of type $\{2^n\}$. In the general case, we exhibit the generator.

Since any two UHF algebras having the same type are isomorphic [7], it suffices to construct a convenient representation $\mathcal{U}$ of a type $\{2^n\}$ algebra and then exhibit a generator for $\mathcal{A} \otimes \mathcal{U}$. To this end, let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space, and let $E_{11}, E_{12}$ be infinite-rank projections such that $E_{11}E_{12} = 0, E_{11} + E_{12} = I$. Choose a partial isometry $U_1$ such that $U_1U_1^* = E_{11}, U_1^*U_1 = E_{12}$. Now choose infinite-rank projections $E_{21}$ and $E_{22}$ such that $E_{21}E_{22} = 0$ and $E_{21} + E_{22} = E_{12}$, and a partial isometry $U_2$ such that $U_2U_2^* = E_{21}$ and $U_2^*U_2 = E_{22}$. Continuing, we obtain partial isometries $U_3$, $U_4, \ldots$ . Let $\mathcal{U}$ be the $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $U_1, U_2, \ldots$; then $\mathcal{U}$ is a UHF algebra of type $\{2^n\}$ [15, p. 81].

The algebra $\mathcal{A}$ is generated by a countable set $\{a_1, a_2, \ldots\}$ of selfadjoint elements. As in previous proofs, we can assume that each $A_j$ is positive and invertible, and that $\sigma(a_n) \subset [2^{-2n-1}, 2^{-2n}]$ for each $n$. Our generator for $\mathcal{A} \otimes \mathcal{U}$ is

$$T = \sum_{n=1}^{\infty} [(a_n \otimes U_nU_n^*) + (2^{-n}1_{\mathcal{A}} \otimes U_n)];$$

this series is absolutely convergent since

$$\|(a_n \otimes U_nU_n^*) + (2^{-n}1_{\mathcal{A}} \otimes U_n)\| \leqslant \|a_n\| \, \|U_nU_n^*\| + 2^{-n}\|U_n\| \leqslant 2^{-2n} + 2^{-n}.$$

The element $T$ is like the infinite operator matrix of Figure 1, although to interpret this literally as a representation of $T$ would lead to confusion.

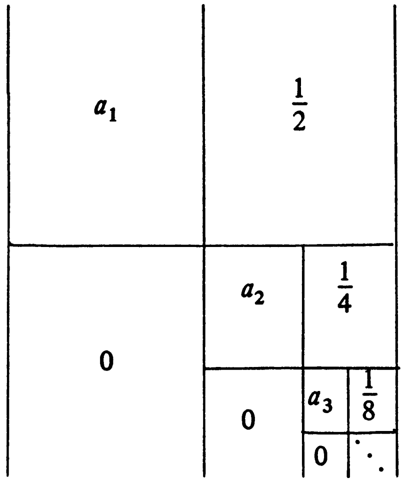


FIGURE 1

In order to show that $T$ generates $\mathfrak{A} \otimes \mathfrak{U}$, it suffices to show that $C^*(T)$ contains $1_{\mathfrak{A}} \otimes U_i$ and $a_j \otimes 1_{\mathfrak{A}}$ for each $i, j$.

Now, $P_1 = 1_{\mathfrak{A}} \otimes U_1 U_1^*$ and $Q_1 = 1_{\mathfrak{A}} \otimes U_1^* U_1$ are orthogonal projections in $\mathfrak{A} \otimes \mathfrak{U}$ with sum $1_{\mathfrak{A}\otimes\mathfrak{U}}$. Furthermore, $P_1 T P_1 = a_1 \otimes U_1 U_1^*$, $P_1 T Q_1 = 2^{-1} 1_{\mathfrak{A}} \otimes U_1$ and $Q_1 T P_1 = 0$. In effect, $T$ is "upper triangular" relative to the decomposition $1_{\mathfrak{A}\otimes\mathfrak{U}} = P_1 + Q_1$. This implies that

$$\begin{aligned}\sigma(T) &\subset \sigma(a_1 \otimes U_1 U_1^*) \cup \sigma(Q_1 T Q_1)\\ &\subset [2^{-3}, 2^{-2}] \cup \sigma(Q_1 T Q_1).\end{aligned}$$

Now

$$Q_1 T Q_1 = \sum_{n=2}^{\infty} [(a_n \otimes U_n U_n^*) + (2^{-n} 1_{\mathfrak{A}} \otimes U_n)]$$

and $P_2 = 1_{\mathfrak{A}} \otimes U_2 U_2^*$ and $Q_2 = 1_{\mathfrak{A}} \otimes U_2^* U_2$ are orthogonal projections in $\mathfrak{A} \otimes \mathfrak{U}$, with sum $Q_1 = 1_{\mathfrak{A}} \otimes U_1^* U_1$. Furthermore,

$$\begin{aligned}P_2 Q_1 T Q_1 P_2 &= P_2 T P_2 = a_2 \otimes U_2 U_2^*,\\ Q_2 Q_1 T Q_1 P_2 &= Q_2 T P_2 = 0,\end{aligned}$$

so that

$$\begin{aligned}\sigma(Q_1 T Q_1) &\subset \sigma(a_2 \otimes U_2 U_2^*) \cup \sigma(Q_2 T Q_2)\\ &\subset [2^{-5}, 2^{-4}] \cup \sigma(Q_2 T Q_2).\end{aligned}$$

Thus

$$\sigma(T) \subset [2^{-3}, 2^{-2}] \cup [2^{-5}, 2^{-4}] \cup \sigma(Q_2 T Q_2).$$

If we define $P_n = 1_A \otimes U_n U_n^*$, $Q_n = 1_A \otimes U_n^* U_n$ for each $n$ and continue in this way, we get

$$\sigma(T) \subset \bigcup_{k=1}^{n} [2^{-2k-1}, 2^{-2k}] \cup \sigma(Q_n T Q_n).$$

Since $\|Q_n T Q_n\| \longrightarrow 0$, we have

$$\sigma(T) \subset \bigcup_{k=1}^{\infty} [2^{-2k-1}, 2^{-2k}] \cup \{0\}.$$

We can also conclude that, for each $n$,

$$\sigma(Q_n T Q_n) \subset \bigcup_{k>n}^{\infty} [2^{-2k-1}, 2^{-2k}] \cup \{0\} \subset [0, 2^{-2k}].$$

Now, since $T = a_1 \otimes U_1 U_1^* + 2^{-1} 1_A \otimes U_1 + Q_1 T Q_1$, direct computation shows that, for any polynomials $q$,

$$q(T) = q(a_1) \otimes U_1 U_1^* + b \otimes U_1 + q(Q_1 T Q_1),$$

where this is also "upper triangular", and $b$ is some element of $A$. Choose a sequence $\{q_n\}$ of polynomials such that $q_n \longrightarrow 1$ uniformly on a neighborhood of $[2^{-3}, 2^{-2}]$ and such that $q_n \longrightarrow 0$ uniformly on a neighborhood of $[0, 2^{-4}]$. Then $q_n(T)$ and $q_n(a_1)$ converge to elements of $A \otimes U$ and $U$ respectively, and $q_n(Q_1 T Q_1)$ converges to zero. Thus

$$q_n(T) \longrightarrow S = 1_A \otimes U_1 U_1^* + c \otimes U_1$$

in $C^*(T)$, for some $c \in A$. Then $SS^* = (1_A + cc^*) \otimes U_1 U_1^*$, so that $C^*(SS^*)$ contains $1_A \otimes U_1 U_1^*$. Hence $C^*(T)$ also contains $a_1 \otimes U_1 U_1^*$, $1_A \otimes U_1$ and $Q_1 T Q_1$. Therefore $C^*(T)$ contains

$$a_1 \otimes U_1^* U_1 = (1_A \otimes U_1^*)(a_1 \otimes U_1 U_1^*)(1_A \otimes U_1)$$

and

$$a_1 \otimes 1_A = a_1 \otimes U_1^* U_1 + a_1 \otimes U_1 U_1^*.$$

We now apply the same analysis to $Q_1 T Q_1$ to conclude that $C^*(T)$ contains $1_A \otimes U_2$, $a_2 \otimes U_1^* U_1 = a_2 \otimes U_2^* U_2 + a_2 \otimes U_2 U_2^*$, and $Q_2 T Q_2$. But then $C^*(T)$ contains

$$a_2 \otimes U_1 U_1^* = (1_A \otimes U_1)(a_2 \otimes U_1^* U_1)(1_A \otimes U_1^*),$$

so $a_2 \otimes 1_A \in C^*(T)$. To complete the proof, we need only analyze $Q_2 T Q_2$, $Q_3 T Q_3, \ldots$ in turn.

For the general case, we construct a UHF algebra of type $\{p_n\}$. On a separable Hilbert space $H$, choose infinite-rank projections $E_{11}, E_{12}, \ldots, E_{1p_1}$ such that $E_{1j} E_{1k} = 0$ if $j \neq k$ and $\Sigma E_{ij} = I$. Choose partial isometries $U_{12}, U_{13}, \ldots$, $U_{1p_1}$ such that $U_{1j}^* U_{1j} = E_{1j}$ and $U_{1j} U_{1j}^* = E_{11}$ for each $j$. Set $q_2 = p_2/p_1$ and choose infinite-rank projections $E_{21}, \ldots, E_{2q_2}$ such that $E_{2j} E_{2k} = 0$ if $j \neq$

$k$ and $\Sigma E_{2j} = E_{1p_1}$. Choose partial isometries $U_{22}, U_{23}, \ldots, U_{2q_2}$ such that $U_{2j}^* U_{2j} = E_{2j}$ and $U_{2j} U_{2j}^* = E_{21}$ for each $j$. Continuing in this fashion, we obtain a family $\{U_{ij}\}$ of partial isometries where $1 \leqslant i < \infty$, $1 < j \leqslant q_i$ ($q_1 = p_1$). Then the $C^*$-subalgebra $\mathcal{U}$ of $\mathcal{B}(\mathcal{H})$ generated by $\{U_{ij}\}$ is a UHF algebra of type $\{p_n\}$. Let $\{a_{ij}$: $1 \leqslant i < \infty$, $1 \leqslant j < q_i\}$ be a set of positive, invertible elements that generates $\mathcal{A}$, and with $\sigma(a_{ij}) \subset [2^{-2i-2j-1}, 2^{-2i-j}]$, each $i, j$. Then our generator for $\mathcal{A} \otimes \mathcal{U}$ is

$$T = \sum_{\substack{1 \leqslant i < \infty \\ 1 \leqslant j < q_i}} (a_{ij} \otimes E_{ij}) + \sum_{\substack{1 \leqslant i < \infty \\ 1 < j \leqslant q_i}} (2^{-i-j} 1_{\mathcal{A}} \otimes U_{ij}).$$

We omit the proof that $C^*(T) = \mathcal{A} \otimes \mathcal{U}$.

REMARK 10. The existence of a single generator for a UHF algebra was established by D. Topping [14]. For $\mathcal{A} = \mathbf{C}$, Theorem 10 yields an explicit construction of such a generator; as far as we know, this is the first explicit construction of such a generator.

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