

SOME C^* -ALGEBRAS WITH A SINGLE GENERATOR⁽¹⁾

BY

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ABSTRACT. This paper grew out of the following question: If X is a compact subset of C^n , is $C(X) \otimes M_n$ (the C^* -algebra of $n \times n$ matrices with entries from $C(X)$) singly generated? It is shown that the answer is affirmative; in fact, $A \otimes M_n$ is singly generated whenever A is a C^* -algebra with identity, generated by a set of $n(n+1)/2$ elements of which $n(n-1)/2$ are selfadjoint. If A is a separable C^* -algebra with identity, then $A \otimes K$ and $A \otimes U$ are shown to be singly generated, where K is the algebra of compact operators in a separable, infinite-dimensional Hilbert space, and U is any UHF algebra. In all these cases, the generator is explicitly constructed.

1. Introduction. This paper grew out of a question raised by Claude Schochet and communicated to us by J. A. Deddens: If X is a compact subset of C^n , is $C(X) \otimes M_n$ (the C^* -algebra of $n \times n$ matrices with entries from $C(X)$) singly generated? We show that the answer is affirmative; in fact, $A \otimes M_n$ is singly generated whenever A is a C^* -algebra with identity, generated by a set of $n(n+1)/2$ elements of which $n(n-1)/2$ are selfadjoint. Working towards a converse, we show that $A \otimes M_2$ need not be singly generated if A is generated by a set consisting of four elements. If A lacks an identity, our results are weaker, and we obtain them only in the commutative case.

Informally, one might say that there are enough degrees of freedom in M_n to allow a small generating set for A to be combined into a single generator for $A \otimes M_n$. For countably generated A we prove two natural infinite analogs: If A is any separable C^* -algebra with identity, then $A \otimes K$ and $A \otimes U$ are singly generated, where K is the algebra of compact operators on a separable, infinite-dimensional Hilbert space and where U is any UHF algebra. In all these cases, we explicitly construct a generator.

Single generators for C^* -algebras and for von Neumann algebras have been studied by R. G. Douglas, C. Pearcy, T. Saitô, N. Suzuki, D. Topping, W. Wogen

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and others (see [5], [8]–[10], [12]–[16]). Our results generalize some of this work.

Throughout, we write M_n for the C^* -algebra of $n \times n$ scalar matrices. If A is any C^* -algebra then the elements of $A \otimes M_n$ may be viewed as $n \times n$ matrices with entries from A . If B_1, \dots, B_k are elements of the C^* -algebra \mathcal{B} , then by $C^*(B_1, \dots, B_k)$ we mean the C^* -subalgebra generated by B_1, \dots, B_k . We write $\sigma(B)$ for the spectrum of $B \in \mathcal{B}$. For general facts about C^* -algebras, the reader is referred to [4] or [11].

2. Matrix algebras.

THEOREM 1. *Let A be a C^* -algebra with identity which is generated by a set of $n(n + 1)/2$ elements, of which $n(n - 1)/2$ are selfadjoint. Then $A \otimes M_n$ has a single generator.*

PROOF. Suppose that $\{a_1, a_2, \dots, a_n, b_1, \dots, b_{n(n-1)/2}\}$ is a generating set for A , where each b_j is selfadjoint. By translating with scalar multiples of 1_A , we can assume that each b_j is positive, that all the generators are invertible, and that their spectra are contained in disjoint discs in \mathbb{C} . Our generator for $A \otimes M_n$ will be the upper triangular matrix:

$$T = \begin{pmatrix} a_1 & b_1 & b_2 & \dots & b_{n-1} \\ 0 & a_2 & b_n & \dots & b_{2n-3} \\ 0 & 0 & a_3 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \dots & b_{n(n-1)/2} \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}.$$

It suffices to show that $C^*(T)$ contains all the matrices of the form:

$$A_i = \begin{pmatrix} a_i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} b_j & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

(for $1 \leq i \leq n, 1 \leq j \leq n(n - 1)/2$) and all the elementary matrices E_{km} , where E_{km} has 1_A in the k th row and m th column and zeros elsewhere (for $1 \leq k \leq m, 1 \leq m \leq n$).

Observe first that $\sigma(T) \subset \bigcup \sigma(a_i)$. Since the spectra of the a_i are in disjoint discs in \mathbb{C} , we can find [5, p. 22] a sequence $\{p_s\}$ of complex polynomials which converge uniformly on some neighborhood of $\sigma(T)$ to a function f which is 1 near $\sigma(a_1)$ and 0 near $\sigma(a_i)$ for $i \neq 1$. Using the analytic functional calculus, we conclude that $\{p_s(T)\}$ converges uniformly to an element $f(T) \in C^*(T)$. Also,

$\{p_s(a_i)\}$ converges in A to $f(a_i)$, where $f(a_1) = 1_A$, $f(a_i) = 0$ for $i \neq 1$. Now, for each s , $p_s(T)$ has the form

$$p_s(T) = \begin{pmatrix} p_s(a_1) & & & & & \\ & p_s(a_2) & & * & & \\ & & \cdot & & & \\ & & & \cdot & & \\ 0 & & & & \cdot & \\ & & & & & p_m(a_n) \end{pmatrix},$$

with some elements of A above the diagonal. Thus

$$f(T) = \begin{pmatrix} f(a_1) & & & & & \\ & f(a_2) & & * & & \\ & & \cdot & & & \\ & & & \cdot & & \\ 0 & & & & \cdot & \\ & & & & & f(a_n) \end{pmatrix} = \begin{pmatrix} 1_A & & & & & \\ & 0 & & * & & \\ & & \cdot & & & \\ & & & \cdot & & \\ 0 & & & & \cdot & \\ & & & & & 0 \end{pmatrix}$$

A direct calculation shows that $[f(T)]^n [f(T)^*]^n$ has a nonzero entry only in the upper left-hand corner, and that entry is of the form $1_A + p$ where p is a positive element of A . Since $1_A + p$ is invertible in A , it follows that $1_A \in C^*(1_A + p) \subset A$. Hence $E_{11} \in C^*(T)$. This means that

$$T - E_{11}T = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & a_2 & b_n & \dots & b_{2n-3} \\ 0 & 0 & a_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & b_{n(n-1)/2} \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}$$

is in $C^*(T)$. Now we can imitate the preceding argument to get $E_{22} \in C^*(T)$. Then $T - E_{11}T - E_{22}T \in C^*(T)$, and continuing in this fashion, we get $E_{ii} \in C^*(T)$, $1 \leq i \leq n$.

From these, we obtain the following elements of $C^*(T)$:

$$E_{11}TE_{22} = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$(E_{11}TE_{22})(E_{11}TE_{22})^* = \begin{pmatrix} b_1^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

and

$$[(E_{11}TE_{22})(E_{11}TE_{22})^*]^{1/2} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = B_1.$$

Since b_1 is invertible in A , $b_1^{-1} \in C^*(b_1)$; by multiplying

$$\begin{pmatrix} b_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = E_{12},$$

we conclude that $E_{12} \in C^*(T)$.

Similar arguments show that each A_i, B_j , and E_{km} is in $C^*(T)$. We conclude that $C^*(T) = A \otimes M_n$, as desired.

As an immediate corollary, we have an affirmative answer to the question raised by Schochet.

COROLLARY 2. *If X is a compact subset of C^n , then $C(X) \otimes M_n$ is singly generated.*

PROOF. This follows immediately from Theorem 1. We note that, if X is appropriately translated, then a simple generator for $C(X) \otimes M_n$ is

$$T = \begin{pmatrix} z_1 & 1 & 0 & 0 & \dots & 0 \\ 0 & z_2 & 1 & 0 & \dots & 0 \\ 0 & 0 & z_3 & \cdot & \cdot & \vdots \\ \cdot & \cdot & \cdot & \ddots & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & z_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 0 & z_n \end{pmatrix}$$

REMARK 3. The proof of Theorem 1 works equally well when A is a von Neumann algebra and we interpret generation in the sense of von Neumann algebras. This result for von Neumann algebras and the following corollary generalize theorems of W. Wogen [16].

COROLLARY 4. *If A is a finitely generated C^* -algebra with identity (resp. von Neumann algebra) and $A \simeq A \otimes M_n$ for some n , then A has a single generator.*

We do not know whether these are the best possible results. This issue is settled for $C(X) \otimes M_2$ by the following theorem, first proved by Donald Hadwin.

THEOREM 5. *If $C(X) \otimes M_2$ is singly generated, then X is homeomorphic to a subset of $C^2 \times \mathbf{R}$, and $C(X)$ is generated by a set of 3 elements of which one is selfadjoint.*

PROOF. We will view $C(X) \otimes M_2$ as the algebra of continuous functions from X into M_2 (with the usual topology). Let T be a generator for $C(X) \otimes M_2$. If x and y are distinct points of X then there is a polynomial in T and T^* whose value at x is close to the zero matrix and whose value at y is close to the identity matrix; it follows that the matrices $T(x)$ and $T(y)$ are not unitarily equivalent. Moreover, since T generates $C(X) \otimes M_2$ it follows that $T(z)$ generates M_2 for each $z \in X$; i.e., $T(z)$ is an irreducible matrix. It is well known [1] that each irreducible 2×2 complex matrix is unitarily equivalent to a matrix of the form

$$A = \begin{pmatrix} \alpha & p \\ 0 & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbf{C}$, $p > 0$, and that this representation is unique up to the interchange of α and β . Let $\text{tr}(A)$ denote the trace of A . Note that $\alpha + \beta = \text{tr}(A)$, $\alpha^2 + \beta^2 = \text{tr}(A^2)$, $|\alpha|^2 + |\beta|^2 + p^2 = \text{tr}(A^*A)$. Since the trace is invariant under unitary equivalence, it follows that the map $\varphi: X \rightarrow C^2 \times \mathbf{R}$ defined by

$$\varphi(x) = (\text{tr}(T(x)), \text{tr}(T(x)^2), \text{tr}(T(x)^*T(x)))$$

is a continuous one-one mapping and thus a homeomorphism. Since the coordinate functions generate $C(\varphi(X))$, which is isomorphic to $C(X)$, the proof is complete.

For a C^* -algebra without identity, the situation is more complicated and we obtain results only in the commutative case. If A is a commutative C^* -algebra, we identify A with $C_0(Y)$ for some locally compact Hausdorff space Y . Suppose that $A = C_0(Y)$ is generated by the set $\{a_1, \dots, a_k\}$ of k functions. Since these functions must separate the points of Y from 0, $f = \sum |a_i|$ is a strictly positive function in $C_0(Y)$. Set $b_i = \text{Re}(a_i) + \|a_i\| + 1$, $c_i = \text{Im}(a_i) + \|a_i\| + 1$ for each i ; each b_i and c_i is then a strictly positive, continuous function on Y (although not in $C_0(Y)$ of course). The Stone-Weierstrass theorem now shows that the set $\{f, fb_1, fc_1, \dots, fb_k, fc_k\}$ of $2k + 1$ strictly positive functions generates $C_0(Y)$. (In general it will not be possible to obtain a set of $2k$ strictly positive generators:

consider the closed unit disc minus the origin.) In view of these observations we formulate our next result as follows.

THEOREM 6. *If Y is a locally compact Hausdorff space and $C_0(Y)$ is generated by a set of $2 + n(n - 1)/2$ strictly positive functions, then $C_0(Y) \otimes M_n$ is singly generated.*

PROOF. Write $2 + n(n - 1)/2 = m$ and let $\{a_1, a_2, \dots, a_m\}$ be a set of strictly positive functions which generates $C_0(Y)$. Set $b = a_1 + ia_2$. We will show that the matrix

$$T = \begin{pmatrix} b & a_3 & a_4 & \cdots & a_{n+1} \\ 0 & 0 & a_{n+2} & \cdots & a_{2n-1} \\ 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_m \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

generates $C_0(Y) \otimes M_n$.

To begin, we remark that, if f is any strictly positive function in $C_0(Y)$, then the closed subalgebra generated by f is a C^* -subalgebra and contains a sequence $\{h_k\}$ of strictly positive functions such that $\{h_k f\}$ is an approximate identity for $C_0(Y)$. Now,

$$T^n (T^n)^* = \begin{pmatrix} g & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{pmatrix}$$

where g is a strictly positive function. The above remark implies that $C^*(T)$ contains matrices

$$U_k = \begin{pmatrix} u_k & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{pmatrix}$$

where $\{u_k\}$ is an approximate identity for $C_0(Y)$. Hence, $U_k T$ converges to an element S of $C^*(T)$ where

$$S = \begin{pmatrix} b & a_3 & a_4 & \cdots & a_{n+1} \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then

$$(T - S)^{n-2} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & h \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$[(T - S)^{n-2}(T^* - S^*)^{n-2}]^{1/2} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & h & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where h is some strictly positive function in $C_0(Y)$. Arguing as above, we see that $C^*(T)$ contains matrices

$$V_k = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & v_k & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $\{v_k\}$ is an approximate identity for $C_0(Y)$. Then, $\{SV_k\}$ converges to $R \in C^*(T)$ where

$$R = \begin{pmatrix} 0 & a_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and thus $C^*(T)$ contains

$$A_3 = (RR^*)^{1/2} = \begin{pmatrix} a_3 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Using our initial remark, we conclude that $C^*(T)$ contains matrices

$$H_k = \begin{pmatrix} h_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where $\{h_k a_3\}$ is an approximate identity for $C_0(Y)$. Then

$$H_k A_3 = \begin{pmatrix} 0 & h_k a_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is also in $C^*(T)$. We continue this process and complete the proof as for Theorem 1, with a sequence of matrices which have the elements of an approximate identity for A in the i th row and j th column (such as $\{V_k\}$ and $\{H_k A_3\}$) playing the role of the elementary matrices used in Theorem 1.

COROLLARY 7. *If Y is a locally compact subset of C^n , then $C_0(Y) \otimes M_n$ has a single generator whenever $n \geq 5$.*

PROOF. For a locally compact $Y \subset C^n$, $C_0(Y)$ can always be generated by a set of $2n + 1$ strictly positive functions: let f be a strictly positive function in $C_0(Y)$, and translate Y so that the real coordinate functions x_1, \dots, x_{2n} are strictly positive. Then $\{f, fx_1, \dots, fx_{2n}\}$ generates $C_0(Y)$. If $n \geq 5$, then $2n + 1 \leq 2 + n(n - 1)/2$.

3. Infinite-dimensional algebras. In this section we establish two infinite analogs of the results of §2. Let K be the C^* -algebra of compact operators on a separable, infinite-dimensional Hilbert space H .

THEOREM 8. *If A is a separable C^* -algebra with identity, then the C^* tensor product $A \otimes K$ has a single generator.*

PROOF. The algebra A is generated by a countable set $\{a_n\}$ of elements, and we can assume these are selfadjoint. Using scalar multiplication and translating by scalar multiples of 1_A , we assume without loss of generality that $\sigma(a_n) \subset [2^{-2n-1}, 2^{-2n}]$, for each n . In particular, each a_n is positive and invertible.

We first describe a generating set for $A \otimes K$ consisting of two elements A and B . Let $\{e_n\}$ be an orthonormal basis for H . Let $E_{ij} \in K$ be defined by $E_{ij}(e_j) = e_i$, and $E_{ij}(e_k) = 0$ if $k \neq j$. Now, $\|a_n \otimes E_{nn}\| = \|a_n\| \|E_{nn}\| \leq 2^{-2n}$, so the series $\sum_{n=1}^\infty a_n \otimes E_{nn}$ converges in norm to a positive element $A \in A \otimes$

K . Let $S \in K$ be the weighted backward shift defined by $Se_n = n^{-1}e^{n-1}$, for $n \geq 2$, and $Se_1 = 0$. Set $B = 1_A \otimes S$. We claim that $C^*(A, B) = A \otimes K$.

Observe first that since S is irreducible, $C^*(S) = K$. Thus $1_A \otimes K \in C^*(A, B)$ for each $K \in K$. Since

$$a_m \otimes E_{11} = (1_A \otimes E_{1m})(1_A \otimes E_{mm})A(1_A \otimes E_{m1}),$$

we conclude that $a_m \otimes E_{11} \in C^*(A, B)$, each m , and therefore $a \otimes E_{11} \in C^*(A, B)$ for each $a \in A$. But then, for any $a \in A$,

$$a \otimes E_{ij} = (1 \otimes E_{i1})(a \otimes E_{11})(a \otimes E_{1j})$$

is in $C^*(A, B)$, and linear combinations of such elements are dense in $A \otimes K$. Thus our claim is proved.

We now note the following isometric $*$ -isomorphisms of C^* tensor products:

$$(A \otimes K) \otimes M_2 \simeq A \otimes (K \otimes M_2) \simeq A \otimes K.$$

Thus it suffices to exhibit a single generator for this first algebra, which we will identify with the 2×2 matrices whose entries lie in $A \otimes K$. Our generator is

$$T = \begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix}.$$

Now

$$TT^* = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} AA^* + BB^* &= \sum a_n^2 \otimes E_{nn} + 1_A \otimes \sum n^{-2} E_{nn} \\ &= \sum [(a_n^2 + n^{-2} 1_A) \otimes E_{nn}]. \end{aligned}$$

Notice that

$$\sigma(a_n^2 + n^{-2} 1_A) \subset [2^{-4n-2} + n^{-2}, 2^{-4n} + n^{-2}]$$

and that these are disjoint intervals. Furthermore,

$$\sigma(AA^* + BB^*) = \bigcup \sigma(a_n^2 + n^{-2} 1_A) \cup \{0\}.$$

Thus the characteristic function of $\sigma(a_n^2 + n^{-2} 1_A)$ is continuous on $\sigma(AA^* + BB^*)$; so using the functional calculus, we obtain $1_A \otimes E_{nn} \in C^*(AA^* + BB^*)$, for each n .

In particular, this means that $U_k = 1_A \otimes \sum_{n=1}^k E_{nn}$ is in $C^*(AA^* + BB^*)$. Furthermore, $\{U_k\}$ forms a positive approximate identity for $A \otimes K$; so we conclude that

$$\lim \begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

is in $C^*(T)$. Thus $C^*(T)$ contains $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} A & A^* \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ since A is positive. Hence, if $X_k = \sum_{n=1}^k a_n^{-1} \otimes E_{nn}$ then

$$\begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U_k \\ 0 & 0 \end{pmatrix}$$

is in $C^*(T)$ for each k . Finally, as in previous proofs, it is clear that the 2×2 matrices over $A \otimes K$ are generated by the set

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & U_k \\ 0 & 0 \end{pmatrix}, k = 1, 2, \dots \right\}$$

and the proof is complete.

Recall that a uniformly hyperfinite (UHF) algebra of type $\{p_n\}$ is a C^* -algebra U , which is the closure of the union of an increasing sequence $M_1 \subset M_2 \subset \dots$ of C^* -subalgebras containing 1_U , where each M_n is isomorphic to M_{p_n} . For each n , p_n divides p_{n+1} [7].

THEOREM 9. *Let A be a separable C^* -algebra with identity and let U be a UHF algebra. Then $A \otimes U$ is singly generated.*

PROOF. We will give a detailed proof only when U is of type $\{2^n\}$. In the general case, we exhibit the generator.

Since any two UHF algebras having the same type are isomorphic [7], it suffices to construct a convenient representation U of a type $\{2^n\}$ algebra and then exhibit a generator for $A \otimes U$. To this end, let H be a separable, infinite-dimensional Hilbert space, and let E_{11}, E_{12} be infinite-rank projections such that $E_{11}E_{12} = 0, E_{11} + E_{12} = I$. Choose a partial isometry U_1 such that $U_1U_1^* = E_{11}, U_1^*U_1 = E_{12}$. Now choose infinite-rank projections E_{21} and E_{22} such that $E_{21}E_{22} = 0$ and $E_{21} + E_{22} = E_{12}$, and a partial isometry U_2 such that $U_2U_2^* = E_{21}$ and $U_2^*U_2 = E_{22}$. Continuing, we obtain partial isometries U_3, U_4, \dots . Let U be the C^* -subalgebra of $B(H)$ generated by U_1, U_2, \dots ; then U is a UHF algebra of type $\{2^n\}$ [15, p. 81].

The algebra A is generated by a countable set $\{a_1, a_2, \dots\}$ of selfadjoint elements. As in previous proofs, we can assume that each A_j is positive and invertible, and that $\sigma(a_n) \subset [2^{-2n-1}, 2^{-2n}]$ for each n . Our generator for $A \otimes U$ is

$$T = \sum_{n=1}^{\infty} [(a_n \otimes U_n U_n^*) + (2^{-n} 1_A \otimes U_n)];$$

this series is absolutely convergent since

$$\|(a_n \otimes U_n U_n^*) + (2^{-n} 1_A \otimes U_n)\| \leq \|a_n\| \|U_n U_n^*\| + 2^{-n} \|U_n\| \leq 2^{-2n} + 2^{-n}.$$

The element T is like the infinite operator matrix of Figure 1, although to interpret this literally as a representation of T would lead to confusion.

0			

FIGURE 1

In order to show that T generates $A \otimes U$, it suffices to show that $C^*(T)$ contains $1_A \otimes U_i$ and $a_j \otimes 1_A$ for each i, j .

Now, $P_1 = 1_A \otimes U_1 U_1^*$ and $Q_1 = 1_A \otimes U_1^* U_1$ are orthogonal projections in $A \otimes U$ with sum $1_{A \otimes U}$. Furthermore, $P_1 T P_1 = a_1 \otimes U_1 U_1^*$, $P_1 T Q_1 = 2^{-1} 1_A \otimes U_1$ and $Q_1 T P_1 = 0$. In effect, T is "upper triangular" relative to the decomposition $1_{A \otimes U} = P_1 + Q_1$. This implies that

$$\begin{aligned} \sigma(T) &\subset \sigma(a_1 \otimes U_1 U_1^*) \cup \sigma(Q_1 T Q_1) \\ &\subset [2^{-3}, 2^{-2}] \cup \sigma(Q_1 T Q_1). \end{aligned}$$

Now

$$Q_1 T Q_1 = \sum_{n=2}^{\infty} [(a_n \otimes U_n U_n^*) + (2^{-n} 1_A \otimes U_n)]$$

and $P_2 = 1_A \otimes U_2 U_2^*$ and $Q_2 = 1_A \otimes U_2^* U_2$ are orthogonal projections in $A \otimes U$, with sum $Q_1 = 1_A \otimes U_1^* U_1$. Furthermore,

$$\begin{aligned} P_2 Q_1 T Q_1 P_2 &= P_2 T P_2 = a_2 \otimes U_2 U_2^*, \\ Q_2 Q_1 T Q_1 P_2 &= Q_2 T P_2 = 0, \end{aligned}$$

so that

$$\begin{aligned} \sigma(Q_1 T Q_1) &\subset \sigma(a_2 \otimes U_2 U_2^*) \cup \sigma(Q_2 T Q_2) \\ &\subset [2^{-5}, 2^{-4}] \cup \sigma(Q_2 T Q_2). \end{aligned}$$

Thus

$$\sigma(T) \subset [2^{-3}, 2^{-2}] \cup [2^{-5}, 2^{-4}] \cup \sigma(Q_2 T Q_2).$$

If we define $P_n = 1_A \otimes U_n U_n^*$, $Q_n = 1_A \otimes U_n^* U_n$ for each n and continue in this way, we get

$$\sigma(T) \subset \bigcup_{k=1}^n [2^{-2k-1}, 2^{-2k}] \cup \sigma(Q_n T Q_n).$$

Since $\|Q_n T Q_n\| \rightarrow 0$, we have

$$\sigma(T) \subset \bigcup_{k=1}^{\infty} [2^{-2k-1}, 2^{-2k}] \cup \{0\}.$$

We can also conclude that, for each n ,

$$\sigma(Q_n T Q_n) \subset \bigcup_{k>n}^{\infty} [2^{-2k-1}, 2^{-2k}] \cup \{0\} \subset [0, 2^{-2k}].$$

Now, since $T = a_1 \otimes U_1 U_1^* + 2^{-1} 1_A \otimes U_1 + Q_1 T Q_1$, direct computation shows that, for any polynomials q ,

$$q(T) = q(a_1) \otimes U_1 U_1^* + b \otimes U_1 + q(Q_1 T Q_1),$$

where this is also “upper triangular”, and b is some element of A . Choose a sequence $\{q_n\}$ of polynomials such that $q_n \rightarrow 1$ uniformly on a neighborhood of $[2^{-3}, 2^{-2}]$ and such that $q_n \rightarrow 0$ uniformly on a neighborhood of $[0, 2^{-4}]$.

Then $q_n(T)$ and $q_n(a_1)$ converge to elements of $A \otimes U$ and U respectively, and $q_n(Q_1 T Q_1)$ converges to zero. Thus

$$q_n(T) \rightarrow S = 1_A \otimes U_1 U_1^* + c \otimes U_1$$

in $C^*(T)$, for some $c \in A$. Then $SS^* = (1_A + cc^*) \otimes U_1 U_1^*$, so that $C^*(SS^*)$ contains $1_A \otimes U_1 U_1^*$. Hence $C^*(T)$ also contains $a_1 \otimes U_1 U_1^*$, $1_A \otimes U_1$ and $Q_1 T Q_1$. Therefore $C^*(T)$ contains

$$a_1 \otimes U_1^* U_1 = (1_A \otimes U_1^*)(a_1 \otimes U_1 U_1^*)(1_A \otimes U_1)$$

and

$$a_1 \otimes 1_A = a_1 \otimes U_1^* U_1 + a_1 \otimes U_1 U_1^*.$$

We now apply the same analysis to $Q_1 T Q_1$ to conclude that $C^*(T)$ contains $1_A \otimes U_2, a_2 \otimes U_1^* U_1 = a_2 \otimes U_2^* U_2 + a_2 \otimes U_2 U_2^*$, and $Q_2 T Q_2$. But then $C^*(T)$ contains

$$a_2 \otimes U_1 U_1^* = (1_A \otimes U_1)(a_2 \otimes U_1^* U_1)(1_A \otimes U_1^*),$$

so $a_2 \otimes 1_A \in C^*(T)$. To complete the proof, we need only analyze $Q_2 T Q_2, Q_3 T Q_3, \dots$ in turn.

For the general case, we construct a UHF algebra of type $\{p_n\}$. On a separable Hilbert space H , choose infinite-rank projections $E_{11}, E_{12}, \dots, E_{1p_1}$ such that $E_{1j} E_{1k} = 0$ if $j \neq k$ and $\sum E_{ij} = I$. Choose partial isometries $U_{12}, U_{13}, \dots, U_{1p_1}$ such that $U_{1j}^* U_{1j} = E_{1j}$ and $U_{1j} U_{1j}^* = E_{11}$ for each j . Set $q_2 = p_2/p_1$ and choose infinite-rank projections E_{21}, \dots, E_{2q_2} such that $E_{2j} E_{2k} = 0$ if $j \neq$

k and $\Sigma E_{2j} = E_{1p_1}$. Choose partial isometries $U_{22}, U_{23}, \dots, U_{2q_2}$ such that $U_{2j}^* U_{2j} = E_{2j}$ and $U_{2j} U_{2j}^* = E_{21}$ for each j . Continuing in this fashion, we obtain a family $\{U_{ij}\}$ of partial isometries where $1 \leq i < \infty$, $1 < j \leq q_i$ ($q_1 = p_1$). Then the C^* -subalgebra U of $B(H)$ generated by $\{U_{ij}\}$ is a UHF algebra of type $\{p_n\}$. Let $\{a_{ij}: 1 \leq i < \infty, 1 \leq j < q_i\}$ be a set of positive, invertible elements that generates A , and with $\sigma(a_{ij}) \subset [2^{-2i-2j-1}, 2^{-2i-j}]$, each i, j . Then our generator for $A \otimes U$ is

$$T = \sum_{\substack{1 \leq i < \infty \\ 1 \leq j < q_i}} (a_{ij} \otimes E_{ij}) + \sum_{\substack{1 \leq i < \infty \\ 1 \leq j < q_i}} (2^{-i-j} 1_A \otimes U_{ij}).$$

We omit the proof that $C^*(T) = A \otimes U$.

REMARK 10. The existence of a single generator for a UHF algebra was established by D. Topping [14]. For $A = C$, Theorem 10 yields an explicit construction of such a generator; as far as we know, this is the first explicit construction of such a generator.

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