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THE QUASI-ORBIT SPACE OF CONTINUOUS C^*-DYNAMICAL SYSTEMS

BY

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ABSTRACT. Let (A, G, α) be a separable continuous C^*-dynamical system. Suppose G is amenable and α is free on the dual $\hat{A}$ of A. Then the quasi-orbit space $(\operatorname{Prim} A/\alpha)^\sim$ of the primitive ideal space $\operatorname{Prim} A$ of A by α is homeomorphic to the induced primitive ideal space which is dense in the primitive ideal space $\operatorname{Prim} C^*(A; \alpha)$ of the C^*-crossed product $C^*(A; \alpha)$ of A by α.

1. **Introduction.** The theory of crossed products of operator algebras has been developed by a number of people since von Neumann constructed the examples of factors. Among its various studies, Effros-Hahn [3] topologically characterized under certain conditions the primitive ideal space of C^*- crossed products as the quasi-orbit space of transformation groups for separable abelian discrete C^*-dynamical systems. Succeedingly, Zeller-Meier [7] generalized their result in separable discrete C^*-dynamical systems. However, several different aspects come out in the continuous case. For instance, the original C^*-algebra is never imbedded in its crossed product as a sub C^*-algebra, therefore it has no associated conditional expectations from the crossed product. Moreover, it is unclear that any element in crossed products can be described as a Fourier expansion in certain fashion.

In this paper, we shall discuss the quasi-orbit space of separable continuous C^*-dynamical systems, specifically, given a separable continuous C^*-dynamical system (A, G, α). Suppose G is amenable and α is free on the dual $\hat{A}$ of A. Then the quasi-orbit space $(\operatorname{Prim} A/\alpha)^\sim$ of the primitive ideal space $\operatorname{Prim} A$ of A by α is homeomorphic to the induced primitive ideal space $\{(\operatorname{Ind} \rho)^{-1}(0)\}_{\rho \in \hat{A}}$ which is dense in the primitive ideal space $\operatorname{Prim} C^*(A, \alpha)$ of the C^*-crossed product $C^*(A, \alpha)$ of A by α. This can be considered as a generalization of Zeller-Meier's result.

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2. **Preliminaries.** According to Effros-Hahn [3], Dang-Ngoc-Guichardet [2] Takai [5], [6], and Zeller-Meier [7], we shall briefly review several basic notions and fundamental results which will be used later. A triple (A, G, α) of a C^*-algebra A, a locally compact group G, and a *-homomorphism α of G into $\mathrm{Aut}(A)$ is called a *continuous C^*-dynamical system* if α is pointwise norm continuous, where $\mathrm{Aut}(A)$ is the set of all *-automorphisms of A. It is said to be *separable* if A and G are separable. Moreover, the action α is said to be *free* on a subset S of the representation space $\mathrm{Rep}\,A$ of A if given a $g \neq e \in G$, and a $\rho \neq 0 \in S$, any subrepresentation $\rho' \neq 0$ of ρ dominates a representation $\rho'' \neq 0$ with $\alpha_g \cdot \rho'' \mathrel{\delta} \rho''$, where $(\alpha_g \cdot \rho'')(a) = \rho'' \circ \alpha_g^{-1}(a)$ for $a \in A, g \in G$. Let $\mathrm{Fac}\,A$ be the set of all factor representations of A. Since any nonzero subrepresentation of ρ is quasi-equivalent to ρ if $\rho \in \mathrm{Fac}\,A$, freeness of α on S for a subset S of $\mathrm{Fac}\,A$ is equivalent to saying that $\alpha_g \cdot \rho \mathrel{\delta} \rho$ for every $\rho \neq 0 \in S$. Furthermore, due to the fact that freeness is congruent with quasi-equivalence (resp. equivalence), it can be defined on the quasi-dual $\hat{A}$ (resp. the dual $\hat{A}$) of A.

Now let (A, G, α) be a continuous C^*-dynamical system. According to Effros-Hahn [3], and Zeller-Meier [7], the quasi-orbit space $(\mathrm{Prim}\,A/\alpha)^\sim$ of the primitive ideal space $\mathrm{Prim}\,A$ of A by α is defined by the set of all orbit closures in $\mathrm{Prim}\,A$. It can be identified with the quotient space of $\mathrm{Prim}\,A$ by the following equivalence relation: $p \sim q$ if and only if $(G \cdot p)^- = (G \cdot q)^-$, where $(G \cdot p)^-$ is the closure of the orbit of $p \in \mathrm{Prim}\,A$ by α. Then, it is easily seen that $(\mathrm{Prim}\,A/\alpha)^\sim$ is a T_0-space with respect to the quotient topology of $\mathrm{Prim}\,A$. Define a Borel structure on $(\mathrm{Prim}\,A/\alpha)^\sim$ by the quotient map of $\mathrm{Prim}\,A$. Then, it is saturated with the quotient topology. Suppose A is separable, each element of $(\mathrm{Prim}\,A/\alpha)^\sim$ is Borel since $(\mathrm{Prim}\,A/\alpha)^\sim$ is a T_0-space (cf. [3]).

The following lemma has the key role in finding a mapping from the primitive ideal space $\mathrm{Prim}\,C^*(A, \alpha)$ of the crossed product $C^*(A; \alpha)$ into $(\mathrm{Prim}\,A/\alpha)^\sim$:

LEMMA 2.1. *Let (A, G, α) be a separable continuous C^*-dynamical system. Suppose there exists a projection valued measure on* $(\mathrm{Prim}\,A/\alpha)^\sim$ *whose values are zero or identity operators. Then it is concentrated in one point* (*cf.* [3]). *Notice that it is unnecessary to assume the separability of G to show the above lemma.*

Next, we shall define the C^*-crossed product (resp. the reduced C^*-crossed product) $C^*(A; \alpha)$ (resp. $C_r^*(A; \alpha)$) of C^*-algebra A by a continuous action α of a locally compact group as follows: Let $L^1(A; G)$ be the set of all A-valued Bochner integrable measurable functions on G with respect to the left Haar measure

α_g of G with the *-algebraic structure given by

$$(xy)(g) = \int_G x(h)\,\alpha_h[y(h^{-1}g)]\,dh$$

and

$$x^*(g) = \Delta(g)^{-1}\alpha_g[x(g^{-1})]^*,$$

where Δ is the modular function of G. Then, $C^*(A;\alpha)$ (resp. $C_r^*(A;\alpha)$) is the completion of $L^1(A;\alpha)$ with respect to the C^*-norm $\|\cdot\|$ (resp. $\|\cdot\|_r$) defined by

$$\|x\| = \sup\{\|\pi(x)\| : \pi \in \operatorname{Rep} L^1(A;G)\}$$

and

$$\|x\|_r = \sup\{\|(\operatorname{Ind}\rho)(x)\| : \rho \in \operatorname{Rep} A\},$$

where $\operatorname{Ind}\rho$ is the representation of $C^*(A;\alpha)$ induced by ρ (cf. [5], [6]). It is clear that $C_r^*(A;\alpha)$ is a quotient algebra of $C^*(A;\alpha)$. Let us define a positive linear functional $\widetilde{\varphi}_f$, on $C^*(A;\alpha)$ determined by

$$\widetilde{\varphi}_f(x) = \iint_{G\times G} f(g^{-1}h)\overline{f(h)}\varphi \circ \alpha_h^{-1}[x(g)]\,dg\,dh$$

for $x \in L^1(A;G)$, where φ is in the set $(A^*)_+$ of all positive linear functionals on A, and f is in the *-algebra $K(G)$ of all continuous functions on G with compact support. Then, one can estimate $\|(\operatorname{Ind}\rho)(x)\|$ by $\widetilde{\varphi}_f$ as follows:

$$\|(\operatorname{Ind}\rho)(x)\| = \sup\left\{\frac{\widetilde{\varphi}_f(y^*x^*xy)^{1/2}}{\widetilde{\varphi}_f(y^*y)^{1/2}} : y \in L^1(A;G), f \in K(G), \varphi \in \Omega(y;f) \cap W(\rho)\right\} \tag{2.1}$$

where $\Omega(y;f) = \{\varphi \in (A^*)_+ : \widetilde{\varphi}_f(y^*y) \neq 0\}$, and $w(\rho) = \{\varphi \in (A^*)_+ : \pi_\varphi \in \operatorname{Rep} A$ is weakly contained in $\rho\}$ (cf. [5]).

We now state a sufficient condition under which $C^*(A;\alpha)$ is equal to $C_r^*(A;\alpha)$.

LEMMA 2.2. *Let (A, G, α) be a continuous C^*-dynamical system. Suppose G is amenable as a topological group, then we have*

$$C^*(A;\alpha) = C_r^*(A;\alpha) \quad (cf.\ [5], [6]).$$

Moreover, we present a condition under which Ind ρ *is faithful on* $C_r^*(A;\alpha)$.

LEMMA 2.3. *Let (A, G, α) be as in the preceding, and ρ in* Rep A. *Then the following conditions are equivalent*:

(I) $\Sigma^{\oplus}_{g\in G}\alpha_g \cdot \rho$ *is faithful on* A,

(II) Ind ρ *is faithful on* $C_r^*(A;\alpha)$ (*cf.* [5]).

By the above lemma, one knows without difficulty that faithfulness of ρ implies that of Ind ρ.

Finally, we shall write down a couple of special notations for later use. Let (A, G, α) be a continuous C^*-dynamical system. Then given $X \in C^*(A; \alpha)$, $a \in A$, there exists an element $aX \in C^*(A; \alpha)$, with $(aX)(g) = aX(g)$ if $X \in L^1(A; G)$. Let $\mathcal{I}$ be an ideal of $C^*(A; \alpha)$ where "ideal" involves closed and two-sided throughout this paper. Denote by $[A : \mathcal{I}]$ the set of all elements a in A such that $aX \in \mathcal{I}$ for all $X \in C^*(A; \alpha)$. It is not evident whether $[A : \mathcal{I}]$ is an ideal of A. However, the following observation tells us that it is affirmative:

LEMMA 2.4. *Let (A, G, α) be a continuous C^*-dynamical system, and $\pi \in$* Rep $C^*(A; \alpha)$. *Suppose ρ is in* Cov-rep A *corresponding to π, then we have*

$$[A : \pi^{-1}(0)] = \rho^{-1}(0) \quad (cf.\ [3])$$

where Cov-rep A *is the set of all covariant representations of A, and $\pi^{-1}(0)$ (resp. $\rho^{-1}(0)$) is the kernel of π (resp. ρ). From this fact, $[A : \mathcal{I}]$ for an ideal $\mathcal{I}$ of $C^*(A; \alpha)$ is an invariant ideal of A under the action α. (It is called α-invariant.)*

3. **The quasi-orbit space of continuous C^*-dynamical systems.** In this section, we shall discuss the topological relation between the quasi-orbit space of continuous C^*-dynamical systems and the primitive ideal space of C^*-crossed products under certain conditions. Let (A, G, α) be a separable continuous C^*-dynamical system. Given a π in Fac $C^*(A; \alpha)$, there exists a unique ρ in Cov-rep A corresponding to π. Due to Glimm [4], there exists a unique projection valued measure μ_ρ on the Borel structure of Prim A such that

$$\mu_\rho(\overline{U}) = \mathrm{Proj}[\rho(k(\overline{U}^c))\mathfrak{H}_\rho] \tag{3.1}$$

for every open set $\overline{U}$ in Prim A, where Proj means projection and $\overline{U}^c$ is the complement of $\overline{U}$ in Prim A, and $k(\overline{U}^c)$ is the kernel of $\overline{U}^c$. Moreover, the range of μ_ρ is contained in the center z_ρ of the von Neumann algebra $\rho(A)^-$ generated by $\rho(A)$. Since π is primary and ρ is covariant, the measure μ_ρ is ergodic. Namely, the values of α-invariant Borel sets under μ_ρ are zero or identity operators on $\mathfrak{H}_\rho$. Let η be the natural mapping from Prim A onto $(\mathrm{Prim}\, A/\alpha)^\sim$, and let ν_ρ be the image of μ_ρ by η. Then the range of ν_ρ is zero or identity operators on $\mathfrak{H}_\rho$. By Lemma 2.1, there exists an element $\eta(p)$ in $(\mathrm{Prim}\, A/\alpha)^\sim$ at which ν_p is concentrated. We claim that $(G \cdot p)^-$ is the support Supp μ_ρ of μ_ρ. In fact, since $\eta^{-1} \circ \eta(p) \subset (G \cdot p)^-$, it follows that Supp $\mu_\rho \subset (G \cdot p)^-$. By the fact that Supp μ_ρ is α-invariant closed in Prim A, it contains $(G \cdot p)^-$, or must be disjoint from $\eta^{-1} \circ \eta(P)$. However, the latter case does not occur since μ_ρ is concentrated in $\eta^{-1} \circ \eta(P)$.

Consequently, one gets $\operatorname{Supp}\mu_\rho = (G\cdot p)^-$. Moreover, it can be verified that $\operatorname{Supp}\mu_\rho = h(\rho^{-1}(0))$ where $h(\rho^{-1}(0))$ is the hull of $\rho^{-1}(0)$. Actually, it is clear by definition that $h(\rho^{-1}(0))^c$ is an open null set with respect to μ_ρ. Therefore, $\operatorname{Supp}\mu_\rho \subset h(\rho^{-1}(0))$. Suppose $\operatorname{Supp}\mu_\rho$ is not equal to $h(\rho^{-1}(0))$. Then, there exists an open neighborhood $\bar{U}$ of an element q in $h(\rho^{-1}(0))$ from which $\operatorname{Supp}\mu_\rho$ is disjoint. Hence, it follows that $k(\bar{U}^c) \subset \rho^{-1}(0)$. Taking hull, one obtains that $\bar{U}$ is disjoint from $h(\rho^{-1}(0))$, which is a contradiction. Summing up the argument discussed above, we have the following lemma:

LEMMA 3.1. *Let (A, G, α) be a separable continuous C^*-dynamical system, and let π be in* $\operatorname{Fac} C^*(A;\alpha)$. *Suppose ρ is in* Cov-rep A *corresponding to π. Then there exist a unique (up to equivalence) projection valued Borel measure μ_ρ on* Prim A, *and an element P in* Prim A *such that*

$$\operatorname{Supp}\mu_\rho = (G\cdot p)^- = h(\rho^{-1}(0)).$$

REMARK. The measure μ_ρ also can be realized in the following way: Let $\rho = \int_{\hat{A}}^{\oplus}\rho_\zeta\,d\nu(\zeta)$ be the central decomposition of ρ. Using the canonical mapping ϕ from $\hat{A}$ onto Prim A, one gets the image μ of ν under ϕ. It is nothing but μ_ρ up to equivalence.

By Lemmas 2.4 and 3.1, we can define a mapping Φ of Prim $C^*(A;\alpha)$ into $(\operatorname{Prim} A/\alpha)^\sim$ by

$$\Phi(\mathfrak{P}) = h([A:\mathfrak{P}]). \tag{3.2}$$

Then, we have the following:

LEMMA 3.2. *The mapping Φ defined above is continuous from* Prim $C^*(A;\alpha)$ *with the hull kernel topology into* $(\operatorname{Prim} A/\alpha)^\sim$ *with the quotient topology.*

PROOF. Though the proof goes on as in the abelian case, we shall trace it for completeness. Given a closed set F in $(\operatorname{Prim} A/\alpha)^\sim$, it suffices to show that there exists a subset $\mathfrak{I}$ of $C^*(A;\alpha)$ such that $\Phi^{-1}(F) = h(\mathfrak{I})$.

Define a set $\mathfrak{I}$ by $\{aX: a\in k(F_1), X\in C^*(A;\alpha)\}$ where $F_1 = \eta^{-1}(F)$. Then, this set enjoys the property cited above. In fact, for any $\mathfrak{P}\in\Phi^{-1}(F)$, there exists a π in Irr $C^*(A;\alpha)$ whose kernel is $\mathfrak{P}$. Let ρ be in Cov-rep A corresponding to π. By Lemma 3.1, there exists a P in Prim A such that $h([A:\mathfrak{P}]) = (G\cdot p)^-$. Since $\Phi(\mathfrak{P})\in F$, it follows by (3.2) that $p\in\eta^{-1}(F) = F_1$. Since F_1 is α-invariant closed in Prim A, it contains $(Gp)^-$, which means that $h([A:\mathfrak{P}]) = F_1$. Taking the kernel, one gets that $k(F_1)\subset [A:\mathfrak{P}] = \rho^{-1}(0)$. Since $\pi(ax) = \rho(a)\pi(x)$ for $a\in A$ and $X\in C^*(A;\alpha)$, it is verified that $\mathfrak{I}\subset\pi^{-1}(0) = \mathfrak{P}$. Thus, one obtains that $\mathfrak{P}\in h(\mathfrak{I})$. Since one can trace the above argument back until its starting point, one gets the conclusion. Q.E.D.

From now on, let us assume that the action α is free on $\hat{A}$. Then, the next lemma guarantees that Φ is surjective:

LEMMA 3.3. *Suppose α is free on $\hat{A}$. Then, one has for every $\rho \in \operatorname{Irr} A$,*
(i) $\operatorname{Ind} \rho \in \operatorname{Irr} C^*(A; \alpha)$, *and*
(ii) $\Phi[(\operatorname{Ind} \rho)^{-1}(0)] = (G \cdot \rho^{-1}(0))^-$,
where Irr *means the set of all irreducible representations.*

PROOF. (i) Let $G_{\hat{\rho}}$ be the stabilizer of the equivalence class $\hat{\rho} \in \hat{A}$ of ρ, and let $(\bar{\rho}, \lambda)$ be in Cov-rep(A, G) corresponding to $\operatorname{Ind} \rho$. Let $\bar{U}$ be a strongly measurable cocycle representation of $G_{\hat{\rho}}$ such that $\bar{U}_g^* \rho(a) \bar{U}g = \alpha_g \cdot \rho(a)$ for $a \in A$, $g \in G_{\hat{\rho}}$. Define $\tilde{U}g, g \in G_{\hat{\rho}}$, by

$$(\tilde{U}g\xi)(h) = \Delta(\rho)^{1/2} \bar{U}g\xi(hg) \quad \text{for } \xi \in L^2(\mathfrak{h}_\rho; G),\ h \in G.$$

Due to Busby-Smith [1], it can be shown that the commutant $\bar{\rho}(A)'$ of $\bar{\rho}$ is generated by the $\tilde{U}g, g \in G_{\hat{\rho}}$, and the diagonalizable algebra. Since α is free on $\hat{A}$, it follows that $\bar{\rho}(A)'$ is the diagonal algebra. Since the only diagonals which commute with λ are the constant, one gets the statement (i).

(ii) By definition, it suffices to show that $h(\bar{\rho}^{-1}(0)) = (G \cdot \rho^{-1}(0))^-$. Since $\bar{\rho}^{-1}(0) = \bigcap_{g \in G} (\alpha_g \cdot \rho)^{-1}(0)$, it follows that

$$h(\bar{\rho}^{-1}(0)) = h\left(\bigcap_{g \in G} (\alpha_g \cdot \rho)^{-1}(0)\right) \supset \left(\bigcup_{g \in G} h[(\alpha_g \cdot \rho)^{-1}(0)]\right)^-,$$

which implies that $(G \cdot \rho^{-1}(0))^- \subset h(\bar{\rho}^{-1}(0))$. On the other hand, let $\mu_{\bar{\rho}}$ be the Borel measure determined by $\bar{\rho}$ as (3.1). Then one gets by simple computation that $\operatorname{Supp} \mu_{\bar{\rho}} \subset (G \cdot \rho^{-1}(0))^-$. Thus, the statement (ii) follows by Lemma 3.1. Q.E.D.

In general, it is not clear whether the mapping Φ is injective. However, one can get a partial answer under the amenability of G. Let $\mathcal{E}$ be the set of all $(\operatorname{Ind} \rho)^{-1}(0)$, $\rho \in \operatorname{Irr} A$. Then the following lemma tells us that Φ is injective on $\mathcal{E}$:

LEMMA 3.4. *Let (A, G, α) be a separable continuous C^*-dynamical system, and let α be free on $\hat{A}$. Suppose G is amenable as a topological group. Then one has for every $\rho \in \operatorname{Irr} A$,*

$$(\operatorname{Ind} \rho)^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha)$$

where $\bar{\rho} \in$ Cov-rep A corresponding to $\operatorname{Ind} \rho$.

PROOF. By definition, one easily gets that $(\operatorname{Ind} \rho)^{-1}(0) \supset C^*(\bar{\rho}^{-1}(0); \alpha)$. To show the converse inclusion, one first observes that $C^*(\bar{\rho}^{-1}(0); \alpha)$ is a primitive ideal of $C^*(A; \alpha)$. In fact, one estimates by Lemmas 2.2 and 2.3 that

$$C^*(A;\alpha)/(\mathrm{Ind}\,\rho)^{-1}(0) \cong (\mathrm{Ind}\,\rho)[C^*(A;\alpha)]$$
$$\cong (\mathrm{Ind}\,\iota)[C_r^*(\bar\rho(A);\bar\alpha)] \cong C_r^*(\bar\rho(A);\bar\alpha) = C^*(\bar\rho(A);\bar\alpha)$$
$$\cong C^*(A/\bar\rho^{-1}(0);\tilde{\alpha}) \cong C^*(A;\alpha)/C^*(\bar\rho^{-1}(0);\alpha),$$

where $\bar\alpha$ (resp. $\tilde\alpha$) is the continuous action of G on $\bar\rho(A)$ (resp $A/\bar\rho^{-1}(0)$) determined by $\bar\rho$ (resp. the natural map of A onto $A/\bar\rho^{-1}(0)$), and ι is the trivial automorphism of $\bar\rho(A)$. Therefore, there exists a π in Irr $C^*(A;\alpha)$ with $\pi^{-1}(0) = C^*(\bar\rho^{-1}(0);\alpha)$. Since every nonzero vector ξ in $\mathfrak{H}_\pi$ is cyclic for π, it suffices to show that $\pi_\xi^{-1}(0) \supset (\mathrm{Ind}\,\rho)^{-1}(0)$ for π_ξ corresponding to $\omega_\xi \circ \pi$. Let (L, ∇) be in Cov-rep(A, G) associated with π. Then, one can see by Lemma 2.4 and $\pi^{-1}(0) = C^*(\bar\rho^{-1}(0);\alpha)$ that

$$L^{-1}(0) = [A : \pi^{-1}(0)] = \bar\rho^{-1}(0). \tag{3.3}$$

Since $\bar\rho^{-1}(0) = (\Sigma^{\oplus}_{g\in G}\alpha_g \cdot \rho)^{-1}(0)$, the equality (3.3) means that L is weakly equivalent to $\Sigma^{\oplus}_{g\in G}\alpha_g \cdot \rho$. On the other hand, the amenability of G gives us a sequence $(f_n)_n$ of $K(G)$ such that $\tilde{f}_n * f_n$ converges to 1 with respect to compact open topology where $\tilde{f}_n(g) = \overline{f_n(g^{-1})}$. Let us take a positive linear functional Ψ_n on $C^*(A;\alpha)$ determined by $\Psi_n(x) = \int_G(\tilde{f}_n * f_n)(g)\langle L[x(g)]\nabla(g)\,\xi|\xi\rangle dg$ for $x \in L^1(A;G)$. Then, it can be shown by [5] that

$$\Psi_n(y^*x^*xy) \leqslant \|y\|^2\,\|\tilde{f}_n \otimes\ \xi\|^2\,\|(\mathrm{Ind}\,L)(x)\|^2 \tag{3.4}$$

for every $x, y \in C^*(A;\alpha)$, and $n = 1, 2, \ldots$. Since L is weakly equivalent to $\Sigma^{\oplus}_{g\in G}\alpha_g \cdot \rho$, it follows by (2.1) that

$$\|(\mathrm{Ind}\,L)(x)\| = \|\mathrm{Ind}(\Sigma^{\oplus}_{g\in G}\alpha_g \cdot \rho)(x)\|. \tag{3.5}$$

Since $\mathrm{Ind}(\Sigma^{\oplus}_{g\in G}\alpha_g \cdot \rho)$ (resp. $\mathrm{Ind}\,\alpha_g \cdot \rho$) is equivalent to $\Sigma^{\oplus}_{g\in G}\mathrm{Ind}\,\alpha_g \cdot \rho$ (resp. $\mathrm{Ind}\,\rho$), the equality (3.5) implies that

$$\|(\mathrm{Ind}\,L)(x)\| = \|\Sigma^{\oplus}_{g\in G}(\mathrm{Ind}\,\alpha_g \cdot \rho)(x)\| = \sup_{g\in G}\ \|(\mathrm{Ind}\,\alpha_g \cdot \rho)(x)\|$$
$$= \|(\mathrm{Ind}\,\rho)(x)\|.$$

Thus, one obtains by (3.4) that

$$\Psi_n(y^*x^*xy) \leqslant \|y\|^2\,\|\tilde{f}_n \otimes\ \xi\|^2\,\|(\mathrm{Ind}\,\rho)(x)\|^2$$

for every $x, y \in C^*(A;\alpha)$ and $n = 1, 2, \ldots$. Now suppose x is in $(\mathrm{Ind}\,\rho)^{-1}(0)$. Then, it follows that $\Psi_n(y^*x^*xy) = 0$ for all $y \in C^*(A;\alpha)$ and $n = 1, 2, \ldots$.

Since Ψ_n converges to $w_\xi \circ \pi$ weakly, $w_\xi \circ \pi(y^*x^*xy) = 0$ for all $y \in C^*(A; \alpha)$, which means that $x \in \pi_\xi^{-1}(0)$. Q.E.D.

By the above lemma, the mapping Φ is bijective from $\mathcal{E}$ onto $(\mathrm{Prim}\, A/\alpha)^\sim$. Combining all the lemmas obtained above, we shall show the following theorem which is a generalization of the result which is obtained by Effros-Hahn, and Zeller-Meier (cf. [3], [7]):

THEOREM 3.5. *Let (A, G, α) be a separable continuous C^*-dynamical system. Suppose G is amenable and α is free on the dual $\hat{A}$ of A. Then, the quasi-orbit space* $(\mathrm{Prim}\, A/\alpha)^\sim$ *of the primitive ideal space* $\mathrm{Prim}\, A$ *of A by α is topologically imbedded in the primitive ideal space* $\mathrm{Prim}\, C^*(A; \alpha)$ *of the C^*-crossed product $C^*(A; \alpha)$ of A by α.*

PROOF. Using the same notations denoted before, we shall show that Φ is a closed mapping from $\mathcal{E}$ onto $(\mathrm{Prim}\, A/\alpha)^\sim$. Let F be a closed set in $\mathcal{E}$. Define F_0 by $\eta^{-1}[\Phi(F)]$. It has to be shown that F_0 is closed in $\mathrm{Prim}\, A$. Take a $p \in F_0$, and an $L \in \mathrm{Irr}\, A$ with $L^{-1}(0) = p$. To prove $p \in F_0$, it suffices to show that

$$(\mathrm{Ind}\, L)^{-1}(0) \supset \bigcap\{(\mathrm{Ind}\, \rho)^{-1}(0) \colon \mathrm{Ind}\, \rho \in F\}. \tag{3.6}$$

Since $p \in F_0^-$, it follows that

$$\begin{aligned} L^{-1}(0) &\supset \bigcap\{\eta^{-1}[(G \cdot \rho^{-1}(0))^-] \colon \mathrm{Ind}\, \rho \in F\} \\ &\supset \bigcap\{(\alpha_g \cdot \rho)^{-1}(0) \colon \mathrm{Ind}\, \rho \in F, g \in G\}, \end{aligned}$$

which implies that L is weakly contained in the set $\{\alpha_g \cdot \rho \colon \mathrm{Ind}\, \rho \in F, g \in G\}$. By the similar method used in the previous lemma, one has that

$$\begin{aligned} \|(\mathrm{Ind}\, L)(x)\| &\leqslant \sup\{\|(\mathrm{Ind}\, \alpha_g \cdot \rho)(x)\| \colon \mathrm{Ind}\, \rho \in F, g \in G\} \\ &= \sup\{\|(\mathrm{Ind}\, \rho)(x)\| \colon \mathrm{Ind}\, \rho \in F\} \end{aligned}$$

for all $x \in C^*(A; \alpha)$. Therefore, the inclusion (3.8) holds. Thus, the mapping Φ is a homeomorphism from $\mathcal{E}$ onto $(\mathrm{Prim}\, A/\alpha)^\sim$. Q.E.D.

Since $k(\mathcal{E}) = (0)$, it follows by the above theorem that $(\mathrm{Prim}\, A/\alpha)^\sim$ is dense in $\mathrm{Prim}\, C^*(A; \alpha)$. Moreover, it is true that $\mathrm{Prim}\, C^*(A; \alpha) = \bigcup\{\mathfrak{P}^- \colon \mathfrak{P} \in \mathcal{E}\}$. In fact, let $\pi^{-1}(0) \in \mathrm{Prim}\, C^*(A; \alpha)$ where $\pi \in \mathrm{Irr}\, C^*(A; \alpha)$. Taking $L \in$ Cov-rep A corresponding to π, there exists a ρ in $\mathrm{Irr}\, A$ such that $\mathrm{Supp}\, \mu_L =$ $(G \cdot \rho^{-1}(0))^-$. Then, it follows that $L^{-1}(0) = \bar{\rho}^{-1}(0)$. By Lemma 3.4, one has $(\mathrm{Ind}\, \rho)^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha)$. Therefore, one obtains that $(\mathrm{Ind}\, \rho)^{-1}(0) \subset \pi^{-1}(0)$, which means that $\pi^{-1}(0) \in \{(\mathrm{Ind}\, \rho)^{-1}(0)\}^-$.

Summing up the argument, we get the following:

COROLLARY 3.6. *Let (A, G, α) be as the preceding. Then* $(\mathrm{Prim}\, A/\alpha)^\sim$ *is dense in* $\mathrm{Prim}\, C^*(A; \alpha)$ *and*

$$\operatorname{Prim} C^*(A;\alpha) = \bigcup\{\mathfrak{P}^-:\ \mathfrak{P} \in (\operatorname{Prim} A/\alpha)^\sim\}.$$

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