A PRODUCT FORMULA FOR GENERALIZATIONS OF THE KERVAIRE INVARIANT

BY

EDGAR H. BROWN, JR. (1)

ABSTRACT. Formulas are developed for the Arf invariant of the product of two manifolds in terms of invariants of the factors. If the Wu orientations are carefully chosen the formula is \( \sigma(M \times N) = \sigma(M)\sigma(N) \).

1. Introduction. Recall, if \( M \) is a smooth \( m \)-manifold, \( f: M \to BO \) is the classifying map of its normal bundle and \( v_i \in H^i(BO; \mathbb{Z}/2) \) is the \( i \)th Wu class, \( f_*v_i = 0 \) for \( 2i > m \). Thus if we form a fibration over \( BO \) by killing \( v_i, 2i > m \), \( f \) lifts to the total space of this fibration. We call such a lift a trivialization of \( v_i \) on \( M \).

In \([4]\) we introduced an invariant \( \sigma(M) \in \mathbb{Z}/8 \) \((\mathbb{Z}/8 \cong \mathbb{Z}/2)\) for closed, compact smooth \( 2n \)-manifolds \( M \) with a trivialization of \( v_{n+1} \). Let \( \sigma(M) = 0 \) if \( \text{dim } M \) is odd. \( \sigma \) generalizes various formulations of the Kervaire invariant \(([6], [3], \text{ and } [1])\) and it plays a role analogous to the Hirzebruch index for surgery problems involving manifolds of dimension \( 2n \) where \( n \) is odd. In this paper we give formulas relating \( \sigma(M \times N), \sigma(M) \) and \( \sigma(N) \). In \([2]\), W. Browder gave a special case of our formulas ((4.7) below). Also, D. Sullivan's product formula for surgery obstructions follows from our results ((4.8)). The results of this paper were announced in \([5]\).

To deal with \( \sigma(M \times N) \) we need a product procedure for handling the \( v_{n+1} \) trivializations. In \( \S 2 \) we define a Wu orientation of \( M \) to be (essentially) a trivialization of each of the classes \( v_i \) for \( 2i > \text{dim } M \) and we define the product of two Wu orientations as in \([2]\). In the remainder of this section we assume all manifolds have Wu orientations and \( M \times N \) denotes the cartesian product with the product orientation.

By analogy with the index, one might hope that \( \sigma(M \times N) = \sigma(M)\sigma(N) \). Although this holds if the Wu orientations are carefully chosen (see below), it does not hold in general. The following is an interesting example of when this formula fails. \( \sigma(M) \) is an algebraic invariant associated to a quadratic function \( \varphi: H^m(M) \to \mathbb{Z}/4 \) where \( \text{dim } M = 2n \). (All cohomology is with \( \mathbb{Z}/2 \) coefficient.) \( S^m \) has a nontrivial Wu orientation \( \nu \). Let \( S^m = (S^m, \nu) \). \( \sigma(S^m \times S^m) = 4 \) while \( \sigma(S^m) = 0 \) since \( S^m \) has no middle dimensional cohomology.

To get around this difficulty we extend \( \varphi \), by a homotopy construct, to \( \varphi: H^n(M) \to \Lambda \) where \( \Lambda \) is a rather complicated ring (see \( \S 3 \)) containing \( \mathbb{Z}/4 \). \( \varphi \) has the property that

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\begin{equation}
\varphi(u \otimes v) = \varphi(u)\varphi(v)
\end{equation}

where \( u \otimes v \in H^*(M \times N) \). We then proceed by algebra, as in [4], to associate to \( \varphi \) an invariant \( \Sigma(\varphi) \) essentially lying in the Grothendieck ring of such gadgets. We then define \( \Sigma(M) \) to be \( \Sigma(\varphi) \). \( \Sigma \) satisfies \( \Sigma(M \times N) = \Sigma(M)\Sigma(N) \). Furthermore, \( \Sigma(M) \) is determined by \( \sigma(M) \) and \( \sigma_k(M) \in \mathbb{Z}/2 \), \( k = 1, 2, \ldots \), where \( \sigma_k \) is characterized by \( 4\sigma_k(M) = \sigma(S^k \times M) \) where \( 4: \mathbb{Z}/2 \to \mathbb{Z}/8 \) is the inclusion. In terms of \( \sigma \) and \( \sigma_k \) our product formulas are

\begin{align*}
\sigma(M \times N) &= \sigma(M)\sigma(N) + 4 \left( \sum \sigma_k(M)\sigma_k(n) \right), \\
\sigma_k(M \times N) &= \sigma(M)\sigma_k(N) + \sigma_k(M)\sigma(N).
\end{align*}

In defining product Wu orientations, one chooses liftings of certain maps on universal examples. Furthermore, as in [4], one chooses certain homomorphisms, in the language of [4], \( h_n: \pi_{2n+k}(N \wedge K_n) \to \mathbb{Z}/4 \). In our present treatment, we again choose the homomorphism \( h_n \) and these choices, together with (1.1), determine the above liftings. Having made these choices and thus defined \( \varphi \) for all manifolds, we define a preferred Wu orientation of \( M \) to be a Wu orientation such that \( \varphi(H^p(M)) = 0 \) for \( 2p < \dim M \). By (1.1) the product of preferred orientations is preferred. Every manifold has a preferred orientation unique up to a trivialization of \( v_{n+1} \), \( n = [\dim M/2] \) and \( \sigma_k(M) = 0 \) for \( M \) with a preferred orientation. Thus for such orientations \( \sigma(M \times N) = \sigma(M)\sigma(N) \).

This paper is organized as follows: §2 deals with Wu orientations and their products. In §3, \( \varphi: H^*(M) \to \Lambda \) is constructed, modulo the main work of this paper which consists in computing pairings between certain homotopy groups. These later results are developed in §§5, 6, 7 and 8. In §4, \( \Sigma(M) \) is defined and our results are stated and proved.

Throughout we use smooth manifolds. Our results apply equally well to Poincaré complexes. To formulate the results in terms of Poincaré complexes one replaces smooth manifolds by triples \((X, \eta, \alpha)\) where \( X \) is a Poincaré complex, \( \eta \) is its Spivak normal bundle and \( \alpha \in \pi_*(T(\eta)) \) is a normal invariant.

All homology and cohomology will be with \( \mathbb{Z}/2 \) coefficients. \( K_n \) denotes \( K(\mathbb{Z}/2, n) \). \( l: \mathbb{Z}/k \to \mathbb{Z}/kl \) denotes the inclusion and \( l(n) \) is denoted by \( l \cdot n \). Although we formulate many of our constructions in terms of spectra, we make no very serious use of spectra theory. The main point of this is to reduce the number of indices we have to carry around. Spectra in the sense of [8] is quite adequate for our purposes. All manifolds will be closed, compact and smooth. We will be talking endlessly about \( u \in H^p(M) \) and \( v \in H^q(N) \). \( m \) and \( n \) will denote the dimensions of \( M \) and \( N \) respectively and \( p \) and \( q \) the dimensions of \( u \) and \( v \).

2. Wu Orientations. In this section we essentially reproduce the material on Wu orientations in [2]. Suppose \( pt = B_0 \subset B_1 \subset \cdots \subset B_k \subset \cdots \subset B \) is one of the families of classifying spaces \( BO_k, BSO_k, BPL_k, BG_k, \) etc., and suppose \( \xi_k \) is the universal bundle over \( B_k \). The essential feature that we require is that Whitney sums make sense. Let \( p: B^m \to B \) be the fibration induced from the contractible fibration
by the map

$$
\prod_{2i > m} v_i : B \rightarrow \prod_{2i > m} K_i
$$

where $v_i \in H^{i}(B)$ is the Wu class. We define $p_m : B^m_k \rightarrow B_k$ and $\xi^m_k$ to be the pull back of $p$ by the inclusion $B_k \subset B$ and $\xi^m_k = p_m \star \xi_k$, respectively. Recall, under the Whitney sum map

$$
B_k \times B_l \rightarrow B_{k+l}, \quad v_i \rightarrow \sum v_i \otimes v_{i-j}.
$$

Hence the Whitney sum maps lift to maps $\mu : B^m_k \times B^l_i \rightarrow B_{k+l}^{m+n}$ which in turn give maps $\mu : \xi^m_k \times \xi^n_l \rightarrow \xi_{k+l}^{m+n}$. Let $TB^m$ be the Thom spectra $(T(\xi^m_k))$, $T = \text{Thom}$ space. With a little effort one can choose the maps $\mu$ consistently with respect to $k$ and $l$ so as to give maps $\mu : TB^m \land TB^n \rightarrow TB^{m+n}$. We call the collection, $\{\xi^m_k, \mu\}$ a $B$-Wu system. Note that the maps $\mu$ involve the choice of liftings $\bar{\mu}$.

Suppose $M$ is a manifold. We assume our manifolds come with smooth embeddings, $i : M \subset \mathbb{R}^{m+k}$, $k$ large. Let $v = \nu_M$ be the normal bundle of $i$ and let $T(v)$ denote the Thom spectrum whose $l$th term is $T(v + O^{l-k})$. The Thom-Pontrjagin construction yields an element $\alpha_M \in \pi_m(T(\nu_M))$.

**Definition (2.1).** A $B$-Wu orientation of an $m$-manifold $M$ is a bundle map $V : \nu_M \rightarrow \xi^m_k$. Two such orientations are equivalent if they are homotopic. If $U$ and $V$ are orientations of $M$ and $N$, respectively, $U \times V$ is the orientation on $M \times N$ given by

$$
\nu_{M \times N} = \nu_M \times \nu_N U \times V \xi^m_k \times \xi^n_l \rightarrow \xi_{k+l}^{m+n}.
$$

Since $v_i$ is zero on $M$ for $2i > m$, we have

**Lemma (2.2).** If $\nu_M$ has a $B$-structure, $M$ has a $B$-Wu orientation.

The standard trivialization of the normal bundle of $S^m$ gives a map $V : \nu_{S^m} \rightarrow \xi^m_k$ over the constant map of $S^m$ into $B_k$. Let $g : S^m \rightarrow B^m_k$ represent the element of $\pi_m(B^m_k)$ coming from $K_m$ in the fibre of $B^m_k \rightarrow B_k$. $g$ and $V$ give a $B$-Wu orientation $U$ of $S^m$. Let $S^m = (S^m, U)$.

**Remark.** When $n = 1, 3$ or $7$, $g : S^m \rightarrow B^m_k$ is homotopic to the constant map, but under this homotopy, $U$ is transformed into the nontrivial framing of $\nu_{S^m}$.

(See proof of (4.5).)

**Remark (2.3).** An alternate and technically somewhat simpler approach to Wu orientations may be done as follows: Let $Y^m$ be the spectrum whose $l$th term is the two stage Postnikov system with $k$-invariant $\Pi_{2i > m} \chi(Sq^i) : K_i \rightarrow \Pi K_{i+j}$; $\chi$ is the canonical antiautomorphism of the Steenrod algebra. The cup product maps $K_{l_1} \land K_{l_2} \rightarrow K_{l_1 + l_2}$ lift to maps $\mu : Y^m \land Y^n \rightarrow Y^{m+n}$. $A = \{Y^m, \mu\}$ orientation of an $m$-manifold is a map $V : T(\nu_M) \rightarrow Y^m$ such that the generator of $H^0(Y^m)$ pulls back to the Thom class. Product orientations are defined as above and everything in this paper can be done with $Y^m$ replacing $TB^m$.

3. The main geometric construction. Suppose $M$ is a manifold, $T(\nu_M)$ is its normal bundle Thom spectrum, $\alpha_M \in \pi_m(T(\nu_M))$ is the Thom Pontrjagin element and
$TB^m$ are all as in $\S 2$. Let $V: T(v_M) \to TB^m$ come from a $B$-Wu orientation for $M$.

Let $G_p^m = \pi_m(TB^m) \land K_p$. The cup product map $c: K_p \land K_q \to K_{p+q}$ and $\mu: TB^m \land TB^n \to TB^{m+n}$ define a pairing $G_p^m \otimes G_q^n \to G_{p+q}^{m+n}$ by

$$\alpha \otimes \beta \mapsto (\mu \land c)(\text{id} \land t \land \text{id})(\alpha \land \beta),$$

where $t$ permutes the factors of $K_p \land TB^m$. Let $G$ be the possibly nonassociative, bigraded ring $G = \sum G_p^m$. Denote multiplication by $\alpha \cdot \beta$.

We construct a function $\theta = \theta_V: H^*(M) \to G$ as follows: Recall, there is a diagonal map $\Delta: T(v) \to T(v) \land M^+$. On a vector $x \in v$, $x \mapsto (x, p(x))$ where $p$ is the projection of $v$. If $u \in H^p(M) = [M^+, \mathcal{E}^p]$, $\theta(u) = ((V \land u)\Delta)_* (\alpha_M) \in G_p^m$.

By simply writing down the definitions one verifies

**Lemma (3.1).** If $u \otimes v \in H^*(M \times N)$ and $U$ and $V$ are $B$-Wu orientations of $M$ and $N$, then

$$\theta_{U \times V}(u \otimes v) = \theta_U(u) \cdot \theta_V(v).$$

The main work of this paper is in analyzing $G$. We postpone this to later sections and, at this point, give a theorem ((3.2)) which summarizes the results of this analysis.

Let $\Lambda$ be the polynomial ring over $\mathbb{Z}/4$ in indeterminants $t$ and $\alpha_k$, $k = 0, 1, 2, \ldots$, modulo the relations

$$ta_k = \alpha_{k-1}, \quad \alpha_0 = 2, \quad 2t = 2\alpha_k = \alpha_k\alpha_0 = 0.$$

Suppose $\lambda_p^{2p}: G_p^{2p} \to \mathbb{Z}/4$ and $\lambda_p^m: G_p^m \to \mathbb{Z}/2$, $m \neq 2p$, are homomorphisms. They give an additive map $\lambda: G \to \Lambda$ as follows: If $\beta \in G_p^m$,

$$\lambda(\beta) = \lambda_p^m(\beta) \quad \text{if } m = 2p,$$

$$= \lambda_p^m(\beta)t^{2p-m} \quad \text{if } 2p > m,$$

$$= \lambda_p^m(\beta)\alpha_{m-2p} \quad \text{if } 2p < m.$$

Let $\varphi = \varphi_V: H^*(M) \to \Lambda$ be defined by $\varphi = \lambda \theta$. If we let $\varphi_p^m = \lambda_p^m \theta$,

$$\varphi(u) = \varphi_{m/2}^m(u) + \sum_{2p > m} \varphi_p^m(u) \alpha_{m-2p} + \sum_{2p < m} \varphi_p^m(u) \alpha_{m-2p},$$

$\varphi_{m/2}^m = 0$ if $m$ is not even. Let $\text{Sq} = \text{Sq}^0 + \text{Sq}^1 + \cdots$ be the Steenrod squaring operation. In $\S 8$ we prove

**Theorem (3.2).** Suppose $\{f_k, \mu\}$ is a $B$-Wu system as in $\S 2$ and $\lambda$ is as above. The $\mu$ and $\lambda$ may be chosen so that for all $B$-Wu oriented manifolds $M$, $\varphi_M$ satisfies:

(i) $\varphi_p^m(u) = (\text{Sq} u)(M)$ if $2p > m$,

$$\varphi_p^m(u) = (\text{Sq} u)(M) \mod (2) \text{ if } 2p = m.$$

(ii) $\varphi_p^m(u + v) = \varphi_p^m(u) + \varphi_p^m(v) + 2 \cdot (u \cup v)(M)$ if $2p \leq m$ and $u, v \in H^p(M)$.

(Note $u \cup v)(M) = 0$ if $2p \neq m$.)

(iii) If $f_k$ are oriented bundles, in the usual sense, and hence $M$ has an orientation $\tilde{M} \in H_m(M; \mathbb{Z})$ via its $B$-Wu orientation, and $m = 4l$, then $\varphi^A_2(u) = (\mu u)(\tilde{M})$, where $\mu$ is the Pontrjagin square.
(iv) \( \varphi_{2p}^2 \) is the function giving rise to \( o(M) \) in [4].

(v) If \( u \otimes v \in H^*(M \times N) \), \( \varphi(u \otimes v) = \varphi(u)\varphi(v) \).

**Remark.** The ring \( \Lambda \) arises from dividing \( G \) by the largest ideal possible without losing the quadratic property of \( \varphi \). One may easily check that any proper ideal of \( \Lambda \) intersects \( Z/4 \) nontrivially and hence \( \varphi \) is linear modulo such an ideal.

**Remark.** As we will see in §8, the choice of \( \mu \) and \( \lambda \) are not independent. In fact, the choice of \( \lambda_{2p}^p \) for all \( p \), determines the remaining choices.

We must next examine how \( \varphi \) depends on \( B \)-Wu orientation of \( M \). Suppose \( U_1 \) and \( U_2 : \nu_M \to \xi_k^m \) are two \( B \)-Wu orientations of \( M \) which are equivalent as \( B \) structures, say to \( V : \nu_M \to \xi_k^m \). Then \( U_1 \) is equivalent to \( (f_1, V) \) where \( f_i : M \to B_k^m \) are liftings of \( \nu_M : M \to B_k^m \). \( f_1 \) and \( f_2 \) differ by a map \( x : M \to \Pi_{1 \geq i} K_{i-1} \). Conversely, \( U_1 \) and an \( x \) as above give a \( U_2 \) equivalent to \( U_1 \) as a \( B \) structure. In §9 we prove

**Theorem (3.3).** If \( U_1 \) and \( U_2 \) are \( B \)-Wu orientations of \( M \) which are equivalent as \( B \) structures on \( \nu_M \) and \( x = \{x_i\} \), \( x_i \in H^i(M) \), is as above, then

\[
\varphi_{U_1}(u) - \varphi_{U_2}(u) = 2 \cdot (u \cup x_{[m/2]})(M) + \sum_{i \geq p} (u \cup x_i)(M)\xi_{i-p}
\]

where \( u \in H^p(M) \).

**Definition (3.4).** A \( B \)-Wu orientation \( U \) is preferred if \( \varphi_U(H^p(M)) = 0 \) for \( 2p < m \).

**Corollary (3.5).** If \( V : \nu_M \to \xi_k \) is a \( B \) structure on \( \nu_M \), \( V \) lifts to a preferred \( B \)-Wu orientation on \( M \). The product of preferred orientations is preferred. Two preferred orientations coming from \( V \) differ by a trivialization of \( u_s \), \( s = [m/2] + 1 \).

**Proof.** Suppose \( U_1 \) is any \( B \)-Wu orientation associated to \( V \). By Poincaré duality, if \( 2p < m \), \( \varphi_{U_1}(u) = (u \cup x_{m-p})(M) \) for some \( x_{m-p} \in H^{m-p}(M) \). Change \( U_1 \) by \( x = \{x_i\} \) to form \( U \). By (3.4), \( U \) is preferred. The assertion about products follows from (3.2)(v). Note in the construction of \( U \) we have not used \( x_{[m/2]} \). Uniqueness up to a trivialization of \( u_s \) now follows.

**4. A generalization of the Kervaire invariant.** We first recall some algebra from [4]. Suppose \( V \) is a vector space over \( Z/2 \) and \( \varphi : V \to Z_4 \) is quadratic in the sense that \( \varphi(u + v) - \varphi(u) - \varphi(v) \) is a nonsingular bilinear form with values in \( Z/2 = \{0, 2\} \subset Z/4 \). \( \sigma(\varphi) \in Z/8 \) is defined by the equation

\[
\sum_{u \in V} \varphi(u) = \sqrt{2} \dim V \left( \frac{i + 1}{\sqrt{2}} \right)^{\sigma(\varphi)}
\]

where \( i = \sqrt{-1} \). In [4] it was proved:

**Theorem (4.0).** \( \sigma \) is additive on direct sums of quadratic functions and multiplicative on tensor products. If \( \varphi = 2\psi \), where \( \psi : V \to Z/2 \), is nonsingular quadratic, then \( \sigma(\varphi) = 4 \cdot \text{Arf}(\psi) \). \( \text{Arf} \theta \) is \( \sum_i \psi(\lambda_i)\psi(\mu_i) \) where \( \{\lambda_i, \mu_i\} \) is a symplectic basis for \( V \) with respect to the bilinear form associated to \( \psi \). \( \sigma(\varphi) = \dim V \mod(2) \).
Suppose $M$ and $\varphi: H^*(M) \to \Lambda$ are as in (3.2). We utilize $\sigma$ and $\varphi$ to define invariants $\sigma(M) \in \mathbb{Z}/2$ and $\sigma_k(M) \in \mathbb{Z}/2$, $k = 1, 2, \ldots$. Let

$$V^k(M) = H^m(M)$$

if $m = 2n$ and $k = 0$,

$$= H^m(M) + H^{m-n}(M)$$

if $m - k = 2n$ and $k > 0$,

$$= 0$$

otherwise.

Let $\theta^k: V^k(M) \to \mathbb{Z}/2$, for $k > 0$, be given by

$$\theta^k(u, v) = \varphi_k(u) + (Sq^v)(M) + (u \cup v)(M)$$

for $m - k$ even and $\theta^k = 0$ for $m - k$ odd.

Let $A = \mathbb{Z}/8[\sigma_k], k = 1, 2, \ldots$, modulo the relations $a_k^2 = 4, 2a_k = a_k a_i = 0, k \neq i$.

**Definition (4.1).**

$$\sigma(M) = \sigma(\varphi_n) \quad \text{if } m = 2n,$$

$$= 0 \quad \text{if } m \text{ is odd}.$$  

$$\sigma_k(M) = \text{Arf}((\theta^k), \quad \sum(M) = \sigma(M) + \sum_k \sigma(M) a_k.$$  

Our main result is

**Theorem (4.2).** If $M$ and $N$ are B-Wu oriented and $M \times N$ has the product orientation with respect to Wu system $\{\xi^m_\mu, \mu\}$ where $\mu$ is as in (3.2), then $\Sigma(M \times N) = \Sigma(M) \Sigma(N)$, or equivalently

$$\sigma(M \times N) = \sigma(M) \sigma(N) + 4 \left( \sum_k \sigma(M) \sigma_k(N) \right),$$

$$\sigma_k(M \times N) = \sigma(M) \sigma_k(N) + \sigma_k(M) \sigma(N)$$

$$(\sigma_k(M) \sigma(N) = \sigma_k(M) (\sigma(M) \mod(2))), \sigma(M) = \chi(M) \mod(2) \text{ where } \chi = \text{Euler characteristic.}$. If $M$ and $N$ have preferred orientations, $\sigma_k(M) = \sigma_k(N) = 0, \text{all } k, \text{ and } \sigma(M \times N) = \sigma(M) \sigma(N).$

Before proving (4.2) we give some related results. $\sigma$ is related to the Kervaire invariant and the index as follows:

**Theorem (4.3).** If the B-Wu orientation $U$ of $M$ has the property that $U_M: M \to B_k^m$ is the constant map, that is, $U$ is a framing and $m = 4l + 2$, then

$$\sigma(M) = 4 \cdot (\text{Kervaire invariant of } M).$$

If $\{\xi_1\}$ are oriented in the usual sense and $m = 4l$, then

$$\sigma(M) = \text{Index}(M) \mod(8).$$

For any $M$, $\sigma(M) = \chi(M) \mod(2)$, where $\chi(M)$ is the Euler characteristic.

The first and third parts of (4.3) are proved in [4] and the second part is a result of S. Morita [7].

**Theorem (4.4).** If $k > m$ or $m - k$ is odd, $\sigma_k(M) = 0$. If $m - k = 2n$, $\sigma_k(M)$
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$V^k(M) = \phi_n(v_n)$ where $v_n$ is the nth Wu class. For any $k$, $4 \cdot \sigma_k(M) = \sigma(S^k \times M)$ where $S^k$ is $S^k$ with the nontrivial B-Wu orientation as in §2.

**Proof.** The first part follows from the fact that $V^k(M) = 0$ for $k > m$ or $m - k$ odd.

Suppose $m - k = 2n$ and $u \in H^{m-n}(M)$. $(\text{Sq} u)(M) = (u \cup v_n)(M)$. Choose a basis $\lambda_i$ for $H^p(M)$ and $\mu_i$ for $H^{m-n}(M)$ such that $\lambda_i \cup \mu_i = \delta_{ij}$. $\{\lambda_i, \mu_i\}$ is a symplectic basis for $V^k(M)$. If $v_n = 0$, $\theta^k(\mu_i) = (\text{Sq} \mu_i)(M) = 0$, and hence $\text{Arf} \theta^k = 0$. If $v_n \neq 0$, choose $\lambda_i = v_n$. Then

$$\text{Arf} \theta^k = \sum \theta^k(\mu_i) \lambda_i = \theta^k(v_n) = \phi_n(v_n).$$

In §9 we will prove

**Lemma (4.5).** For $1 \in H^0(S^k)$, $\phi(1) \neq 0$. Hence $\sigma_k(S^k) = \phi_k(v_0) = \phi_k(1) = 1$. Then by (4.3), $\sigma(S^k \times M) = 4 \cdot \sigma_k(M)$.

Combining the above results yields:

**Corollary (4.6).** $\sigma_k(S^k) = 1$, $\sigma(S^k \times S^k) = 4$, $\sigma_k(S^k \times P_{2n}) = 1$, where $P_{2n}$ is real projective 2n-space.

The last equation in (4.6) is proved as follows:

$$\sigma_k(S^k \times P_{2n}) = \sigma_k(S^k) \sigma(P_{2n}) = \sigma_k(S^k) \chi(P_{2n}) = 1.$$

In our language, W. Browder's formula [2] is:

**Corollary (4.7).** If the Wu classes $v_i$, $i > 0$, are zero on $M$ and $\phi_M(1) = 0$, then $\sigma(M \times N) = \sigma(M) \chi(N)$.

**Proof.** By (4.4) $\sigma_k(M) = 0$ for all $k$. (4.7) is obvious if $M$ is odd-dimensional.

Suppose $m = 2l$. Then by (3.2)(ii) on $M$,

$$2\phi(u) = 2 \cdot (u \cup u)(M) = 2 \cdot (v_n \cup u)(M) = 0.$$

Hence $\phi(u) \in \{0, 2\}$ and $\phi_l = 2\theta$, where $\theta: H^1(M) \rightarrow \mathbb{Z}/2$. Thus $\sigma(M) = 4 \cdot \text{Arf} \theta \in \{0, 4\}$. Then

$$\sigma(M \times N) = \sigma(M) \sigma(N) = \sigma(M)(\sigma(N) \mod (2)) = \sigma(M) \chi(N).$$

Finally, we sketch a proof of Sullivan's product formula for surgery obstructions. Suppose $X = (x, \eta, \alpha)$ is a Poincaré complex as in §1, $X$ is simply connected and has formal dimension $4 \ell + 2$, $M$ is a manifold and $f: v_M \rightarrow \eta$ is a map such that $T(f)^* \alpha_M = \alpha$. Choose a $BG$-Wu orientation of $X$ and orient $M$ by pulling back this orientation via $f$. In [4] it is shown that the surgery obstruction to making $f_M: M \rightarrow X$ into a homotopy equivalence, $\sigma(f) \in \mathbb{Z}/2$, is given by

$$4 \cdot \sigma(f) = \sigma(M) - \sigma(X) \in \{0, 4\} \subset \mathbb{Z}/8.$$

Suppose $N$, $Y$ and $g$ satisfy the same hypotheses as $M$, $X$ and $f$ except that $\dim N = \dim Y = 4k$.

**Corollary (4.8).** $\sigma(f \times g) = \sigma(f) \chi(Y)$. 
Proof. Recall,
\[ \sigma(Y) - \sigma(N) = I(Y) - I(N) \mod(8), \]
\[ = 0 \mod(8), \]
since \( I(Y) - I(N) \), the surgery obstruction of \( g \), is divisible by 8. As in (4.4), \( \sigma_k(X) = \varphi_n(v_n(X)) \). \( \varphi_X = \varphi_M f_M^k \) (see [4]). Thus \( \sigma_k(M) = \sigma_k(X) \) and \( \sigma_k(N) = \sigma_k(Y) \). Hence
\[ 4 \cdot \sigma(f \times g) = \sigma(M \times N) - \sigma(X \times Y) = \sigma(M)\sigma(N) - \sigma(X)\sigma(Y) \]
\[ = (\sigma(M) - \sigma(X))\sigma(Y) + \sigma(M)(\sigma(N) - \sigma(Y)) \]
\[ = (4 \cdot \sigma(f))\sigma(Y) = 4 \cdot (\sigma(f)\chi(Y)), \]
since \( \sigma(Y) = \chi(Y) \mod(2) \). Hence \( \sigma(f \times g) = \sigma(f)\chi(Y) \).

Proof of (4.2). We first prove
\[ \sigma_k(M \times N) = \sigma(M)\sigma_k(N) + \sigma_k(M)\sigma(N). \]
Since \( \sigma(M) = \chi(M) \mod(2), (4.9) \) is equivalent to
\[ \sigma_k(M \times N) = \chi(M)\sigma_k(N) + \sigma_k(M)\chi(N). \]
If \( m + n - k \) is odd, one easily checks that both sides of (4.10) are zero. Suppose \( 2p = m + n - k \). Note that the proof in (4.4) that \( \sigma_k(M) = \varphi_l(v_l) \) for \( 2l = m - k \) does not depend on (4.2).
\[ v_p(M \times N) = \sum v_l(M) \otimes v_{p-l}(N) \]
and
\[ \sigma_k(M \times N) = \varphi_p(v_p(M \times N)). \]
Since \( v_l(M) = 0, 2l \geq m, \) the above sum for \( v_p(M \times N) \) has three types of nonzero terms:
(a) \( i = m/2; \) (b) \( i < m/2, p - i = n/2; \) (c) \( i < m/2, p - i < n/2. \) In case (c),
\[ \varphi(v_i \otimes v_{p-i}) = \varphi(v_i)\sigma_{m-2i}v_{p-i-2}p_{n-2(p-i)} = 0, \]
since \( \alpha_k\alpha_l = 0 \) in \( \Lambda. \) In case (a),
\[ \varphi(v_i \otimes v_{p-i}) = \varphi_{m/2}(v_{m/2})\sigma_{p-i}v_{n-2(p-i)}. \]
When \( i = m/2, p - i = (n - k)/2 \) and hence \( \varphi_{p-i}(v_{p-i}) = \sigma^k(N) \). Let \( l = m/2. \) By (3.2)(i),
\[ \varphi_l(v_l) = (v_l^2)(M) \mod(2). \]
But \( (v_l^2)(M) = \chi(M) \mod(2). \) The same argument applies to case (b), and (a) and (b) give (4.10).

We complete the proof of (4.2) by showing
\[ \sigma(M \times N) = \sigma(M)\sigma(N) + 4 \cdot \left( \sum \sigma_k(M)\sigma_k(N) \right). \]
Let \( V_1^k = V^k(M), V_2^k = V^k(N) \) and \( V_k = V^k(M \times N). \) Let \( v_1^0 = \varphi_M|V^0(M) \) and \( \varphi_k^k = 2\delta_M^k \) for \( k > 0. \) Let \( \varphi_k^k \) and \( \varphi^k \) be similarly defined for \( N \) and \( M \times N \) and let \( \varphi = \varphi^k. \) Since \( \sigma \) is additive,
\[ \sigma(\varphi) = \sum \sigma(\varphi^k) = \sigma(M \times N) + 4 \cdot \sum \sigma^k(M \times N). \]
On the other hand, \( H^*(M \times N) = \Sigma V^k_1 \otimes V^l_2 \) and hence
\[
\sigma(\phi) = \sum \sigma(\phi_1 V^k_1 \otimes V^l_2);
\]
\( \sigma \phi V^0_1 \otimes V^0_2 = \psi_1 \psi_2 \) and therefore
\[
\sigma(\phi V^0_1 \otimes V^0_2) = \sigma(\phi_1 \psi_2) = \sigma(\phi_1) \sigma(\psi_2) = \sigma(M) \sigma(N).
\]
We show that for \( k \) and \( l \) not both zero,
\[
(4.15) \quad \sigma(\phi V^0_1 \otimes V^0_2) = 4 \cdot \sigma(M) \sigma(N) \quad \text{if} \quad k = l,
\]
\[
= 4 \cdot (\sigma(M) \sigma(N)) \quad \text{if} \quad k = 0,
\]
\[
= 4 \cdot (\sigma(M) \chi(N)) \quad \text{if} \quad l = 0,
\]
\[
= 0 \quad \text{otherwise}.
\]
Combining this with (4.12), (4.13), and (4.14) we obtain (4.11). To prove (4.15) one chooses symplectic bases for \( V^k_1 \) and \( V^l_2 \) as in the proof of 4.4 for \( k \) and \( l > 0 \). One then forms a symplectic basis for \( V^k_1 \otimes V^l_2 \) and computes the Arf invariant of \( (\phi V^0_1 \otimes V^0_2)/2 \). We leave the details to the reader.

5. The ring \( G \). Let \( TB^m \) and \( \mu \) be as in §2. Let \( G_{n,p}^m = \pi_n(TB^m \land K_p) \), \( G = \Sigma G^m, \) where \( G^m_p = G_{m,p}^m \). As in §3, \( \mu \) and the cup product maps define a pairing \( \mu^*: G^m_{n,p} \otimes G^m_{s,q} \to G^m_{n+s,p+q}. \) Let \( \alpha \cdot \beta = \mu(\alpha \otimes \beta). \) The main objective of this section is to compute \( G^m_{n,p} \) for \( 2p > n \) and to describe \( \mu. \) To do this we introduce a variety of gadgets.

For any space or spectrum \( X \) let \( \rho: \pi_m(X \land K_p) \to H_{m-p}(X) \) be defined by \( u(\rho(\alpha)) = (u \otimes t_p)(\alpha) \) for all \( u \in H^{m-p}(X) \) (\( \rho \) is the Hurewicz map followed by \( l_p \)). \( t_p \) is the generator of \( H^p(K_p). \) \( \rho \) is the map which gives an isomorphism \( \lim_{p \to \infty} \pi_{n+1}(X \land K_p) \approx H_m(X). \) Also let \( \tau(l): \pi_m(X \land K_p) \to \pi_{m+l}(X \land K_{p+l}) \) be the maps used to define the above limit, that is \( \tau(l) \) is the composition of the obvious maps
\[
\pi_m(X \land K_p) \approx \pi_{m+l}(X \land S^1 K_p) \to \pi_{m+l}(X \land K_{p+l}).
\]

Let \( s: G^m_{n,p} \to \mathbb{Z}/2 \) be given by \( s(\alpha) = (1 \otimes Sq t_p)(\alpha) \) where \( 1 \in H^0(TB^m) \) is the Thom class. Let \( S \) be the sphere spectrum. Recall \( \pi_{2p}(S \land K_p) \approx \mathbb{Z}/2. \) Let \( \beta_p^m \in G_{2p,p}^m \) be the image of the generator under
\[
(1 \land \text{id})_*: \pi_{2p}(S \land K_p) \to \pi_{2p}(TB^m \land K_p)
\]
where \( i: S \to TB^m \) is the “inclusion of a fibre” and \( p > 0. \) Let \( \beta_0^m = 2[i] \in \pi_0(TB^m) = G^m_{0,0}. \)

**Lemma (5.1).**

\[
\tau(l)(\alpha) \cdot \tau(k)(\beta) = \tau(l+k)(\alpha \cdot \beta), \quad s(\alpha \cdot \beta) = s(\alpha)s(\beta),
\]
\[
(i) \quad s(\alpha) = 0 \quad \text{if} \quad \alpha \in G^m_{n,p} \text{ and } n > 2p,
\]
\[
\tau_l(\alpha) = s(\alpha), \quad s(\alpha) = (\chi(Sq)1)(\rho(\alpha)), \quad \rho(1) = \rho.
\]
\[
(ii) \text{ If } 2p > n, \rho: G^m_{n,p} \to H_{n-p}(TB^m) \text{ is an isomorphism. If } 2p = n, \rho \text{ is an}
\]
epimorphism and the kernel is \( \{0, \beta^m_p\} \). \( \beta^m_p \neq 0 \) if \( m < 2(p + 1) \).

(iii) If \( \beta \in G_{2q,q}^n, \beta^m_p = \beta \cdot \beta^m_p = s(\beta)^{n+1} \). Note \( G_{00}^0 = \pi_0(S) = Z \) and \( n \cdot \beta = \beta n, n \in Z \). Hence a special case of (iii) is \( 2\beta = \beta \cdot \beta^m_p = s(\beta)^{n+1} \).

**Proof.** (i) follows directly from the definitions. For large \( l, \rho: \pi_{n+l}(TB^m \wedge K_{p+1}) \rightarrow H_{n-l}(TB^m) \) is an isomorphism since \( TB^m \) is connected. \( TB^m = (-1) \) connected and \( K_{p+1} = S^lK_p \cup e^{2(p+1)+1} \cup \cdots \). Hence, \( \tau(l): \pi_{n}(TB^m \wedge K_p) \approx \pi_{n+l}(TB^m \wedge K_{p+1}) \) for \( 2p > n \). Therefore \( \rho \) is an isomorphism on \( G_{n,p}^m \) for \( 2p > n \).

Let \( i = S_{ip}: SK_p \rightarrow K_{p+1} \) be an inclusion, \( D = K_{p+1}/SK_p \), and \( j: K_{p+1} \rightarrow D \) the quotient map. \( D \) is \( 2p + 1 \) connected, kernel of \( H^{2p+2}(i) \) is \( S^{2p+1}p_{p+1} \) and \( H^{2p+2}(D) \approx \mathbb{Z}/2 \) is generated by a class \( \gamma_p \) such that \( H^{2p+2}(i)\gamma_p = S^{2p+1p+1}p_{p+1} \).

Consider the commutative diagram: Denote \( TB^m \) by \( T^m \).

\[
\begin{array}{cccccccc}
\pi_{2p+2}(T^m \wedge K_{p+1}) & \rightarrow & \pi_{2p+2}(T^m \wedge D) & \xrightarrow{3^*} & \pi_{2p+1}(T^m \wedge SK_p) & \rightarrow & \pi_{2p+1}(T^m \wedge K_{p+1}) & \rightarrow & 0 \\
\cong \rho & & \cong \tau & & \cong S & & \cong \rho & & \\
H_{p+1}(T^m) \xrightarrow{\chi(S^{p+1}q)} H_q(T^m) & & \pi_{2p}(T^m \wedge K_p) \xrightarrow{p} H_q(T^m) & & \\
\end{array}
\]

where \( u(\tau(a)) = (u \otimes \gamma_p(\alpha)) \) for all \( u \in H^0(T^m) \). The top row is exact and the squares commute. Since \( D \) is \( 2p + 1 \) connected and \( \pi_{2p+2}(D) \approx \mathbb{Z}/2 \) for \( p > 0 \), the Hurewicz theorem gives that \( \gamma_p \) is an isomorphism for \( p > 0 \). Therefore the kernel of \( \rho: \pi_{2p}(T^m \wedge K_p) \rightarrow H_q(T^m) \) is image(\( S^{-1}\partial_e \)). The above diagram is functorial in \( T^m \) so \( i: S \rightarrow T^m \) carries the diagram for \( S \) into the diagram for \( T^m \). For the \( S \) diagram, image \( S^{-1}\partial_e = \pi_{2p}(S \wedge K_p) \). Hence for \( T^m \), ker \( \rho \) is \( \{0, \beta_p^m\}, p > 0 \). In the above diagram

\[
\chi(S^{p+1}q)(c) = (\chi(S^{p+1}q)1)(c) = \nu_{p+1}(c).
\]

Therefore (ii) is proved for \( p > 0 \). For \( p = 0 \), \( \pi_{2p+2}(T^m \wedge D) \approx \pi_{2p}(T^m) \) and \( S^{-1}\partial_e \) is multiplication by 2. Thus ker \( \rho \) is \( 2\{1\} = \beta_p^m \) and (ii) is proved for \( p = 0 \).

To prove (iii) consider the commutative diagram: \( (D_p = K_{p+1}/SK_p) \).

\[
\begin{array}{cccccccc}
K_q \wedge SK_p \xrightarrow{id \times i} K_q \wedge K_{p+1} & \xrightarrow{id \times j} & K_q \wedge D_p \\
\downarrow a & & \downarrow b & & \downarrow c & & \\
SK_{p+q} & \xrightarrow{i} & K_{p+q+1} & \xrightarrow{j} & D_{p+q} & & \\
\end{array}
\]

where \( a \) is the suspension of the cup product map, \( b \) is cup product and \( c \) is defined by \( b \). (Choose \( b, a \) to be an inclusion of pairs.) \( H^*(j)(c)(\gamma_{p+q}) = Sq^q t_q \otimes Sq^{p+1} \gamma_{p+1} \).

Hence,

\[
H^*(c)\gamma_{p+q} = Sq^q t_q \otimes \gamma_p.
\]

(5.2)

Let \( \gamma_p^m \in \pi_{2p+2}(T^m \wedge D_p) \) be the generator. \( \partial_e \gamma_p^m = \beta_p^m \). \mu and \( c \) define a map \( T^n \wedge K_q \wedge T^m \wedge D_p \rightarrow T^{n+m} \wedge D_{p+q} \). By (5.2), this map takes \( \beta \wedge \gamma_p^m \) into \( s(\beta)^{n+m} \gamma_{p+q} \) for \( \beta \in G_{2q,q}^n \). Smash the above diagram with
and apply \( \pi_* \) to obtain a commutative ladder of exact sequences with connecting homomorphisms \( \partial_* \)
\[
\mu(\beta \otimes \beta_p^m) = \mu(\partial_* (\beta \otimes \beta_p^m)) = \partial_* (s(\beta)\beta_{p+q}^n) = s(\beta)\beta_{p+q}^n.
\]
This completes the proof of (5.1).

6. The ring \( L \). The ring \( G \) is unnecessarily complicated for our purposes. For example \( G_p^m \) for low values of \( p \) is very messy and \( \theta: H^p(M) \to G_p^m \) is neither linear nor quadratic. In this section we introduce a simpler ring \( L \). Let
\[
L_p^m = G_{m-p}^m = G_p^m \quad \text{if } 2p > m,
\]
\[
= G_{2(m-p),m-p}^m \quad \text{if } 2p < m,
\]
\[
L = \sum L_p^m.
\]
Let \( \alpha_p^m \in L_p^m \) be defined by \( \alpha_p^m = \beta_{m-p}^m \) if \( 2p < m \) and \( \alpha_p^m = 0 \) if \( 2p > m \). Let \( g: G \to L \) by
\[
g(\beta) = \beta \quad \text{if } \beta \in G_p^m \text{ and } 2p > m,
\]
\[
= \tau(m - 2p)(\beta) \quad \text{if } 2p < m.
\]
We define a multiplication of \( L \) so that \( g \) is a ring homomorphism as follows: Suppose \( \alpha \in L_p^m \) and \( \beta \in L_q^n \). If \( 2p > m \) and \( 2q > n \) or \( 2p < m \) and \( 2q < n \), \( \alpha \beta = \alpha \cdot 0 \). Otherwise \( \alpha \beta \) is defined as follows: Let
\[
\tau_1(\beta) = \beta \quad \text{if } 2p < m,
\]
\[
= \tau(2p-m)^{-1}\beta \quad \text{if } 2p > m.
\]
If \( \gamma \in G_{a,b}^m \), let \( \tau(m, p)\gamma = \tau(l)\gamma \) where \( l \) is chosen, if possible, so that \( \tau(l)\gamma \in L_p^m \); then \( \alpha \beta = \tau(m + n, p + q)(\tau(\alpha) \cdot \tau(\beta)) \). It is extremely tedious but straightforward, using (5.1)(ii) and (iii), to check that \( \alpha \beta \) is well defined. The formula \( \tau(l)(\alpha) \cdot \tau(k)(\beta) = \tau(l + k)(\alpha \cdot \beta) \) immediately yields:

**Lemma (6.1).** \( g(\alpha \cdot \beta) = \alpha \beta \).

**Lemma (6.2).** \( 0 \to \{ \alpha_p^m \} \to L_p^m \xrightarrow{-p} H_{m-p}(T^m) \to 0 \) is exact. \( \alpha_p^m \neq 0 \) if \( 2p \leq m \). \( \beta \in L_p^m \),
\[
\alpha_q^n \beta = \beta \alpha_p^m = s(\beta)\alpha_{p+q}^n, \quad 2\beta = s(\beta)\alpha_p^m.
\]

**Proof.** (5.1)(ii) gives the exactness above. Suppose \( 2q > n \) and hence \( \alpha_q^n = 0 \). If \( 2p < m, s(\beta) = 0 \). If \( 2p \geq m, \alpha_p^{n+m} = 0 \). Suppose \( 2q \leq n \). If \( 2p \leq m \),
\[
\alpha_q^n \beta = \beta_{n-q} \cdot \beta = s(\beta)\alpha_{n-q}^{n+m} = s(\beta)\alpha_{p+q}^n.
\]
If \( 2p > m \),
\[
\alpha_q^n \beta = \tau(m + n, p + q)(\alpha_q^n \cdot \tau(\beta)).
\]
If \(2(p + q) > m + n\) this is zero and also \(a_{p+q}^{n+m} = 0\). If \(2(p + q) \leq m + n\), \(t(m + n, p + q) = \text{id}\) and

\[a_{q}^{n} \cdot r(\beta) = s(r(\beta))a_{p+q}^{n+m} = s(\beta)a_{p+q}^{n+m}.\]

If \(2p > m\), \(2\beta = 0\) since \(L_{p}^{m} \approx H_{m-p}(T^{m})\). If \(2p \leq m\), \(2\beta = s(\beta)a_{p}^{m}\) by (5.1)(iii).

7. Quadratic functions with values in \(L\). Let \(M\) be a \(B\)-Wu oriented manifold and let \(\theta: H^{*}(M) \to G\) be as in §3. Define \(\psi: H^{*}(M) \to L\) by \(\psi = g\theta\) where \(g: G \to L\) is the map given in (6.1).

**Proposition (7.1).** \(\psi\) has the following properties:

(i) \(\psi(u + v) = \psi(u) + \psi(v) + (u \cup v)(M)a\) where \(u, v \in H^{p}(M)\), \(a = 0\) if \(m \neq 2p\) and \(a = \alpha_{2p}\) if \(m = 2p\).

(ii) If \(u \otimes v \in H^{*}(M \times N)\), \(\psi(u \otimes v) = \psi(u)\psi(v)\).

(iii) If \(U\) is the \(B\)-Wu orientations of \(M\), then the diagram below is commutative.

\[
\begin{array}{ccc}
H^{p}(M) & \xrightarrow{\psi} & L_{p}^{m} \\
\downarrow M & & \downarrow t \\
H_{m-p}(M) & \xrightarrow{(U \alpha t)\ast} & H_{m-p}(B^{m})
\end{array}
\]

\(t = \rho d\) and \(d: H_{*}(T^{B^{m}}) \to H_{*}(B^{m})\) is the Thom isomorphism.

**Proof.** The definitions of \(t\) and \(\psi\) immediately yield (iii). (ii) follows from (3.1) and (6.2). (i) is true for \(2p > m\) by (iii) and the fact that \(t\) is an isomorphism for \(2p > m\).

To prove (i) for \(2p \leq m\) consider

\[
A: H^{p}(M) \approx H^{m-p}(S^{m-2pM+}) = [S^{m-2pM+}, K_{m-p}] \to \{S^{m-2pM+}, K_{m-p}\}
\]

where \([, ,]\) denotes stable maps. In [4] it is shown that \(A\) is quadratic and its associated bilinear form is cup product in \(H^{*}(S^{m-2pM+})\). Thus \(A\) is linear if \(m \neq 2p\).

\[
S \xrightarrow{\alpha_{M}} T(v_{M}) \xrightarrow{\Delta} T(v_{M}) \wedge M^{+}
\]

is an \(S\) duality giving an \(S\) duality isomorphism

\[
B: \{S^{m-2pM+}, K_{m-p}\} \approx \pi_{m}(T(v_{M}) \wedge K_{m-p}).
\]

The \(B\)-Wu orientation \(U\) on \(M\) gives a map

\[
(U \wedge \text{id})_{\ast}: \pi_{m}(T(v_{M}) \wedge K_{m-p}) \to \pi_{m}(TB^{m} \wedge K_{m-p}) = L_{p}^{m},
\]

\(\psi = (U \wedge \text{id})_{\ast}BA\). Hence \(\psi\) is linear for \(2p < m\). In [4] it is shown that \(\psi\) satisfies (i) when \(m = 2p\).

8. Proof of (3.2). Recall, in §2, \(B^{m}\) was constructed by killing \(v_{i}\) for \(2i > m\). We may arrange these constructions so that \(pt = B^{0} \subset B^{1} \subset \cdots \subset B^{m} \subset \cdots\). These maps define inclusion maps \(j_{i}: TB^{m} \to TB^{m+i}\). For \(2p \leq m\), \(j_{m-2p}\) induces a
map 
\[ r: L^m_p = \pi_{2(m-p)}(TB^m \wedge K_{m-p}) \rightarrow \pi_{2(m-p)}(TB^{2(m-p)} \wedge K_{m-p}) = L^{2(m-p)}_{m-p}. \]
Furthermore, since \( \alpha^m_p \) comes from \( S \wedge K_{m-p} \rightarrow TB^m \wedge K_{m-p}, r(\alpha^m_p) = \alpha^{2(m-p)}_{m-p}. \)

**Lemma (8.1).** If the Wu system \( \{ t^m_\kappa, \mu \} \) is oriented in the usual sense, 
\[ (X \otimes p_{\mu})(s^2_t) = 2 \] 
where \( p \) is the Pontrjagin square and \( X \in H^0(TB^m; Z) \) is the Thom class.

**Proof.** Put the trivial \( B \)-Wu orientation on \( M = S^{2l} \times S^{2l} \) and let \( u = x \otimes 1 + 1 \otimes x \), where \( x \in H^{2l}(S^{2l}) \) is the generator. Since \( u \) comes from an integer class, \( pu = u^2 \) and \( pu(M) = 2(x \otimes x)(M) = 2 \). \( M \rightarrow B^m \) is trivial and therefore, by (7.1)(iii),
\[ \psi(u) = ca^t_{4l}, c \in Z/2. \] 
\( (X \otimes p_{\mu})(\psi(u)) = \psi(u)(M) = 2. \) Thus \( c = 1 \) and (8.1) is proved.

Choose homomorphisms \( h_n: L^{2l}_n \rightarrow Z/4 \) such that \( h_n(\alpha^m_n) = 2 \). This is possible by (6.2). If \( \{ t^m_\kappa \} \) are oriented, choose \( h_{2l} = (X \otimes p_{2l}) \). Recall, by (6.2), if \( \beta \in L^m_p \), \( 2\beta = s(\beta) \alpha^m_p \) and \( s(\beta) = 0 \) for \( 2p < m \). Thus every element of \( L^m_p \), for \( 2p < m \), has order two. We define \( \lambda^m_p \) as in §3 by the formulas:
\[
\lambda^m_p = s \quad \text{if } 2p > m, \\
= h_p \quad \text{if } 2p = m, \\
2 \cdot \lambda^m_p = h_{m-p}r_g \quad \text{if } 2p < m,
\]
where \( r \) is as above and \( g: G \rightarrow L \) as in §6. Equivalently stated,
\[
\varphi_p(u) = s\psi(u) \quad \text{if } 2p > m, \\
= h_p\psi(u) \quad \text{if } 2p = m, \\
2 \cdot \varphi_p(u) = h_{m-p}r\psi(u) \quad \text{if } 2p < m.
\]

**Lemma (8.2).** \( \varphi_p \) satisfies (3.2)(i)-(iv).

**Proof.** Proof of (i): If \( 2p > m \),
\[
\varphi_p(u) = s\psi(u) = (1 \otimes Sq t_p)(\psi(u)) \\
= (1 \otimes Sq t_p)((U \wedge u)\Delta)(\alpha_M) \\
= (S\mu u) \cup X_2(\alpha_M) = (S\mu u)(M).
\]
\( X_2 = \text{Thom class of } T(v_M). \) To show that \( \varphi_p(u) = (S\mu u)(M) \mod(2) \) for \( m = 2p \), we show that \( h_p = s \mod(2) \) and hence the above argument gives the desired result.

\[ 2h_p(\beta) = h_p(2\beta) = h_p(s(\beta)\alpha^2_{p}p). \]

Since \( h_p(\alpha^2_{p}) = 2, h_p = s \mod(2). \)

(3.2)(ii) follows from (7.1)(i). (iii) is true by construction. (iv) holds because our construction of \( \varphi_p \) for \( 2p = m \) is exactly the same as in [4].

To complete the proof of (3.2) we show that \( \mu \) can be chosen so that \( \varphi(u \otimes v) = \varphi(u)\varphi(v) \).
\[ \varphi = \lambda \theta \text{ and } \theta(u \otimes v) = \theta(u) \cdot \theta(v) \text{ by (3.1).} \]
Also \( \lambda = x_g \) where \( x: L \rightarrow \)
\( g \) is multiplicative. Denote \( \tilde{X} \) by \( \lambda \). \((3.2)(\nu)\) will be proved if we prove

**Lemma (8.3).** The maps \( \mu \) for the B-Wu system \( \{ \Lambda^m_k, \mu \} \) may be chosen so that \( \lambda: L \to \Lambda \) satisfies \( \lambda(a\beta) = \lambda(a)\lambda(\beta) \).

In the remainder of this section \( \alpha \in L^m_p \) and \( \beta \in L^n_q \); \( r = p + q \) and \( l = n + m \). We first prove that (8.3) holds for \( 2r > l \) and for \( \alpha = \alpha^m_p \) or \( \beta = \alpha^q_q \) independently of the choice of \( \mu \). If \( 2r > l \),

\[
\lambda(a\beta) = s(\alpha\beta)t^{2r-l} = s(\alpha)s(\beta)t^{2r-l}.
\]

If \( 2p < m \), \( s(\alpha) = 0 \) and

\[
\lambda(\alpha)\lambda(\beta) = (\lambda^m_p(\alpha)\alpha_{m-2p})(s(\beta)t^{2q-n}) = 0,
\]

since \( m - 2p < 2q - n \) and thus \( \alpha_{m-2p}t^{2q-n} = 0 \). If \( 2p = m \),

\[
\lambda(\alpha)\lambda(\beta) = \lambda^m_p(\alpha)(s(\beta)t^{2q-n}) = (\lambda^m_p(\alpha) \mod(2))s(\beta)t^{2q-n}.
\]

As shown in the proof of (8.2), \( \Lambda^m_p = h_p = s \mod(2) \). Hence \( \lambda(\alpha)(\beta) = s(\alpha)s(\beta)t^{2q-n} \).

The same argument holds for \( 2q < n \). If both \( 2p > m \) and \( 2q > n \), \( \lambda(\alpha)\lambda(\beta) = s(\alpha)s(\beta)t^{2r-l} \).

Suppose \( \alpha = \alpha^m_p \cdot \alpha^q_q = s(\beta)a_{r} \). If \( 2p < m \) or \( 2l > r \), \( s(\beta)a_{r} = 0 \) and \( \lambda(\alpha^m_p)\lambda(\beta) = \alpha_{m-2p}t^{2q-n} = 0 \). If \( 2p = m \) and \( 2r < l \), \( s(\beta) = 0 \) and \( \lambda(\alpha^m_p)\lambda(\beta) = 2\lambda^m_q(\beta)\alpha_{n-2q} = 0 \). If \( 2p = m \) and \( 2r = l \), \( \lambda(s(\beta)a_{r}) = 2 \cdot s(\beta) \). \( \lambda(\alpha^m_p)\lambda(\beta) = 2 \cdot s(\beta) \) because \( \lambda(\alpha^m_p) = 2 \) and \( \lambda(\beta) = s(\beta) \mod(2) \).

\( \{a_{r}^m \} \) is the kernel of \( \rho: \Lambda^m_p \to H_{m-p}(TB^m) \). Hence we have proved

**Lemma (8.4).** There is a class \( d \in \Sigma H^i(TB^m \wedge TB^n) \) where the sum is over \( n \) and \( m \) and \( i \geq [n + m/2] \), such that

\[
\lambda(\alpha\beta) - \lambda(\alpha)\lambda(\beta) = d(\rho(\alpha) \otimes \rho(\beta))a_{l-2r}.
\]

(If \( l = 2r \), \( a_{0} = 2 \) means \( 2 \cdot \cdot \cdot \).)

Let \( \mu' \) be some choice of \( \mu \). With respect to \( \mu' \) we obtain a \( d \) as in (8.4). Let

\[
\tilde{d} \in \Sigma H^*(B^m \times B^n),
\]

be the class corresponding to \( d \) under the Thom isomorphism.

\( B^{m+n} \) is a principal fibration over \( B \) with fibre \( \Pi K_i, 2(i + 1) > m + n \). Modify

\( B^m \times B^n \to B^{m+n} \) by \( d \) to obtain a new \( \mu: TB^m \wedge TB^n \to TB^{m+n} \). Consider the bilinear form from \( L^m_p \otimes L^q_q \to \Lambda \) defined by \( \tilde{A}(\alpha, \beta) = \lambda(\alpha\beta) - \lambda(\alpha \ast \beta) \) where \( \alpha \beta \) comes from \( \mu \) and \( \alpha \ast \beta \) from \( \mu' \). As proved above, \( \lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta) \) for \( 2r > l \) and for \( \alpha = \alpha^m_p \) or \( \beta = \alpha^q_q \), independently of the choice of \( \mu \). Hence \( \tilde{A}(\alpha, \beta) = 0 \) if \( 2r > l \) and \( \tilde{A}(\alpha, \beta) = A(\alpha(\rho), \rho(\beta)) \) where \( A \) is a bilinear form from \( H_{m-p}(TB^m) \otimes H_{n-q}(TB^n) \) to \( \Lambda \). Suppose \( 2r \leq l \). Consider the commutative diagram:

\[
\begin{array}{ccc}
L^m_p \otimes L^q_q & \longrightarrow & L^1_r \\
\rho \otimes \rho \downarrow & & \rho \downarrow \\
H_{m-p}(TB^m) \otimes H_{n-q}(TB^n) & \longrightarrow & H_{l-r}(TB^{l-r})
\end{array}
\]
The bottom row is independent of $\mu$ since $H_{l-r}(TB^2(-1)^{(l-r)}) \approx H_{l-r}(B)$. Therefore

$$A(\rho(\alpha), \rho(\beta)) = D(\rho(\alpha), \rho(\beta))\alpha_{l-2r}$$

where $D$ takes values in $Z/2$. To complete the proof of (8.2) we show

**Lemma (8.5).** $D(\rho(\alpha), \rho(\beta)) = d(\rho(\alpha) \otimes \rho(\beta))$.

**Proof.** Let $Y^l$ be the spectrum described in (2.3), that is, $Y^l_k$ is the two stage Postnikov system with $k$-invariant $\Pi_{2i+1} \chi(Sq^i): K_k \to \Pi K_{k+l}$. Let $K_t$ be the spectrum whose $j$th term is $K_{t+j}$ and $d_j \in H^t(K_t)$ be the generator. Also denote the generator of $H^0(Y^l)$ by $\delta_0$. Let $f^l: TB^l \to Y^l$ be a map such that $f \ast \delta_0$ is the Thom class. $f$ exists because $\chi(Sq^l)1 = v_l \cup 1$. One may check that for any space $X$, if $G': X \to B'$ and one modifies $G'$ by $d = \{d_j\}$, $d_j \in H^t(X), 2i > l$ to form $G: X \to B'$, then $T(G \ast \chi^t) \to TB^l \to Y^l$ is modified by $d_j \in H^t(TG \ast \chi^t)$ where $d_j$ corresponds to $d_j$ under the Thom isomorphism. We may choose $f^l$ and maps $g_2: Y^l \to Y^{l+1}$ such that

\[
\begin{array}{c}
\prod_{2(i+1)+1} K_i \\
\downarrow \\
Y^l \end{array}
\xrightarrow{g_1} \begin{array}{c}
\prod_{2(i+1)+1} K_i \\
\downarrow \\
TB^l \end{array}
\xrightarrow{g_2} \begin{array}{c}
\prod_{2(i+1)+1} K_i \\
\downarrow \\
Y^{2(l-r)} \\
TB^l \\
\end{array}
\xrightarrow{g_2} Y^{2(l-r)}
\]

commutes. $g_1$ is the projection. Let

$$F^l = (f^l \wedge \text{id}): TB^l \wedge K_{l-r} \to Y^l \wedge K_{l-r}$$

and let $\bar{d}_j = f^l \ast d_j$. (5.1) is applicable to $Y^l$ instead of $TB^l$ so that $\bar{d}_j \neq 0$. $(g_2 \wedge \text{id})_* \bar{d}_j = \alpha_{l-2r}^2$. Let $C$ be defined by

$$C(\alpha, \beta) = (g_2 \wedge \text{id})* F^l \wedge \bar{A}(\alpha, \beta) = D(\rho(\alpha), \rho(\beta))\alpha_{l-2r}^2.$$  

On the other hand $C(\alpha, \beta)$ is the image of $S^{l-2r}(\alpha \wedge \beta)$ under the map

$$S^{l-2r}(TB^m \wedge K_p \wedge TB^n \wedge K_q) \to \prod_{2(i+1)+1} K_i \wedge S^{l-2r}K_i$$

$$\to \prod_{2(i+1)+1} K_i \wedge K_{l-r} \to Y^{2(l-r)} \wedge K_{l-r}$$

where the first map is given by $\{d_j\}$ and cup product. $\gamma_{l-r} \otimes \gamma_{l-r}$ evaluated on the image of $S^{l-2r}(\alpha \wedge \beta)$ is $d(\rho(\alpha) \otimes \rho(\beta))$. Thus it is sufficient to prove

**Lemma (8.6).** Under the map $Z/2 \approx \pi_{2s}(\Pi_{2s}K_t \wedge K_s) \to \pi_{2s}(Y^{2s} \wedge K_s), 1$ goes to $0$.

**Proof.** Let $\beta$ be the image of $1$. $\rho(\beta) = 0$ since $\Pi K_t \to Y^{2s}$ is homologically trivial in dimension $s$. Note $\pi_{2s}(Y^{2s} \wedge K_s)$ is the homology of $K_s$ with coefficients in $Y^{2s}$. The coefficient sequence $\Pi K_t \to Y^{2s} \to K_0 \to \Pi K_t$ gives an exact sequence,
where \( a = \sum_{i > s} Sq^i \). \( a = 0 \) since \( Sq^i \mu_s = 0 \) for \( i > s \). Therefore \( b \) is an injection and thus \( \beta = \alpha^{2s} \). The proof of (3.2) is complete.

9. Proof of (3.3) and (4.5).

Proof of (3.3). Suppose \( U_1 \) and \( U_2 \) are two \( B \)-Wu orientations on \( M \) which are equivalent as \( B \) structures on \( v_M \). As in the proof of (3.2)(v), \( U_1, U_2: T(v_M) \to TB^m \) differ by a map \( d: T(v) \to \Pi_{q > m} K_q \). Let \( D(u): \Sigma_{2p < m} H_p(M) \to Z_2 \) be defined by \( \varphi_{U_1}(u) - \varphi_{U_2}(u) = D(u)\alpha_{m-2p} \). We show

\begin{equation}
(9.1) \quad D(u) = (u \cup d_{m-p})(M)
\end{equation}

as in the proof of (3.2): \( (g_2 \wedge \text{id}) \ast F \ast \psi_{t'}(u) \) differ, for \( i = 1 \) and \( 2 \), by the image of \( \alpha_M \) under

\begin{align*}
\pi_m(T(v_M)) & \to \pi_m(T(v_M) \wedge K_p) \\
& \to \pi_2(m-p)(K_{m-p} \wedge K_{m-p}) \\
& \to \pi_2(m-p)(K_{m-p} \wedge K_{m-p})
\end{align*}

where \( a = (d \wedge u) \ast \iota_{m-p} \otimes \iota_{m-p} \) pulls back to \( (d_{m-p} \cup u) \cup 1 \) in \( H^q(T(v_M)) \).

\( \iota_{m-p} \otimes \iota_{m-p} \) is one on the generator of \( \pi_2(m-p)(K_{m-p} \wedge K_{m-p}) \). (9.1) now follows from (8.6) and the proof of 3.3 is complete.

Proof of (4.5). Recall \( S^n \) was \( B \)-Wu oriented to form \( \overline{S^n} \) as follows: Choose a standard framing of \( v_{S^n} \), \( F: v_{S^n} \to R^k \) \( (S^n, F) \) is framed cobordant to zero. This gives a \( B \)-Wu orientation \( U_1: v_{S^n} \to F \to R^k \to \iota_{S^n}^n \) where \( i \) is the inclusion of a fibre.

Let \( U \) be the orientation on \( S^n \) such that \( U \) and \( U_1 \) differ by the nontrivial map \( d: S^n \to K_0 \). Hence by (3.3),

\( d_{S^n}(1) = \varphi_{U_1}(1) + (1 \cup d)(S^n)\alpha_n \),

\( (1 \cup d)(S^n) = 1 \) and \( \varphi_{U_1}(1) = 0 \) since the following diagram is commutative:

\begin{equation}
\begin{array}{ccc}
T(v_{S^n}) & \to & T(v_{S^n}) \wedge S^{n+} \\
\downarrow U_1 & & \downarrow TB^n \wedge K_0 \\
TB^n & \to & TB^n \wedge B^{n+}
\end{array}
\end{equation}

and \( U_1(\alpha_{S^n}) = 0 \). Thus \( \varphi_U(1) = \alpha_n \) and (4.5) is proved.

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MASSACHUSETTS 02154