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A GENERALIZATION OF H. WEYL'S "UNITARY TRICK"

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ABSTRACT. H. Weyl's "unitary trick" is generalized to the context of semisimple symmetric Lie algebras with Cartan subspaces, over fields of characteristic zero. As an illustration of its usefulness, the result is used to transfer to characteristic zero an important theorem in the representation theory of real semisimple Lie algebras.

1. **Introduction.** H. Weyl's "unitary trick", as formalized by C. Chevalley and S. Eilenberg (see [3, Chapter IV, §7]), enables one to transfer certain kinds of theorems from compact real semisimple Lie algebras to semisimple Lie algebras over arbitrary fields of characteristic zero. Here we shall generalize this process to semisimple symmetric Lie algebras with Cartan subspaces, over characteristic zero, in the sense of [1, §1.13] and [5(b)]. This work was mentioned in the Introduction of [4(b)].

As an application, we shall answer an algebraic question left open in [4(a)], by transferring to characteristic zero a well-known result proved by analytic methods over the real numbers; see §2. This solves Problem 35, p. 336, of [1], but an algebraic proof of the result would still be interesting. Using [5(a)], the result implies that if k is an algebraically closed field of characteristic zero, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a semisimple symmetric Lie algebra over k and $\mathfrak{a}$ a Cartan subspace of $\mathfrak{p}$, then the irreducible $\mathfrak{g}$-modules containing a nonzero $\mathfrak{k}$-fixed vector are naturally indexed (up to equivalence) by the Weyl group orbits in the dual of $\mathfrak{a}$ (cf. [1, 9.1.12, 9.5.6 and 9.7.5(b)]); this is a well-known theorem of Harish-Chandra when k is the field of complex numbers.

J. Dixmier has pointed out that the Lefschetz principle can also be used to answer the algebraic question just mentioned. We shall discuss this in §2.

We thank G. McCollum for several helpful conversations on the subject of this paper. A joint approach using real closed fields has been planned, but proved to be unnecessary.

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2. **Formulation of the main results.** In [3, Chapter IV, §7], Weyl's unitary trick is expressed essentially as follows: Let P be a property of Lie algebras over fields. Call P *linear* if whenever $\mathfrak{g}$ is a Lie algebra over a field k and K is any extension of k, then $\mathfrak{g}$ satisfies P if and only if $\mathfrak{g}_K$ $(= \mathfrak{g} \otimes_k K)$ does. The unitary trick is made up of two results: First, any linear property satisfied by all compact real semisimple Lie algebras also holds for all complex semisimple Lie algebras. (This follows from the existence of a compact real form for a complex semisimple Lie algebra.) Second, any linear property valid for all semisimple Lie algebras over one field of characteristic zero also holds for all semisimple Lie algebras over any other field of characteristic zero. (This follows from the existence of a rational form for a semisimple Lie algebra over an algebraically closed field of characteristic zero.) Combining these two results, we can conclude that a linear property valid for all compact real semisimple Lie algebras actually holds for all semisimple Lie algebras of characteristic zero.

In this paper, we shall generalize these results to the following context: A *symmetric Lie algebra* over a field k of characteristic not 2 is a pair $(\mathfrak{g}, \theta)$ where $\mathfrak{g}$ is a (finite-dimensional) Lie algebra over k and θ is an automorphism of $\mathfrak{g}$ such that $\theta^2 = 1$ (see [5(b), §2] and [1, §1.13]). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the ± 1-eigenspace decomposition of $\mathfrak{g}$ with respect to θ. A *Cartan subspace* of $\mathfrak{p}$ is a nil subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $\mathfrak{a}$ is the intersection of $\mathfrak{p}$ and the generalized zero eigenspaces of ad x for all $x \in \mathfrak{a}$ (see [5(b), §3]). If k is infinite, Cartan subspaces exist [5(b), Corollary 1 of Theorem 3.2]. Call $(\mathfrak{g}, \theta, \mathfrak{a})$ a *symmetric triple* (over k). If K is an extension of the infinite field k, then $(\mathfrak{g}_K, \theta_K, \mathfrak{a}_K)$ (where the subscripts have the obvious meanings) is a symmetric triple over K (see [5(b), Theorem 3.4(3)]). Call this triple $(\mathfrak{g}, \theta, \mathfrak{a})_K$.

Now define a property P of symmetric triples over infinite fields to be *linear* if $(\mathfrak{g}, \theta, \mathfrak{a})$ satisfies P if and only if $(\mathfrak{g}, \theta, \mathfrak{a})_K$ does, using the above notation. The condition that $(\mathfrak{g}, \theta, \mathfrak{a})$ be semisimple of characteristic zero (i.e., that $\mathfrak{g}$ be of this type) is such a property. Hence it makes sense to speak of linear properties of semisimple symmetric triples of characteristic zero.

If $(\mathfrak{g}, \theta, \mathfrak{a})$ is semisimple of characteristic zero, then the Cartan subspaces of $\mathfrak{p}$ are the maximal abelian subspaces of $\mathfrak{p}$ consisting of semisimple elements [5(b), Corollary of Theorem 5.2]. If k is the field of real numbers, call $(\mathfrak{g}, \theta, \mathfrak{a})$ *Cartan* if it is semisimple and θ is a Cartan involution in the classical sense (i.e., the Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$); in this case, the Cartan subspaces are the Cartan subspaces of $\mathfrak{p}$ in the classical sense, i.e., the maximal abelian subspaces of $\mathfrak{p}$.

Call a symmetric triple $(\mathfrak{g}, \theta, \mathfrak{a})$ over k *split* if $\mathfrak{a}$ is a splitting Cartan subspace of $\mathfrak{p}$, i.e., for all $x \in \mathfrak{a}$, all the eigenvalues of ad x lie in k. A Cartan symmetric triple is split, as is any symmetric triple over an algebraically closed field.

Also, any extension of a split symmetric triple of characteristic zero is split (see [5(b), Corollary 2 of Theorem 4.2]). We say that a property P of split semisimple symmetric triples of characteristic zero is *split-linear* if such a triple $(\mathfrak{g}, \theta, \mathfrak{a})$ satisfies P if and only if $(\mathfrak{g}, \theta, \mathfrak{a})_K$ does, for any extension K of k.

In the following three theorems, "linear property" means "linear property of semisimple symmetric triples of characteristic zero", and "split-linear property" means "split-linear property of split semisimple symmetric triples of characteristic zero". We shall prove:

THEOREM 2.1. *Any linear or split-linear property satisfied by all Cartan symmetric triples over* **R** *also holds for all semisimple symmetric triples over* **C**.

The context of this result is actually well known (see §3). **R** and **C** of course denote the real and complex fields.

THEOREM 2.2. *Any linear property valid for all semisimple symmetric triples (resp., any split-linear property valid for all split semisimple symmetric triples) over one field of characteristic zero also holds for all semisimple symmetric triples (resp., all split semisimple symmetric triples) over any other field of characteristic zero.*

Combining these two results, we have:

THEOREM 2.3. *Any linear (resp., split-linear) property satisfied by all Cartan symmetric triples over* **R** *also holds for all semisimple symmetric triples (resp., all split semisimple symmetric triples) over any field of characteristic zero.*

REMARK. These results do in fact generalize Weyl's unitary trick as formulated above. In fact, let P be an arbitrary linear property of semisimple Lie algebras of characteristic zero, and apply the above theorems to the linear property "If $\theta = 1$, then P holds for $\mathfrak{g}$" of semisimple symmetric triples $(\mathfrak{g}, \theta, \mathfrak{a})$ of characteristic zero.

REMARK. It is easy to apply Theorem 2.3 to answer the open question discussed in the Introduction of [4(a)], namely: Is the Chevalley polynomial restriction map F_* in Theorem 3.1 of [4(a)] onto $S(\mathfrak{a}^*)^W$, and is the Harish-Chandra map in Theorem 4.1 of [4(a)] onto A^W, in the characteristic-zero setting of that paper? These maps are well known to be onto for Cartan symmetric triples over **R** (see for example [2(a), Chapter X, Theorem 6.10 and Lemma 6.14]). Since the surjectivity of these maps is easily seen to be a split-linear property of split semisimple symmetric triples of characteristic zero, Theorem 2.3 answers the question affirmatively. J. Dixmier has observed that the Lefschetz principle can also be used to answer this question. Cf. the next Remark.

REMARK. The Lefschetz principle can be used to prove Theorem 2.3.

Actually, it enables us to pass directly from Theorem 2.1 to 2.3, bypassing Theorem 2.2. (But Theorem 2.2 is independently interesting.) Specifically, let $(\mathfrak{g}, \theta, \mathfrak{a})$ be a semisimple symmetric triple over the field k of characteristic zero with symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, let $\mathfrak{q}$ be any linear complement of $\mathfrak{a}$ in $\mathfrak{p}$, and form a basis of $\mathfrak{g}$ by combining bases of $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{q}$. Let k' be the subfield of k generated over the rationals by the structure constants of $\mathfrak{g}$ with respect to this basis. Since k' is finitely generated over the rationals, k' is isomorphic to a subfield of $\mathbf{C}$. Thus we get a semisimple symmetric triple $(\mathfrak{g}', \theta', \mathfrak{a}')$ over k' such that $(\mathfrak{g}', \theta', \mathfrak{a}')_k = (\mathfrak{g}, \theta, \mathfrak{a})$ and $(\mathfrak{g}', \theta', \mathfrak{a}')_{\mathbf{C}}$ is a semisimple symmetric triple over $\mathbf{C}$. If $(\mathfrak{g}, \theta, \mathfrak{a})$ is split, let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \coprod \mathfrak{g}^{\varphi}$ be the restricted root space decomposition of $\mathfrak{g}$, where φ ranges through the restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ (see [4(a), §2]). Choose bases of the $\mathfrak{g}^{\varphi}$ for φ ranging through a system of positive restricted roots, apply θ to get bases of the $\mathfrak{g}^{\varphi}$ for negative φ, and combine these with bases of $\mathfrak{m}$ and $\mathfrak{a}$ to get a basis of $\mathfrak{g}$. The subfield k' of k generated over the rationals by the corresponding structure constants for $\mathfrak{g}$ is finitely generated over the rationals, giving rise to a k'-subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. Since $\mathfrak{g}'$ is θ-invariant, we clearly get a split semisimple symmetric triple $(\mathfrak{g}', \theta', \mathfrak{a}')$ over k' such that $(\mathfrak{g}', \theta', \mathfrak{a}')_k = (\mathfrak{g}, \theta, \mathfrak{a})$. (Cf. the proof of Theorem 2.2 in §4.) Thus we see that Theorem 2.3 follows from Theorem 2.1. Of course, the final statement in the above formulation of Weyl's unitary trick follows similarly from the first statement in the formulation (on the existence of compact real forms).

Now we must prove Theorems 2.1 and 2.2.

3. **Proof of Theorem 2.1.** Theorem 2.1 is an immediate consequence of the following well-known result, whose proof we include for completeness:

THEOREM 3.1. *Let* $(\mathfrak{g}, \theta, \mathfrak{a})$ *be a semisimple symmetric triple over* $\mathbf{C}$. *Then* $(\mathfrak{g}, \theta, \mathfrak{a})$ *has a Cartan real form, i.e.,* $(\mathfrak{g}, \theta, \mathfrak{a}) = (\mathfrak{g}', \theta', \mathfrak{a}')_{\mathbf{C}}$ *for some Cartan symmetric triple* $(\mathfrak{g}', \theta', \mathfrak{a}')$ *over* $\mathbf{R}$.

PROOF. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the symmetric decomposition of $\mathfrak{g}$, and let $\mathfrak{l}$ be a Cartan subalgebra of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, and $\mathfrak{t}$ a maximal abelian subspace of $\mathfrak{u}$, so that $\mathfrak{t}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. Conjugating $\mathfrak{t}_{\mathbf{C}}$ to $\mathfrak{h}$, we may assume that $\mathfrak{t}_{\mathbf{C}} = \mathfrak{h}$. Let τ be the involution of $\mathfrak{g}$ which is 1 on $\mathfrak{u}$ and -1 on $(-1)^{1/2}\,\mathfrak{u}$, so that τ is a Cartan involution of $\mathfrak{g}$. By [2(b), p. 29, Theorem 3.1] and its proof, there exists an automorphism ω of $\mathfrak{g}$ (regarded as a real Lie algebra) such that $\omega\tau\omega^{-1}$ commutes with θ, and ω is a real polynomial in $(\theta\tau)^2$ (since ω is the positive definite fourth root of the positive definite operator $(\theta\tau)^2$). Hence ω is complex-linear (since $(\theta\tau)^2$ is), and ω preserves $\mathfrak{h}$ (since θ and τ do).

Replacing $\mathfrak{u}$ by $\omega\mathfrak{u}$, we may thus assume that θ preserves $\mathfrak{u}$. Also, θ preserves $\mathfrak{u} \cap \mathfrak{h} = \mathfrak{t}$. It is now clear that $\mathfrak{g}' = (\mathfrak{k} \cap \mathfrak{u}) \oplus (\mathfrak{p} \cap (-1)^{1/2}\mathfrak{u})$ is a real form of $\mathfrak{g}$, $\theta' = \theta \,|\, \mathfrak{g}'$ is a Cartan involution of $\mathfrak{g}'$, $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{g}'$ is a Cartan subspace of $\mathfrak{p} \cap \mathfrak{g}'$, and $(\mathfrak{g}, \theta, \mathfrak{a}) = (\mathfrak{g}', \theta', \mathfrak{a}')_{\mathbf{C}}$. Q.E.D.

4. **Proof of Theorem 2.2.** Since any field of characteristic zero can be extended to its algebraic closure on the one hand and is an extension of the field **Q** of rational numbers on the other hand, Theorem 2.2 follows immediately from the first part of:

THEOREM 4.1. *Let $(\mathfrak{g}, \theta, \mathfrak{a})$ be a semisimple symmetric triple over an algebraically closed field K of characteristic zero. Then $(\mathfrak{g}, \theta, \mathfrak{a})$ has a split rational form, i.e., $(\mathfrak{g}, \theta, \mathfrak{a}) = (\mathfrak{g}', \theta', \mathfrak{a}')_K$ for some split semisimple symmetric triple $(\mathfrak{g}', \theta', \mathfrak{a}')$ over* **Q**. *Moreover, let $\mathfrak{l}$ be a Cartan subalgebra of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ (the $+1$-eigenspace of θ in $\mathfrak{g}$). Then $(\mathfrak{g}', \theta', \mathfrak{a}')$ may be chosen so that if $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$, then $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$ is a splitting Cartan subalgebra of the centralizer of $\mathfrak{a}'$ in $\mathfrak{k}'$, and $\mathfrak{l} = \mathfrak{l}'_K$.*

PROOF. Let $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{a}$, so that $\mathfrak{h}$ is a (splitting) Cartan subalgebra of $\mathfrak{g}$ (see [1, Proposition 1.13.7] or [5(b), Theorem 5.2]). Let $R \subset \mathfrak{h}^*$ be the corresponding set of roots, $R' \subset R$ the set of roots vanishing on $\mathfrak{a}$, and R'' the complement of R' in R. Then the root spaces corresponding to the roots in R' lie in the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$, and in fact R' may be identified with the set of roots of $\mathfrak{m}$ with respect to $\mathfrak{l}$. Let $\Sigma_+ \subset \mathfrak{a}^*$ be a system of positive restricted roots, R''_+ the set of roots in R'' whose restrictions to $\mathfrak{a}$ lie in Σ_+, and R'_+ a positive system in R'. Then $R_+ = R'_+ \cup R''_+$ is a positive system in R. Let Π be the corresponding simple system, $\Pi' = R' \cap \Pi$ and $\Pi'' = R'' \cap \Pi$.

The automorphism θ of $\mathfrak{g}$ induces a linear automorphism, also called θ, of $\mathfrak{h}^*$, which we identify with $\mathfrak{l}^* \oplus \mathfrak{a}^*$. Then θ is 1 on $\mathfrak{l}^*$ and -1 on $\mathfrak{a}^*$, and θ preserves R. Also let $\sigma = -\theta$ on $\mathfrak{h}^*$. Then (R, σ) is a normal σ-system, in the sense of [6, p. 21] (except that σ might be ± 1), by [4(a), Lemma 2.3] (cf. also [1, 1.14.14]). Thus [6, Lemma 1.1.3.2] is applicable. Hence there is a bijection ω of Π'' such that $\omega^2 = 1$ and for all $\alpha \in \Pi''$,

$$\sigma\alpha = \omega\alpha + \sum \beta_i,$$

where the β_i are in Π' and are not necessarily distinct. Thus

$$(*) \qquad \theta\alpha = -\omega\alpha - \sum \beta_i,$$

and also

$$(**) \qquad \theta\omega\alpha = -\alpha - \sum \beta_i,$$

since $\beta_i \in \mathfrak{l}^*$.

For each $\alpha \in \Pi$, define $x_\alpha \in \mathfrak{h}$ by the condition $B(x, x_\alpha) = \alpha(x)$ for all $x \in \mathfrak{h}$, where B is the Killing form of $\mathfrak{g}$, and let $h_\alpha = 2x_\alpha/(\alpha, \alpha)$, where $(\cdot, \cdot)$ is the nonsingular symmetric bilinear form on $\mathfrak{h}^*$ induced by B. For each α, let e_α be an arbitrary nonzero element of the root space $\mathfrak{g}^\alpha$, and then determine $f_\alpha \in \mathfrak{g}^{-\alpha}$ uniquely by the condition $[e_\alpha, f_\alpha] = h_\alpha$. Then the rational Lie subalgebra $\mathfrak{g}^\wedge$ of $\mathfrak{g}$ generated by $\{h_\alpha, e_\alpha, f_\alpha \mid \alpha \in \Pi\}$ is a rational form of $\mathfrak{g}$, by [3, p. 124, Theorem 2] (i.e., $\mathfrak{g}^\wedge$ is a rational Lie algebra such that $\mathfrak{g} = (\mathfrak{g}^\wedge)_K$). If we can modify the e_α (by multiplying them by scalars in K) so that the corresponding rational form $\mathfrak{g}'$ is θ-invariant, then our theorem will clearly be proved, since we may then take $\theta' = \theta | \mathfrak{g}'$, $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{g}'$ and $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$.

For each $\alpha \in \Pi$, let μ_α be a nonzero element of K. Define $e'_\alpha = \mu_\alpha e_\alpha$ and $f'_\alpha = (\mu_\alpha)^{-1} f_\alpha$, so that $\{h_\alpha, e'_\alpha, f'_\alpha \mid \alpha \in \Pi\}$ generates over $\mathbf{Q}$ a rational form $\mathfrak{g}'$ of $\mathfrak{g}$. We claim that if the μ_α's can be chosen so that $\theta e'_\alpha \in \mathfrak{g}'$ for all $\alpha \in \Pi$, then $\theta \mathfrak{g}' = \mathfrak{g}'$. To see this, note that it is sufficient to show that $\theta f'_\alpha \in \mathfrak{g}'$ for all $\alpha \in \Pi$. But $\theta e'_\alpha$ is a root vector in $\mathfrak{g}'$ for the root $\theta\alpha$, $\theta f'_\alpha$ is a root vector in $\mathfrak{g}$ for the root $-\theta\alpha$, and

$$B(\theta e'_\alpha, \theta f'_\alpha) = B(e'_\alpha, f'_\alpha) = B(e_\alpha, f_\alpha) = 2/(\alpha, \alpha),$$

which is rational. Hence $\theta f'_\alpha$ must be a rational multiple of any root vector in $\mathfrak{g}'$ for the root $-\theta\alpha$. Thus $\theta f'_\alpha \in \mathfrak{g}'$, proving the claim.

For $\alpha \in \Pi'$, $\theta e_\alpha = e_\alpha$, so that we may take $\mu_\alpha = 1$ (and hence $e'_\alpha = e_\alpha$ and $f'_\alpha = f_\alpha$) for these α.

Now let $\alpha \in \Pi''$. Since $\theta R = R$, $-\omega\alpha - \Sigma\beta_i = \theta\alpha \in R$, by (*). Thus there is a permutation $\gamma_1, \ldots, \gamma_m$ of $\omega\alpha$ and the β_i such that for all $j = 1, \ldots, m$, $\gamma_1 + \cdots + \gamma_j \in R$ (see [3, p. 123]). Hence

$$x = [f_{\gamma_m}, [f_{\gamma_{m-1}}, \ldots, [f_{\gamma_2}, f_{\gamma_1}] \ldots]] \neq 0$$

(see [3, p. 123]), and x is a root vector in $\mathfrak{g}^\wedge$ for the root $\theta\alpha$. It follows that there is a nonzero element $c_\alpha \in K$ such that $\theta e_\alpha = c_\alpha x$. If $\omega\alpha = \alpha$, then we may take $\mu_\alpha = (c_\alpha)^{-1/2}$. In fact, with this choice, $\theta e'_\alpha = \theta \mu_\alpha e_\alpha = (c_\alpha)^{1/2} x = (\mu_\alpha)^{-1} x$. But all except one γ_i is in Π', and the remaining γ_i is α. Thus

$$(\mu_\alpha)^{-1} x = [f'_{\gamma_m}, [f'_{\gamma_{m-1}}, \ldots, [f'_{\gamma_2}, f'_{\gamma_1}] \ldots]] \in \mathfrak{g}',$$

and so $\theta e'_\alpha \in \mathfrak{g}'$.

If $\omega\alpha$ is not necessarily α, we must say more. By (**), there is a permutation $\delta_1, \ldots, \delta_m$ of α and the β_i such that

$$y = [f_{\delta_m}, [f_{\delta_{m-1}}, \ldots, [f_{\delta_2}, f_{\delta_1}] \ldots]] \neq 0,$$

and y is a root vector in $\mathfrak{g}^\wedge$ for the root $\theta\omega\alpha$. Thus there is a nonzero element $c_{\omega\alpha} \in K$ such that $\theta e_{\omega\alpha} = c_{\omega\alpha} y$. Applying θ to this equation, and recalling (letting $\alpha = \delta_r$) that $f_{\delta_i} \in \mathfrak{m}$ for $j \neq r$, we have

$$e_{\omega\alpha} = c_{\omega\alpha}[f_{\delta_m}, \ldots, [\theta f_\alpha, [f_{\delta_{r-1}}, \ldots, [f_{\delta_2}, f_{\delta_1}] \ldots]] \ldots].$$

Since $[f_{\delta_{r-1}}, \ldots, [f_{\delta_2}, f_{\delta_1}] \ldots] \neq 0$ and is a root vector in $\mathfrak{g}^\wedge$, we have

$$e_{\omega\alpha} = c_{\omega\alpha}[y_1, [y_2, \ldots, [y_s, \theta f_\alpha] \ldots]],$$

where the y_i are root vectors in $\mathfrak{g}^\wedge$ for various negative roots. Let $x_1, \ldots, x_s$ be root vectors in $\mathfrak{g}^\wedge$ for the corresponding positive roots. Then repeated application of [3, p. 116, formula (20)] shows that there exists $q \in \mathbf{Q}^*$ $(= \mathbf{Q} - \{0\})$ such that

$$\theta f_\alpha = q(c_{\omega\alpha})^{-1}[x_s, [x_{s-1}, \ldots, [x_1, e_{\omega\alpha}] \ldots]],$$

and the bracket expression is a root vector in $\mathfrak{g}^\wedge$ for the root $-\theta\alpha$. But from the above, $\theta e_\alpha = c_\alpha x$, and x is a root vector in $\mathfrak{g}^\wedge$ for the root $\theta\alpha$. Since

$$B(\theta f_\alpha, \theta e_\alpha) = B(f_\alpha, e_\alpha) = 2/(\alpha, \alpha) \in \mathbf{Q}^*,$$

we must have $q(c_{\omega\alpha})^{-1}c_\alpha \in \mathbf{Q}^*$, and so $(c_{\omega\alpha})^{-1}c_\alpha \in \mathbf{Q}^*$.

We may now take $\mu_\alpha = \mu_{\omega\alpha} = (c_\alpha)^{-1/2}$. Indeed, $\theta e'_\alpha = \theta\mu_\alpha e_\alpha = (c_\alpha)^{1/2}x = (\mu_{\omega\alpha})^{-1}x$. But in the above formula defining x, all but one γ_i is in Π', and the remaining γ_i is $\omega\alpha$. Hence

$$(\mu_{\omega\alpha})^{-1}x = [f'_{\gamma_m}, [f'_{\gamma_{m-1}}, \ldots, [f'_{\gamma_2}, f'_{\gamma_1}] \ldots]] \in \mathfrak{g}',$$

so that $\theta e'_\alpha \in \mathfrak{g}'$. Finally (and it is here that we use the fact that $(c_{\omega\alpha})^{-1}c_\alpha \in \mathbf{Q}^*$),

$$\theta e'_{\omega\alpha} = \theta\mu_{\omega\alpha}e_{\omega\alpha} = \mu_{\omega\alpha}c_{\omega\alpha}y = c_{\omega\alpha}(c_\alpha)^{-1/2}y$$

$$= pc_\alpha(c_\alpha)^{-1/2}y = p(c_\alpha)^{1/2}y = p(\mu_\alpha)^{-1}y,$$

where $p \in \mathbf{Q}^*$. In the above formula defining y, all but one δ_i is in Π', and the remaining δ_i is α. Thus

$$(\mu_\alpha)^{-1}y = [f'_{\delta_m}, [f'_{\delta_{m-1}}, \ldots, [f'_{\delta_2}, f'_{\delta_1}] \ldots]] \in \mathfrak{g}',$$

and so $\theta e'_{\omega\alpha} = p(\mu_\alpha)^{-1}y \in \mathfrak{g}'$.

Since we have chosen nonzero elements $\mu_\alpha \in K$ for all $\alpha \in \Pi$ such that $\theta e'_\alpha = \theta\mu_\alpha e_\alpha \in \mathfrak{g}'$ for all $\alpha \in \Pi$, we are finished. Q.E.D.

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