Some locally convex spaces of continuous vector-valued functions over a completely regular space and their duals

by
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Abstract. The strict, superstrict and the $\beta_F$ topologies are defined on a space $A$ of continuous functions from a completely regular space into a Banach space $E$. Properties of these topologies are discussed and the corresponding dual spaces are identified with certain spaces of operator-valued measures. In case $E$ is a Banach lattice, $A$ becomes a lattice under the pointwise ordering and the strict and superstrict duals of $A$ coincide with the spaces of all $\tau$-additive and all $\sigma$-additive functionals on $A$ respectively.

Introduction. The Riesz representation theorem says that any continuous linear functional $F$ on the space of continuous real functions on a compact Hausdorff space $X$ with the uniform topology must have the form $F(f) = \int_X f \, dm$ for some bounded regular Borel measure on $X$. This representation was extended later to other spaces, first in case $X$ is locally compact and later by Aleksandrov [1] for continuous linear functionals on the space $C^b(X)$ of all bounded continuous real functions on a completely regular space. The representation was given by means of integrals with respect to members of the space $M(X)$ of all bounded, finitely-additive, regular with respect to zero sets, measures on the algebra generated by the zero sets. The $\sigma$-additive, $\tau$-additive and tight linear functionals correspond to the $\sigma$-additive, $\tau$-additive and tight members of $M(X)$ respectively (see Varadarajan [24]). Buck [4], for the locally compact case, and Sentilles [23], for the completely regular case, have defined the strict topologies on $C^b(X)$ which yield as dual spaces certain subspaces of $M(X)$. Several others like Hewitt [10], Bogdanowich [2], Wells [25], the author [12] and others have considered the problem of representation of linear functionals on spaces of continuous scalar-valued or vector-valued functions. In this paper we define certain locally convex topologies on spaces of continuous vector-valued functions on a completely regular space. We study some of the properties of these topologies and represent their duals with operator-valued measures on certain $\sigma$-algebras of subsets of $X$. The
integration process employed is a generalization of the process of Aleksandrov to the vector case. It is one of the many integration processes defined by McShane [17].

1. Definition and notation. Throughout this paper $X$ will denote a completely regular Hausdorff space and $Y$ will be a Hausdorff compactification of $X$. We will denote by $B$ the algebra of continuous real-valued functions $f$ on $X$ which have continuous extensions $\hat{f}$ to all of $Y$. Let $E$ be a Banach space over the real field. We will denote by $A$ the space of all continuous functions $f$ from $X$ into $E$ which have continuous extensions $\hat{f}$ to all of $Y$. Let $C = \{\hat{f} : f \in A\}$. If $f \in A$ and $g \in B$, the function $gf$ is defined on $X$ by $(gf)(x) = g(x)f(x)$. For $s \in E$ we will denote also by $s$ the element of $A$ whose value at every $x$ is equal to $s$.

We will consider on $A$ various locally convex topologies.

(a) The uniform topology $\sigma$ generated by the norm

$$f \mapsto \|f\| = \sup\{\|f(x)\| : x \in X\}.$$

(b) The topology $k$ generated by the family of seminorms $p_K$, $K$ compact in $X$, where

$$p_K(f) = \|f\|_K = \sup\{\|f(x)\| : x \in K\}.$$

(c) The topology $\pi$ generated by the family of seminorms $p_x$, $x \in X$, where

$$p_x(f) = \|f(x)\|.$$

Clearly all topologies $\pi, k, \sigma$ are Hausdorff and $\pi \leq k \leq \sigma$. Finally, if $\tau$ is a linear topology on $A$, then $(A, \tau)'$ denotes the topological dual of $(A, \tau)$.

2. The strict and superstrict topologies on $A$. Buck [4] defined the strict topology on the space of bounded continuous functions on a locally compact Hausdorff space. This topology has been studied later by several other authors. Recently Sentilles [23] defined the strict and superstrict topologies on the family of all bounded, continuous, real-valued functions on a completely regular Hausdorff space. In this section we will define the strict and superstrict topologies on the space $A$ defined in §1. Our approach will be analogous to that of Sentilles. Several of our theorems will be generalizations of his results.

A subset $Z$ of $Y$ is called a zero set if $Z = f^{-1}\{0\}$ for some continuous real function $f$ on $Y$. We will denote by $\Omega$ (or $\Omega_1$) the class of all closed (zero) subsets of $Y$ which are disjoint from $X$. For a $Q$ in $\Omega$, let $B_Q = \{f \in B : \hat{f} = 0 \text{ on } Q\}$. It is not hard to see that $B_Q$ is a Banach algebra (under the uniform norm) with an approximate identity of norm $\leq 1$.

Let $Q \in \Omega$. We will denote by $\beta_Q$ the locally convex topology on $A$ generated by the family of seminorms $f \mapsto \|gf\|$, $g \in B_Q$. The space $(A, \sigma)$ is a Banach space and a $B_Q$-module since $gf \in A$ for every $g \in B_Q$ and every $f \in A$. The topology $\beta_Q$ is the strict topology on $A$ as defined by Sentilles [20]. Hence
\( \beta_Q \) is the finest locally convex topology on \( A \) which agrees with \( \beta_Q \) on norm bounded subsets of \( A \) by Sentilles [21, Theorem 2.2]. A convex balanced absorbent set \( W \) in \( A \) is a \( \beta_Q \)-neighborhood of zero iff given \( r > 0 \) there exists a \( \beta_Q \)-neighborhood \( V \) of zero such that \( U_r \cap V \subseteq W \), where \( U_r = \{ f \in A : ||f|| \leq r \} \).

The strict topology \( \beta = \beta(A) \) on \( A \) is defined to be the inductive limit of the topologies \( \beta_Q, Q \in \Omega \). The superstrict topology \( \beta_1 \) is the inductive limit of the topologies \( \beta_Z, Z \in \Omega_1 \). By definition of the inductive limit topology (see Schaefer [19, p. 57]), a convex balanced absorbent subset \( W \) of \( A \) is a \( \beta \) (\( \beta_1 \)) neighborhood of zero iff \( W \) is a \( \beta_Q \)-neighborhood of zero for each \( Q \in \Omega \) (\( Q \in \Omega_1 \)).

**Theorem 2.1.** \( k < \beta < \beta_1 < \sigma \).

**Proof.** It is clear that \( \beta \leq \beta_1 \leq \sigma \). To prove that \( k \leq \beta \) consider an arbitrary compact set \( K \) in \( X \). We want to show that the set \( W = \{ f \in A : ||f||_K \leq 1 \} \) is a \( \beta \)-neighborhood of zero. Since \( W \) is convex balanced and absorbent it suffices to show that \( W \) is a \( \beta_Q \)-neighborhood of zero for every \( Q \) in \( \Omega \). So, let \( Q \in \Omega \). Since \( K \) is compact, there exists \( g \in B \) such that \( g = 1 \) on \( K \) and \( g = 0 \) on \( Q \). Then \( g \in B_Q \) and \( V = \{ f \in A : ||gf|| \leq 1 \} \subseteq W \). Since \( V \) is a \( \beta_Q \)-neighborhood of zero the result follows.

Since, for each \( Q \in \Omega, \beta_Q \) is the finest locally convex topology on \( A \) which agrees with \( \beta_Q \) on norm bounded subsets of \( A \), it follows that \( \beta \) (\( \beta_1 \)) is the finest locally convex topology on \( A \) which agrees with \( \beta \) (\( \beta_1 \)) on norm bounded sets.

**Lemma 2.2.** Let \( \pi \) be a locally convex topology on \( A \) such that \( R \leq \tau \leq \sigma \) and such that \( (A, \tau') = H \) is a norm closed subspace of \( A' = (A, \sigma) \). Then \( \tau \) and \( \sigma \) have the same bounded sets.

**Proof.** Every \( \sigma \)-bounded set is obviously \( \tau \)-bounded. On the other hand, suppose that \( G \) is a \( \tau \)-bounded subset of \( A \). Then \( G \) is \( \sigma(A, H) \)-bounded. By our hypothesis \( H \) is a Banach space under the norm

\[
\phi \rightarrow ||\phi|| = \sup \{|\phi(f)| : f \in A, ||f|| \leq 1\}.
\]

Each \( f \in A \) defines a bounded linear functional \( T_f \) on \( H \) by \( T_f(\phi) = \phi(f) \). Since \( G \) is \( \sigma(A, H) \)-bounded, we have sup \( \{ ||T_f(\phi)|| : f \in G \} \leq K \) for each \( \phi \in H \). By the principle of uniform boundedness there exists \( K > 0 \) such that sup \( \{ ||T_f|| : f \in G \} \leq K \). Let now \( f \in G \) and \( x \in X \). By the Hahn-Banach theorem there is a \( T \) in \( E', ||T|| \leq 1, T(f(x)) = ||f(x)|| \). Define \( \pi_x : A \rightarrow R, \pi_x(g) = T(g(x)) \). Then \( \pi_x \) is in \( H \) since \( \pi_x \in (A, \tau') \subseteq H \). Moreover \( ||\pi_x|| \leq 1 \). Thus \( ||f(x)|| = ||\pi_x(f)|| = T_f(\pi_x) \leq K \). It follows that sup \( \{ ||f|| : f \in G \} \leq K \) which completes the proof.

**Theorem 2.3.** Let \( \tau \) be as in Lemma 2.2. The following are equivalent:

1. \( \tau = \sigma \).
2. \( \tau \) is normable.
(3) \(\tau\) is metrizable.
(4) \(\tau\) is bornological.
(5) \(\tau\) is barrelled.

**Proof.** It is clear that (1) implies (2), (2) implies (3), and (3) implies (4). To prove that (4) implies (5), we first observe that the set \(U_1 = \{f \in A : \|f\| \leq 1\}\) is convex balanced and absorbs every norm (and hence every \(\tau\) bounded set). By (4) \(U_1\) is a \(\tau\)-neighborhood of zero. It follows that \(\tau = \sigma\). Since \((A, \sigma)\) is a Banach space, (5) follows. Finally to prove that (5) implies (1) we observe that the set \(U_1\) is \(\pi\)-closed and hence \(\tau\)-closed. Since \(U_1\) is also convex balanced and absorbent it follows that \(U_1\) is a \(\tau\)-neighborhood of zero and hence \(\tau = \sigma\).

**Theorem 2.4.** The duals of the spaces \((A, \beta)\) and \((A, \beta_1)\) are norm closed subspaces of the dual of \((A, \sigma)\).

**Proof.** Let \(\{T_n\}\) be a sequence in \((A, \beta)'\), \(T \in A'\), such that \(\|T_n - T\| \to 0\). Let \(W = \{f \in A : |T(f)| \leq 1\}\). We need to show that \(W\) is a \(\beta\)-neighborhood of zero. Since \(W\) is convex balanced and absorbent it suffices to show that given \(r > 0\) there exists a \(\beta\)-neighborhood \(V\) of zero such that \(V \cap \{f \in A : \|f\| \leq r\} \subseteq W\). So, let \(r > 0\). Choose \(n\) so that \(\|T_n - T\| < 1/(2r)\). Let \(V = \{f \in A : |T_n(f)| < 1/2\}\). Then \(V\) is a \(\beta\)-neighborhood of zero and \(V \cap \{f \in A : \|f\| \leq r\} \subseteq W\). This proves the result for \(\beta\). The proof for \(\beta_1\) is similar.

**Corollary 2.5.** (a) Theorem 2.3 holds for \(\tau = \beta\) or \(\beta_1\).
(b) \(\beta, \beta_1\) and \(\sigma\) have the same bounded sets.

**Theorem 2.6.** The topologies \(\beta\) and \(\sigma\) coincide iff \(X\) is compact.

**Proof.** Clearly \(\beta = \sigma\) when \(X\) is compact. On the other hand assume that \(X\) is not compact. Then \(Y \neq X\). Let \(x \in Y - X\), \(Q = \{x\}\). Let \(g \in B_Q\) and set \(V = \{f \in A : \|gf\| \leq 1\}\). Choose \(s \in E, \|s\| = 2\) and set \(F = \{y \in Y : |g(y)| \geq \frac{1}{2}\}\). There exists \(h \in B, 0 < h \leq 1\), such that \(\hat{h} = 0\) on \(F\) and \(\hat{h}(x) = 1\). Then the function \(f = hs\) is in \(V\) but not in \(U_1 = \{f \in A : \|f\| \leq 1\}\). Hence \(U_1\) does not contain \(V\). It follows that \(U_1\) is not a \(\beta_Q\)-neighborhood of zero and hence \(\beta \neq \sigma\).

3. The topology \(\beta_F\). Let \(F\) be a collection of compact subsets of \(X\) satisfying the following two conditions:

(1) \(\bigcup F = X\).
(2) \(F\) is directed, i.e. given \(G_1, G_2\) in \(F\) there exists \(G\) in \(F\) containing both \(G_1\) and \(G_2\). We denote by \(\tau_F\) the locally convex topology on \(A\) generated by the family of seminorms \(\{\|\cdot\|_G : G \in F\}\), where \(\|f\|_G = \sup \{|f(x)| : x \in G\}\). The topology \(\beta_F = \beta_F(A)\) is defined to be the mixed topology \(\gamma[\sigma, \tau_F]\) as defined by Wiweger [26]. By Wiweger we have \(\tau_F \leq \beta_F \leq \sigma\) and that \(\beta_F\) is the finest locally
convex topology on $A$ which agrees with $\tau_F$ on each norm bounded subset of $A$ (see Wiweger [26, 2.2.2.]).

**Lemma 3.1.** Let $G_1, \ldots, G_n$ be in $F$, $\epsilon > 0$ and $f$ in $A$. Then there exist $g, h$ in $A$ such that $f = g + h$, $h = 0$ on each $G_i$, and $\|g\| \leq \epsilon + \max \{\|f\|_{G_i} : 1 \leq i \leq n\}$.

**Proof.** Let $G = \bigcup G_i$ and set $d = \epsilon + \|f\|_G$. Then $G \subset V = \{x \in Y : |\hat{f}(x)| < d\}$. Since $G$ is compact and $V$ open there exists $h_0 \in B$, $0 \leq h_0 \leq 1$, $h_0 = 1$ on $G$, $h_0 = 0$ on $Y - V$. Set $g = fh_0$ and $h = f(1 - h_0)$. Then $g$ and $h$ satisfy the requirements.

**Corollary 3.2.** The sets of the form $\bigcap_{i=1}^n \{f \in A : \|f\|_{G_i} < a_i\}$, where $0 < a_i \to \infty$ and $G_i \in F$, constitute a $\beta_F$-neighborhood base at zero.

**Proof.** It follows from the preceding lemma and from Wiweger's Theorem 3.1.1.

We will next give an alternative description of $\beta_F$. Denote by $B_0(F)$ the collection of all bounded real-valued functions $f$ on $X$ with the property that given $\epsilon > 0$ there exists $G$ in $F$ such that $\{x \in X : |f(x)| \geq \epsilon\} \subset G$. We define on $A$ the locally convex topology $\tau(F)$ generated by the family of seminorms $P_g, g \in B_0(F)$, where

$$P_g(f) = \|gf\| = \sup \{\|g(x)f(x)\| : x \in X\}.$$ 

**Lemma 3.3.** The topologies $\tau(F)$ and $\tau_F$ coincide on each $\sigma$-bounded subset of the space $A$.

**Proof.** Let $r > 0$ and set $U_r = \{f \in A : \|f\| \leq r\}$. Suppose that $V$ is a subset of $U_r$ which is closed with respect to the relative topology of $\tau(F)$ on $U_r$. We will show that $V$ is closed with respect to the $\tau_F$ relative topology on $U_r$. Indeed, let $f \in U_r$ be in the $\tau_F$ closure of $V$. Let $g \in B_0(F)$. Given $\epsilon > 0$ there exists $G$ in $F$ such that $|g(x)| < \epsilon/(2r)$ if $x$ is not in $G$. Choose $h$ in $V$ with $\|f - h\|_G < \epsilon/\|g\|$. Then $\|g(f - h)\| \leq \epsilon$. This proves that $f$ is in the $\tau(F)$ closure of $V$ and hence in $V$. It follows that $\tau(F)|_{U_r} \leq \tau_F|_{U_r}$. On the other hand $\tau_F \leq \tau(F)$ because the characteristic function of any set in $F$ belongs to $B_0(F)$. We conclude that $\tau(F)$ and $\tau_F$ agree on $U_r$. The result follows.

**Theorem 3.4.** The topologies $\tau(F)$ and $\beta_F$ coincide.

**Proof.** Since $\beta_F$ is the finest locally convex topology on $A$ which agrees with $\tau_F$ on norm bounded subsets of $A$, we have $\tau(F) \leq \beta_F$ by 3.3. On the other hand, let $W = \bigcap_{i=1}^\infty \{f \in A : \|f\|_{G_i} \leq a_i\}$ where $0 < a_i \to \infty$ and $G_i \in F$. Define $g = \sup a_i^{-1} \chi_{G_i}$ ($\chi_{G_i}$ is the characteristic function of $G_i$). Then $g \in B_0(F)$. Moreover, $\{h \in A : \|gh\| \leq 1\} \subset W$. Hence $W$ is a $\tau(F)$-neighborhood of zero. Now Corollary 3.2 completes the proof.
Theorem 3.5. The topology $\beta_F$ is weaker than $\beta$.

Proof. Let $W$ be a convex balanced absorbent $\beta_F$-neighborhood of zero. Let $Q \in \Omega$ and $r > 0$. Since $\beta_F$ coincides with $\tau_F$ on $U_r$, there exist $G$ in $F$ and $\delta > 0$ such that $0 = \{f \in A : \|f\|_G \leq \delta\} \cap U_r \subset W$. Choose $g \in BQ$, $0 \leq g \leq 1$, $g = 1$ on $G$. Then $V \cap U_r \subset W$, where $V = \{f \in A : \|gf\| \leq \delta\}$. This proves that $W$ is a $\beta_Q$-neighborhood of zero for each $Q$ in $\Omega$. The theorem is proved.

We will next identify the space $(A, \beta_F)'$. Let $L_F(A)$ denote the collection of all linear functionals $T$ in $A'$ such that $T(f_a) \to 0$ whenever $\{f_a\}$ is a net in $U_1$ that converges to zero uniformly on each $G \in F$. We omit the proof of the following easily established

Lemma 3.6. $L_F(A)$ is a norm closed subspace of $A'$.

Lemma 3.7. Let $\phi \in (A, \sigma)'$. Then $\phi$ is in $L_F(A)$ iff for each $\epsilon > 0$ there exist $G$ in $F$ and $\delta > 0$ such that $|\phi(f)| < \epsilon$ for all $f \in A$ with $\|f\|_G \leq \delta$. Proof. The necessity of this condition is clear. To prove the sufficiency assume, by way of contradiction, that there exists $\epsilon > 0$ such that for each $G \in F$ and each $\delta > 0$ there exists $f = f_{(G, \delta)}$ in $A$, with $\|f\| \leq 1$ and $\|f\|_G \leq \delta$, such that $|\phi(f)| > \epsilon$. For $\alpha_1 = (G_1, \delta_1)$, $\alpha_2 = (G_2, \delta_2)$ we write $\alpha_1 \geq \alpha_2$ iff $G_1 \supset G_2$ and $\delta_1 < \delta_2$. In that way we get a net $\{f_{\alpha}\}$ in $U_1$. Moreover, $f_{\alpha} \to 0$ uniformly on each $G \in F$. Since $|\phi(f_{\alpha})| > \epsilon$ for all $\alpha$ we arrive at a contradiction.

Theorem 3.8. The topological dual of the space $(A, \beta_F)$ is the space $L_F(A)$.

Proof. Let $\phi \in L_F(A)$. Set $W = \{f \in A : |\phi(f)| \leq 1\}$. If $r > 0$, there exist $G \in F$ and $\delta > 0$ such that $|\phi(f)| \leq 1/r$ whenever $\|f\| \leq 1$, $\|f\|_G \leq \delta$. Let $V = \{f \in A : \|f\|_G \leq \delta r\}$. Then $V$ is a $\tau_F$-neighborhood of zero and $V \cap U_r \subset W$. This shows that $W$ is a $\beta_F$-neighborhood of zero in view of the fact that $\beta_F$ is the finest locally convex topology on $A$ which agrees with $\tau_F$ on norm bounded subsets of $A$. It follows that $\phi$ is $\beta_F$-continuous. Conversely, assume that $\phi$ is in $(A, \beta_F)'$ and let $\epsilon > 0$. There exist $G$ in $F$ and $\delta > 0$ such that

$$\{f \in A : \|f\|_G \leq \delta\} \cap U_{1/\epsilon} \subset \{f \in A : |\phi(f)| \leq \epsilon\}.$$ 

Thus $|\phi(f)| \leq \epsilon$ whenever $f \in U_1$ and $\|f\|_G \leq \delta \epsilon$. By Lemma 3.7, $\phi \in L_F(A)$.

Corollary 3.9. (a) $\beta_F$ and $\sigma$ have the same bounded sets.
(b) Theorem 2.3 holds if we replace $\tau$ with $\beta_F$.

4. The dual spaces of $(A, \beta)$, $(A, \beta_1)$, and $(A, \beta_F)$. In this section we will represent the dual spaces of $(A, \beta)$, $(A, \beta_1)$ and $(A, \beta_F)$ by means of integrals with respect to operator-valued measures. We will denote by $\text{Bo}(X)$ and $\text{Bo}(Y)$...
the $\sigma$-algebras of Borel subsets of $X$ and $Y$ respectively.

The $\sigma$-algebra of Baire subsets of $Y$ will be denoted by $\mathcal{B}(Y)$, while the $\sigma$-algebra of subsets of $X$, generated by the $B$-zero sets will be denoted by $\mathcal{B}(Z_B)$ (a subset $Z$ of $X$ is called a $B$-zero set if $Z = f^{-1}\{0\}$ for some $f \in B$).

Let $\Sigma$ be a $\sigma$-algebra of sets and let $\Sigma_1 \subseteq \Sigma$. A bounded, countably-additive, real-valued, measure $m$ on $\Sigma$ is called regular with respect to $\Sigma_1$ if for every $G \in \Sigma$ and every $e > 0$ there exists $G_1 \in \Sigma_1$, contained in $G$, such that $|m(H)| < e$ for every $H \in \Sigma$ which is contained in $G - G_1$.

We denote by $M_\sigma(B)$ the space of all bounded, real-valued, countably-additive, regular with respect to the family of all $B$-zero sets, measures on $\mathcal{B}(Z_B)$. The space of all bounded, real-valued countably-additive, regular with respect to the family of zero sets in $Y$, measures on $\mathcal{B}(Y)$ will be denoted by $M_\sigma(Y)$. A regular Borel measure on $\mathcal{B}(X)$ ($\mathcal{B}(Y)$) is a bounded, countably-additive, real-valued, measure on $\mathcal{B}(X)$ ($\mathcal{B}(Y)$) which is regular with respect to the closed sets in $X(Y)$. A regular Borel measure $m$ on $\mathcal{B}(X)$ is called $\tau$-additive if $|m|(F_\alpha) \to 0$ for each net $\{F_\alpha\}$ of closed sets in $X$ which decreases to the empty set. In the case of $Y$, every regular Borel measure is $\tau$-additive (see [23]).

Note. A regular Borel measure $m$ on $\mathcal{B}(X)$ is $\tau$-additive iff $|m|(Z_\alpha) \to 0$ for every net $\{Z_\alpha\}$ of $B$-zero sets which decreases to the empty set. Indeed, assume that the condition is satisfied and let $\{G_\alpha\}$ be a net of closed sets decreasing to the empty set. Since the zero sets in $Y$ form a base for the closed sets, it follows that the family of $B$-zero sets forms a base for the closed sets in $X$. Thus each $G_\alpha$ is an intersection of $B$-zero sets. Let

$$D = \{Z \subseteq X : Z \text{ a } B\text{-zero set, } Z \supseteq G_\alpha \text{ for some } \alpha\}.$$  

Then $D$ is directed (by inclusion) downwards to the empty set. By hypothesis, given $e > 0$, there exists $Z \in D$ with $|m|(Z) < e$. Let $\alpha_0$ be such that $G_{\alpha_0} \subset Z$.

Now for each $\alpha \geq \alpha_0$ we have $|m|(G_\alpha) \leq |m|(Z) < e$ which proves that $|m|(G_\alpha) \to 0$.

We will denote by $M_\tau(X)$ and $M_\tau(Y)$, respectively, the spaces of all $\tau$-additive regular Borel measures on $X$ and $Y$.

Let $M_\sigma(\mathcal{B}(Z_B), E')$ denote the set of all functions $m : \mathcal{B}(Z_B) \to E'$ with the following two properties:

1. For each $s \in E$, the function $ms : \mathcal{B}(Z_B) \to R$, $(ms)(F) = m(F)s$, is in $M_\sigma(B)$.

2. $|m|(X) < \infty$, where $|m|$ is defined on $\mathcal{B}(Z_B)$ by $|m|(G) = \sup \sum m(F_i)s_i$, the supremum being taken over all finite partitions $\{F_i\}$ of $G$ into sets in $\mathcal{B}(Z_B)$ (we will refer to such a partition as a $\mathcal{B}(Z_B)$-partition) and all finite collections $\{s_i\} \subset E$ with $\|s_i\| \leq 1$. We define $M_\sigma(\mathcal{B}(X), E')$ as we did $M_\sigma(\mathcal{B}(Z_B), E')$ by replacing $\mathcal{B}(Z_B)$ with $\mathcal{B}(X)$ and $M_\sigma(B)$ with $M_\sigma(X)$.
spaces $M_0(Ba(Y), E')$ and $M_r(Bo(Y), E')$ are defined analogously. For $m$ in any one of the above spaces, we define its norm $\|m\|$ by $\|m\| = |m|(X)$ or $|m|(Y)$ (depending on the $\sigma$-algebra on which $m$ is defined).

**Theorem 4.1.** If $m \in M_0(Ba(Z_B), E')$, then $|m| \in M_0(B)$.

**Proof.** It is easy to see that $|m|$ is a bounded, monotone, finitely-additive, set function on $Ba(Z_B)$. To prove the regularity, consider a $G \in Ba(Z_B)$ and let $\epsilon > 0$ be given. By the definition of $|m|(G)$, there exist a finite $Ba(Z_B)$-partition $\{F_i\}$ of $G$ and $s_i \in E$, with $\|s_i\| \leq 1$, such that $\sum m(F_i) s_i > |m|(G) - \epsilon$. By the regularity of each $m s_i$, there are $B$-zero sets $Z_i \subset F_i$ such that $\sum m(Z_i) s_i > |m|(G) - \epsilon$. The $B$-zero set $Z = \bigcup Z_i$ is contained in $G$. Moreover, $|m|(Z) \geq \sum m(Z_i) s_i > |m|(G) - \epsilon$ which proves the regularity of $|m|$.

To finish the proof it remains to show that $|m|$ is countably-additive. To this end, consider a sequence $\{F_i\}$ of disjoint members of $Ba(Z_B)$ and let $G = \bigcup_i F_i$. Since $|m|$ is monotone and finitely-additive, we have

$$|m|(G) \geq |m| \left( \bigcup_{i=1}^n F_i \right) = \sum_{i=1}^n |m|(F_i)$$

for all $n$ and hence $|m|(G) \geq \sum_{i=1}^\infty |m|(F_i)$. On the other hand, let $\epsilon > 0$ be given. There exist a $Ba(Z_B)$-partition $G_1, \ldots, G_N$ of $G$, and $s_i \in E$, with $\|s_i\| \leq 1$, such that $\sum_{i=1}^N m(G_i) s_i > |m|(G) - \epsilon$. For each $i$ we have $m(G_i) s_i = \sum_{n=1}^\infty m(G_i \cap F_n) s_i$.

Moreover,

$$\sum_{n=1}^\infty \sum_{i=1}^N |m(G_i \cap F_n) s_i| \leq \sum_{n=1}^\infty |m|(F_n) \leq |m|(G).$$

Hence

$$|m|(G) - \epsilon < \sum_{i=1}^N m(G_i) s_i = \sum_{i=1}^N \sum_{n=1}^\infty m(G_i \cap F_n) s_i$$

$$= \sum_{n=1}^\infty \sum_{i=1}^N m(G_i \cap F_n) s_i < \sum_{n=1}^\infty |m|(F_n) \leq |m|(G).$$

Since $\epsilon > 0$ was arbitrary, we conclude that $|m|(G) = \sum_{n=1}^\infty |m|(F_n)$. This completes the proof.

**Theorem 4.2.** If $m \in M_r(Bo(X), E')$, then $|m| \in M_r(X)$.

**Proof.** Using an argument similar to that of Theorem 4.1, we show that $|m|$ is a regular Borel measure on $Bo(X)$. It remains to show that $|m|$ is $\tau$-additive. By the note at the beginning of §4, it suffices to show that $|m|(Z_\alpha) \rightarrow 0$ for each net $\{Z_\alpha\}$ of $B$-zero sets which decreases to the empty set. So, let $\{Z_\alpha\}$ be such a net. For each $\alpha$ there exists a zero set $Z_\alpha$ in $Y$ such that $Z_\alpha = \bigcup Z_\alpha \cap X$. Define $\bar{m} : Bo(Y) \rightarrow E'$ by $\bar{m}(F) = m(F \cap X)$. For each $s \in E$, the function $\bar{m}s : Bo(Y) \rightarrow R$, $(\bar{m}s)(F) = (ms)(F \cap X)$ is in $M_r(Y)$ since $ms$ is $\tau$-additive. It
now follows easily that \( \tilde{m} \in M_c(\mathcal{B}(Y), E') \) and that \( |\tilde{m}|(F) = |m|(F \cap X) \) for each Borel set \( F \) in \( Y \). Let \( D \) denote the collection of all subsets \( Z \) of \( Y \) which are intersections of a finite number of \( \mathcal{B}_\alpha \)'s. Then \( D \) is directed downwards to \( G = \bigcap \mathcal{B}_\alpha \). Hence \( |\tilde{m}|(G) = \lim_{Z \in D} |\tilde{m}|(Z) \). Since \( G \cap X = \emptyset \) we have \( |\tilde{m}|(G) = 0 \). Therefore, given \( \epsilon > 0 \) there exists a \( Z = \mathcal{B}_{\alpha_1} \cap \cdots \cap \mathcal{B}_{\alpha_n} \) in \( D \) such that \( |\tilde{m}|(Z) < \epsilon \). Now, if \( \alpha \geq \alpha_1, \ldots, \alpha_n \), then

\[
|m|(Z_{\alpha}) \leq |m|(Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_n}) = |\tilde{m}|(Z) < \epsilon.
\]

This proves that \( \lim |m|(Z_{\alpha}) = 0 \) and the proof is complete.

We have analogous theorems for the elements in the spaces \( M_0(\mathcal{B}(Y), E') \) and \( M_c(\mathcal{B}(Y), E') \).

Next we will define integrals with respect to measures belonging to one of the spaces defined above. The integration process which we will employ is a generalization, to the vector case, of the process of Aleksandrov. It is one of the many integration processes defined by McShane [17].

Let \( m \in M_c(\mathcal{B}(Z_B), E'), G \in \mathcal{B}(Z_B) \), and \( f \in A \). Consider the collection \( D \) of all \( \alpha = \{F_i, \ldots, F_n; x_1, \ldots, x_n\} \) where \( \{F_i\} \) is a \( \mathcal{B}(Z_B) \) partition of \( G \) and \( x_i \in F_i \). For \( \alpha_1, \alpha_2 \in D \) we write \( \alpha_1 \geq \alpha_2 \) iff the partition of \( G \) in \( \alpha_1 \) is a refinement of the partition of \( G \) in \( \alpha_2 \). Then \( (D, \geq) \) is a directed set. For each \( \alpha = \{F_i, \ldots, F_n; x_1, \ldots, x_n\} \) in \( D \) we define \( w_\alpha = \sum m(F_i)f(x_i) \). We will show that \( \{w_\alpha\} \) is a Cauchy net in \( R \) and hence convergent. Indeed, let \( \epsilon > 0 \) be given. We may assume, without loss of generality, that \( ||m|| \leq 1 \). For each \( x \in X \), let \( V_x = \{y \in X: ||f(x) - f(y)|| < \epsilon\} \). Then \( V_x \) is a \( \mathcal{B}_c \)-cozero set and hence in \( \mathcal{B}(Z_B) \): If \( W = \{s \in E: ||s|| < \epsilon\} \), then \( V_x = f^{-1}(f(x) + W) \). Since \( f(X) \) is totally bounded, there are \( x_1, \ldots, x_N \) in \( X \) such that \( f(X) \subseteq \bigcup_{i=1}^N (f(x_i) + W) \). Thus \( X = \bigcup_{i=1}^N V_{x_i} \). Let \( G_1 = V_{x_1} \cap G \). Define \( G_i = G_{i-1} \cap G_{i+1} \) for \( i = 1, \ldots, N - 1 \). Keeping those \( G_i \) which are not empty we get a \( \mathcal{B}(Z_B) \) partition \( \{F_1, \ldots, F_n\} \) of \( G \) with the property that \( ||f(x) - f(y)|| \leq 2\epsilon \) if \( x, y \) are in the same \( F_i \). Pick \( x_i \in F_i \) and let \( \alpha_0 = \{F_1, \ldots, F_n; x_1, \ldots, x_n\} \).

If \( \alpha_1, \alpha_2 \) are in \( D \) with \( \alpha_1, \alpha_2 \geq \alpha_0 \), then

\[
|w_{\alpha_1} - w_{\alpha_2}| \leq |w_{\alpha_1} - w_{\alpha_0}| + |w_{\alpha_0} - w_{\alpha_2}| \leq 2\epsilon |m|(G) + 2\epsilon |m|(G) \leq 4\epsilon.
\]

This proves that the net \( \{w_\alpha\} \) is a Cauchy net and hence convergent. We define \( \int_G f dm = \lim w_\alpha \). It can be shown easily that, for disjoint \( F_1 \) and \( F_2 \) in \( \mathcal{B}(Z_B) \) and \( G = F_1 \cup F_2 \), we have \( \int_G f dm = \int_{F_1} f dm + \int_{F_2} f dm \). Moreover we have the following easily established

**Lemma 4.3.** (a) The map \( f \mapsto \int_G f dm \) is linear on \( A \).

(b) \[ |\int_G f dm| \leq \int_G ||f(x)|| d|m|(x) \leq ||f|| |m|(G) \] for all \( f \in A \).
We similarly define integrals of functions in $A$ with respect to members of $M_r(\mathcal{B}_0(X), E')$.

**Lemma 4.4.** Let $m$ be a bounded, real-valued, countably additive measure on $\mathcal{B}(\mathcal{Z}_B)$. Then $m \in M_o(B)$.

**Proof.** We first show that for any $B$-zero set $Z$ and any $\epsilon > 0$ there exists a $B$-cozero set $V$, containing $Z$, such that $|m|(Z) > |m|(V) - \epsilon$. Indeed, if $Z$ is a $B$-zero set, there exists $\alpha > 0$ in $B$ such that $Z = \mathbb{1}_{\{0\}} - \{0\}$. For each positive integer $n$, we set $V_n = \{x : f(x) < 1/n\}$. Then $V_n$ is a $B$-cozero set and the sequence $\{V_n\}$ decreases to $Z$. Since $|m|$ is countably additive, we have $|m|(Z) = \lim |m|(V_n)$ which implies our claim. Thus every $B$-zero set belongs to the family $\Sigma$ of all subsets $G$ of $X$ with the following property: Given $\epsilon > 0$ there exist a $B$-zero set $Z$ and a $B$-cozero set $V$, with $Z \subset G \subset V$, such that $|m|(V - Z) < \epsilon$. The family $\Sigma$ is a $\sigma$-algebra which contains all $B$-zero sets and hence $\mathcal{B}(\mathcal{Z}_B) \subset \Sigma$. This implies that $m$ is regular with respect to the family of all $B$-zero sets. The lemma is proved.

**Lemma 4.5.** Let $m \in M_r(\mathcal{B}_0(X), E')$ and let $\mu$ denote the restriction of $m$ to $\mathcal{B}(\mathcal{Z}_B)$. Then $\mu \in M_o(\mathcal{B}(\mathcal{Z}_B), E')$ and $\int_G f \text{d}m = \int_G f \text{d}\mu$ for each $f \in A$.

**Proof.** In view of the preceding lemma, the restriction $m s|_{\mathcal{B}(\mathcal{Z}_B)}$ belongs to $M_o(B)$ for all $s \in E$. Now, it follows that $\mu \in M_o(\mathcal{B}(\mathcal{Z}_B), E')$. If we look at the proof of the existence of $\int_G f \text{d}m$ and $\int_G f \text{d}\mu$ we can see that $\int_G f \text{d}m$ and $\int_G f \text{d}\mu$ coincide.

Integrals of functions in $C$, with respect to members of $M_o(\mathcal{B}(\mathcal{Y}), E')$ and $M_r(\mathcal{B}(\mathcal{Y}), E')$, are defined similarly.

**Lemma 4.6.** If $m_1, m_2 \in M_r(\mathcal{B}(\mathcal{Y}), E')$ are such that $\int_{\mathcal{Y}} f \text{d}m_1 = \int_{\mathcal{Y}} f \text{d}m_2$ for all $f \in A$, then $m_1 = m_2$.

**Proof.** Let $s \in E$. For each $f \in B$, we have 
\[ \int_{\mathcal{Y}} \hat{f} \text{d}(m_1 s) = \int_{\mathcal{Y}} \hat{f} \text{d}m_1 = \int_{\mathcal{Y}} \hat{f} \text{d}m_2 = \int_{\mathcal{Y}} \text{d}(m_2 s). \]
By the uniqueness part of the Riesz representation theorem, we have $m_1 s = m_2 s$. This, being true for all $s \in E$, implies that $m_1 = m_2$.

For a proof of the following theorem see Wells [25].

**Theorem 4.7.** Let $\phi$ be a linear functional on $C$. Then $\phi$ is continuous with respect to the uniform norm topology iff there exists $m \in M_r(\mathcal{B}(\mathcal{Y}), E')$ such that $\phi(\hat{f}) = \int_{\mathcal{Y}} \hat{f} \text{d}m$ for all $\hat{f} \in C$. Moreover, $\|\phi\| = \|m\|$. If $m \in M_o(\mathcal{B}(\mathcal{Y}), E')$ and $m_1 = m|_{\mathcal{B}(\mathcal{Y})}$ then $m_1 \in M_o(\mathcal{B}(\mathcal{Y}), E')$ and $\int_{\mathcal{Y}} \hat{f} \text{d}m_1 = \int_{\mathcal{Y}} \hat{f} \text{d}m$ for all $f \in A$. Furthermore, $\|m_1\| = \|m\|$. To prove the last equality, consider the linear map $\phi : C \rightarrow R$, $\phi(\hat{f}) = \int_{\mathcal{Y}} \hat{f} \text{d}m = \int_{\mathcal{Y}} \hat{f} \text{d}m_1$. By 4.7
we have \( \|\phi\| = \|m\| \). Also \( \|\phi\| \leq \|m_1\| \) since \( |\phi(f)| = \int f \hat{\phi} dm_1 \leq \|\phi\| \|m_1\| \).

Since \( \|m_1\| \geq \|m_1\| \), it follows that \( \|\phi\| = \|m_1\| = \|m\| \). Moreover, the inequality \( |m_1|\{G\} \leq |m|\{G\} \), together with \( |m_1|\{Y\} = |m|\{Y\} \), implies that \( |m_1| = |m| \). 

Let now \( \phi \in A' \). Define \( \hat{\phi} : C \to R \), \( \hat{\phi}(f) = \phi(f) \). Clearly \( \hat{\phi} \in C' \). Let \( m = \hat{m}_\phi \) be the element of \( M_r(Bo(Y), E') \) that corresponds to \( \hat{\phi} \) by Theorem 4.7.

**Lemma 4.8.** For a \( Q \subset \Omega \), the following are equivalent:

1. \( \phi \in (A, \beta Q)' \).
2. \( |m|(Q) = 0 \).

**Proof.** (1) \( \to \) (2). By regularity it suffices to show that \( m(G)s = 0 \) for each closed set \( G \) in \( Y \) contained in \( Q \) and each \( s \in E \), \( \|s\| \leq 1 \). So, let \( G \) be such a set and \( s \in E \) with \( \|s\| \leq 1 \). There exists an open set \( O \) in \( Y \) containing \( G \) and such that \( |ms|(O - G) < \epsilon \) (\( \epsilon > 0 \) arbitrary). Since \( \phi \) is \( \beta Q \)-continuous, there exist \( g \in B_Q \) and \( K > 0 \) such that \( |\phi(f)| \leq K \) for all \( f \in A \) with \( \|gf\| \leq 1 \). Choose \( n > 0 \) so that \( K/n < \epsilon \). Set

\[ O_1 = \{x \in Y : |g(x)| < 1/n\} \quad \text{and} \quad O_2 = O_1 \cap O. \]

Clearly \( G \subset O_2 \) and \( |ms|(O_2 - G) < \epsilon \). Choose \( h \in B \), \( 0 < h < 1 \), \( \hat{h} = 1 \) on \( G \) and \( \hat{h} = 0 \) on the complement of \( O_2 \). Let \( f = nhs \). Since \( \|gf\| \leq 1 \), we have \( |\phi(hs)| \leq K/n < \epsilon \). But

\[ |\phi(hs)| = \left| m(F)s + \int_{O_2 - G} \hat{h}s\ dm \right| \geq |m(F)s| - \epsilon. \]

Thus \( |m(G)s| \leq 2\epsilon \) which proves that \( m(G)s = 0 \) and (2) follows.

(2) \( \to \) (1). Suppose that \( |m|(Q) = 0 \) and let \( r > 0 \). Choose an open set \( V \) in \( Y \) with \( |m|(V) < 1/(2r) \), \( Q \subset V \). There exists \( g \in B_Q \) such that \( \hat{g} = 1 \) on \( V \). Set \( W = \{f \in A : \|gf\| \leq 1/2\|m\|\} \). Then \( W \cap U_r \subset H \) where \( H = \{f \in A : |\phi(f)| \leq 1\} \) and \( U_r = \{f \in A : \|f\| \leq r\} \). This shows that \( H \) is a \( \beta Q \)-neighborhood of zero and hence \( \phi \) is \( \beta Q \)-continuous.

**Theorem 4.9.** Let \( \phi \in A' \) and let \( m \in M_r(Bo(Y), E') \) be such that \( \phi(f) = \int_Y f \hat{\phi} dm \) for all \( f \in A \). Then:

1. \( \phi \in (A, \beta)' \) iff \( |m|(Q) = 0 \) for all \( Q \in \Omega \).
2. \( \phi \in (A, \beta_1)' \) iff \( |m|(Z) = 0 \) for all \( Z \in \Omega_1 \).

**Proof.** It follows from the preceding lemma and from the fact that \( \phi \) is \( \beta \)-continuous iff \( \phi \) is \( \beta Q \)-continuous for all \( Q \in \Omega \), and \( \phi \) is \( \beta_1 \)-continuous iff \( \phi \) is \( \beta Q \)-continuous for all \( Q \in \Omega_1 \).

Let now \( \hat{m} \in M_r(Bo(Y), E') \) be such that \( |\hat{m}|(Q) = 0 \) for all \( Q \in \Omega \). By the regularity of \( |\hat{m}| \), we have \( |\hat{m}|(G) = 0 \) for each Borel set \( G \) in \( Y \) disjoint from \( X \). Define \( m : Bo(X) \to E' \) by \( m(G \cap X) = \hat{m}(G) \) for each \( G \) in \( Bo(Y) \). This
gives us a well-defined function on \( \text{Bo}(X) \). The proof of the following is straightforward and we omit it.

**Lemma 4.10.**

1. \( m \in M_{r}(\text{Bo}(X), E') \).
2. \( |m|(G \cap X) = |\hat{m}|(G) \) for each \( G \) in \( \text{Bo}(Y) \).
3. \( \int_{X}f\,dm = \int_{Y}f\,d\hat{m} \) for all \( f \in A \).

Similarly, if \( \hat{m}_{1} \in M_{r}(\text{Bo}(Y), E') \) is such that \( |\hat{m}_{1}|(Z) = 0 \) for each \( Z \in \Omega_{1} \), then the function \( m_{1} : \text{Bo}(Z_{B}) \to E' \), \( m_{1}(G \cap X) = \hat{m}_{1}(G) \) for all \( G \in \text{Bo}(Y) \), is well defined and the following is true.

**Lemma 4.11.**

1. \( m_{1} \in M_{r}(\text{Bo}(Z_{B}), E') \).
2. \( |m_{1}|(G \cap X) = |\hat{m}_{1}|(G) \) for each \( G \) in \( \text{Bo}(Y) \).
3. \( \int_{X}f\,dm_{1} = \int_{Y}f\,d\hat{m}_{1} \) for each \( f \in A \).

An element \( \phi \) of the uniform dual \( B' \) of \( B \) is called \( \tau \)-additive iff \( \phi(f_{\alpha}) \to 0 \) for each net \( \{f_{\alpha}\} \) in \( B \) which decreases pointwise to zero. Let \( L_{r}(B) \) denote the collection of all \( \tau \)-additive members of \( B \).

**Lemma 4.12.** The map \( m \to \phi \) defined by the formula \( \phi(f) = \int fdm \) for all \( f \in B \) establishes an isomorphism between the spaces \( M_{r}(X) \) and \( L_{r}(B) \).

**Proof.** By LeCam [16, p. 214], every \( \tau \)-additive member of \( B' \) has a unique extension to a \( \tau \)-additive functional on the space \( C^{b}(X) \) of all bounded continuous real-valued functions on \( X \). By Varandarajan [24] and by Kirk [13, Theorem 1.12], the space of \( \tau \)-additive functionals on \( C^{b}(X) \) is isomorphic to the space \( M_{r}(X) \) via the isomorphism \( m \to \phi, \phi(f) = \int fdm \) for all \( f \in C^{b}(X) \). Hence the result follows.

**Lemma 4.13.** Let \( m \in M_{r}(X) \). Define \( \tilde{m} \) on \( \text{Bo}(Y) \) by \( \tilde{m}(G) = m(G \cap X) \). Then \( \tilde{m} \in M_{r}(Y) \).

**Proof.** By (4.12), the linear functional \( \phi \), defined on \( B \) by \( \phi(f) = \int fdm \), is \( \tau \)-additive. Define \( \hat{\phi} \) on the space \( C(Y) = \{ \hat{f} : f \in B \} \) by \( \hat{\phi}(\hat{f}) = \phi(f) \). Then \( \hat{\phi} \) is in the uniform dual of \( C(Y) \). By the Riesz representation theorem there exists \( \mu \in M_{r}(Y) \) such that \( \hat{\phi}(\hat{f}) = \int_{Y}f\,d\mu \) for all \( f \in B \). By an argument similar to that employed by Knowles [14, Theorem 2.4], we show that \( |\mu|(G) = 0 \) for each Borel subset \( G \) of \( Y \) which is disjoint from \( X \). Define \( m_{1} \) on \( \text{Bo}(X) \) by \( m_{1}(G \cap X) = \mu(G) \) for all \( G \in \text{Bo}(Y) \). It is easy to see that \( m_{1} \) is a well-defined element of \( M_{r}(X) \) and \( \int_{X}f\,dm_{1} = \int_{Y}f\,d\mu = \phi(f) = \int_{X}f\,dm \) for all \( f \in B \). By 4.12 we have \( m = m_{1} \). Since for \( G \in \text{Bo}(Y), \mu(G) = m_{1}(G \cap X) = m(G \cap X) \), it follows that \( \hat{m} = \mu \in M_{r}(Y) \). This completes the proof.

**Lemma 4.14.** If \( m \) and \( m_{1} \) are both in \( M_{r}(B) \) and if \( \int fdm = \int fdm_{1} \) for all \( f \in B \), then \( m = m_{1} \).
Proof. Let \( Z \) be a \( B \)-zero set. There exists a sequence \( \{f_n\} \) in \( B \) which decreases pointwise to the characteristic function \( \chi_Z = g \) of \( Z \). Thus

\[
m(Z) = \int g \, dm = \lim \int f_n \, dm = \lim \int f_n \, dm_1 = m_1(Z).
\]

The result now follows from the regularity of \( m \) and \( m_1 \).

**Lemma 4.15.** Let \( m \in M_o(B) \). Define \( \tilde{m} \) on \( Ba(Y) \) by \( \tilde{m}(G) = m(G \cap X) \) for all \( G \in Ba(Y) \). Then \( \tilde{m} \in M_o(Y) \).

**Proof.** Let \( \mu \in M_+(Y) \) be such that \( \int_Y \hat{f} \, d\mu = \int_X \hat{f} \, dm \) for all \( f \in B \). Let \( \mu_1 = \mu |_{Ba(Y)} \). Then \( \mu_1 \in M_o(Y) \) and \( \int_X f \, dm = \int_Y \hat{f} \, d\mu_1 \) for all \( f \in B \). Since the functional \( f \mapsto \int f \, dm \) is \( \sigma \)-additive on \( B \) (i.e. \( \int f_n \, dm \to 0 \) for each sequence \( \{f_n\} \) in \( B \) which decreases pointwise to zero) it follows, as in the proof of Theorem 2.1 of Knowles [14], that \( |\mu_1|(G) = 0 \) for each Baire set \( G \) in \( Y \) which is disjoint from \( X \). Define \( m_1 : Ba(Z_B) \to R \), \( m_1(G \cap X) = \mu_1(F) \) for each Baire set \( G \) in \( Y \). Then \( m_1 \) is a well-defined member of \( M_o(B) \) and \( \int f \, dm_1 = \int f \, d\mu_1 = \int f \, dm \) for all \( f \in B \). By Lemma 4.14 we have \( m_1 = \mu_1 \). Thus \( m = \mu_1 \in M_o(Y) \).

Now using 4.13 and 4.15 we easily get the following result.

**Lemma 4.16.** Let \( m \in M_+(Bo(X), E') \) and \( m_1 \in M_o(Ba(Z_B), E') \). Define \( \hat{m} \) and \( \hat{m}_1 \) on \( Bo(Y) \), respectively, by \( \hat{m}(Q) = m(Q \cap X) \), \( \hat{m}_1(G) = m_1(G \cap X) \). Then:

1. \( \hat{m} \in M_+(Bo(Y), E') \) and \( \hat{m}_1 \in M_o(Ba(Y), E') \).
2. \( |\hat{m}|(Q) = |m|(Q \cap X) \) for all \( Q \in Bo(Y) \), and \( |\hat{m}_1|(Q) = |m_1|(G \cap X) \) for all \( G \in Ba(Y) \).
3. \( \int_X f \, dm = \int_Y \hat{f} \, d\hat{m} \) and \( \int_X f \, dm_1 = \int_Y \hat{f} \, d\hat{m}_1 \) for all \( f \in A \).

**Lemma 4.17.** Every \( m \in M_o(Ba(Y), E') \) has a unique extension to a \( \mu \) in \( M_+(Bo(Y), E') \).

**Proof.** Define \( \phi \) on \( C \) by \( \phi(\hat{f}) = \int \hat{f} \, dm \). Then \( \phi \in C' \). By 4.7 there exists a unique \( \mu \) in \( M_+(Bo(Y), E') \) such that \( \phi(\hat{f}) = \int \hat{f} \, d\mu \) for all \( f \in A \). Let \( \mu_1 = \mu |_{Ba(Y)} \). We will show that \( \mu_1 = m \). Indeed, let \( s \in E \). Then \( \mu_1 s \) and \( ms \) are both in \( M_o(Y) \). Moreover \( \int f \, dm_1 = \int f \, d\mu_1 = \int f \, d(\mu_1 s) \) for all \( f \in B \). It follows that \( ms = \mu_1 s \) for all \( s \in E \) and hence \( m = \mu_1 \). Since \( \mu \in M_o(Bo(Y), E') \) the result follows.

Combining Lemmas 4.6, 4.17 and 4.16 we get

**Lemma 4.18.** If \( m_1, m_2 \in M_+(Bo(X), E') \) \( [m_1, m_2 \in M_o(Ba(Z_B), E')] \) are such that \( \int_X f \, dm_1 = \int_X f \, dm_2 \) for all \( f \in A \), then \( m_1 = m_2 \).

We are now in a position to identify the dual spaces of \( (A, \beta) \), \( (A, \beta_1) \) and \( (A, \beta_F) \).
Theorem 4.19. Let $\phi \in A'$. Then:

1. $\phi$ is $\beta$-continuous iff there exists $m \in M_{\beta}(Bo(X), E')$ such that $\int f \, dm = \phi(f)$ for all $f \in A$.
2. $\phi$ is $\beta_1$-continuous iff there exists $m \in M_{\alpha}(Ba(Z_B), E')$ such that $\phi(f) = \int f \, dm$ for all $f \in A$.

Furthermore, the $m$ that corresponds to a $\beta$-continuous ($\beta_1$-continuous) member $\phi$ of $A'$ is unique and $\|\phi\| = \|m\|$.

Proof. (1) Suppose that $\phi$ is $\beta$-continuous. Let $\hat{m}$ be the element of $M_{\beta}(Bo(Y), E')$ with the property that $\phi(f) = \int \hat{m} \, f$ for all $f \in A$. Define $m$ on $Bo(X)$ by $m(G \cap X) = \hat{m}(G)$ for all $G \in Bo(Y)$. This gives us an element $m$ of $M_{\beta}(Bo(X), E')$ by 4.9 and 4.10. Moreover, by 4.10, $\int_X f \, dm = \int_Y \hat{m} \, f = \phi(f)$ for all $f \in A$. Also $\|\phi\| = \|\hat{m}\| = \|m\|$. Conversely, let $m \in M_{\beta}(Bo(X), E')$ be such that $\phi(f) = \int f \, dm$ for all $f \in A$. Define $\hat{m}$ on $Bo(Y)$ by $\hat{m}(G) = m(G \cap X)$. By 4.16, we have $m \in M_{\beta}(Bo(Y), E')$ and $\int_X f \, dm = \int_Y \hat{m} \, f$ for all $f \in A$. Since $|\hat{m}(Q)| = |m|(Q \cap X) = 0$ for all $Q \in \Omega$, we have $\phi \in (A, \beta_1)'$ by 4.9. Finally the uniqueness of $m$ follows from 4.18.

(2) The proof is similar to that of (1).

Theorem 4.20. For a $\phi \in A'$ the following are equivalent:

1. $\phi \in (A, \beta_F)'$.
2. There exists $m \in M_{\beta}(Bo(X), E')$ such that
   (a) $\phi(f) = \int f \, dm$ for all $f \in A$,
   (b) given $\epsilon > 0$ there exists $G \in F$ with $|m|(X - G) < \epsilon$.

Proof. (2) $\rightarrow$ (1). Let $\epsilon > 0$ be given. Choose $G$ in $F$ with $|m|(X - G) < \epsilon/2$. Let $\delta > 0$ be such that $2\delta \|m\| < \epsilon$. If $f \in A$, $\|f\| \leq 1$, $\|f\|_G \leq \delta$, then

$$|\phi(f)| \leq \left|\int_G f \, dm\right| + \left|\int_{X - G} f \, dm\right| \leq \delta \|m\|(G) + |m|(X - G) < \epsilon.$$

Hence $\phi \in (A, \beta_F)'$ by 3.7.

(1) $\rightarrow$ (2). Since $\beta_F \leq \beta$, we have $\phi \in (A, \beta)'$. Hence there exists $m \in M_{\beta}(Bo(X), E')$ such that $\phi(f) = \int f \, dm$ for all $f \in A$. Define $\hat{m}$ on $Bo(Y)$ by $\hat{m}(G) = m(G \cap X)$. By 4.16, $\hat{m} \in M_{\beta}(Bo(Y), E')$. Let $\epsilon > 0$ be given. By 3.7 there exist $G$ in $F$ and $\delta > 0$ such that $|\phi(f)| < \epsilon/3$ for all $f \in W = \{h \in A : \|h\| \leq 1, \|h\|_G \leq \delta\}$. By the definition of $|\hat{m}|$ there exist a partition $F_1, \ldots, F_n$ of $Y - G$, into Borel sets, and $s_i \in E$, with $\|s_i\| \leq 1$, such that $\Sigma \hat{m}(F_i) s_i > |\hat{m}|(Y - G) - \epsilon_1 = |m|(X - G) - \epsilon_1$. There are closed sets $G_i$ in $Y$, $G_i \subset F_i$, such that $\Sigma \hat{m}(G_i) s_i > |m|(X - G) - \epsilon_1$. Choose pairwise disjoint open sets $V_i, 1 \leq i \leq n, G_i \subset V_i \subset Y - G$, such that $\Sigma |\hat{m}|(V_i - G_i) < \epsilon_1$. For each $i, 1 \leq i \leq n$, choose $h_i$ in $B$, $0 \leq h_i \leq 1, \hat{h}_i = 1$ on $G_i$ and $\hat{h}_i = 0$ on $Y - V_i$. Let $f = \Sigma h_i s_i$. Then $f \in W$ and hence $|\phi(f)| < \epsilon_1$. Since
it follows that $|m|(X - G) \leq 3\varepsilon_1 = \varepsilon$. The theorem is proved.

**Definition.** A subset $M_0$ of $M_\varepsilon(\text{Bo}(X), E')$ is called F-tight if $M_0$ is norm bounded and given $\varepsilon > 0$ there exists $G$ in $F$ with $|m|(X - G) \leq \varepsilon$ for all $m$ in $M_0$.

**Lemma 4.21.** Let $\phi \in (A, \beta)'$ and let $m$ be the corresponding element of $M_\varepsilon(\text{Bo}(X), E')$. Let $G$ in $F$ and $\varepsilon > 0$. The following are equivalent:

1. $|m|(X - G) \leq \varepsilon$.
2. For all $f \in A$ with $\|f\| \leq 1$ and $\|f\|_G = 0$ we have $|\phi(f)| \leq \varepsilon$.

We omit the proof of this lemma since we can use an argument similar to that used in the implication (1) $\longrightarrow$ (2) of Theorem 4.20.

For $H \subset L_F(A)$, let $M_H = \{m_\phi : \phi \in H\} \subset M_\varepsilon(\text{Bo}(X), E')$ where $m_\phi$ is the measure that corresponds to $\phi$.

**Theorem 4.22.** For $H \subset L_F(A)$ the following are equivalent:

1. $H$ is $\beta_F$-equicontinuous.
2. (a) $H$ is norm bounded.
   (b) Given $\varepsilon > 0$ there exists $G \in F$ such that $|\phi(f)| \leq \varepsilon$ for all $\phi \in H$ and all $f \in A$ with $\|f\| \leq 1$ and $f = 0$ on $G$.
3. $M_H$ is F-tight.

**Proof.** By 4.21, (2) and (3) are equivalent.

(1) $\longrightarrow$ (2). (a) The set $U_1 = \{f \in A : \|f\| \leq 1\}$ is norm bounded and hence $\beta$-bounded. Since $H^0$ (= polar of $H$ with respect to the pair $(L_F(A), A)$) is a $\beta_F$-neighborhood of zero there exists $K > 0$ such that $U_1 \subset KH^0$. It follows that $\|\phi\| \leq K$ for all $\phi$ in $H$.

(b) Let $\varepsilon > 0$ be given. Since $eH^0$ is a $\beta_F$-neighborhood of zero there exist $G$ in $F$ and $\delta > 0$ such that $W = \{f \in A : \|f\| \leq 1, \|f\|_G \leq \delta\} \subset eH^0$. Thus (b) follows.

(3) $\longrightarrow$ (1). Let $d = \sup \{\|m_\phi\| : \phi \in H\} = \sup \{\|m_\phi\| : \phi \in H\}$. Given $r > 0$ there exists $G$ in $F$ such that $|m_\phi|(X - F) \leq 1/(2r)$ for all $\phi$ in $H$. If $V = \{f \in A : \|f\|_G \leq 1/(2d)\}$, then $V \cap U_r \subset H^0$, where $U_r = \{f \in A : \|f\| \leq r\}$. This shows that $H^0$ is a $\beta$-neighborhood of zero and this completes the proof.

5. In this section we will assume that $E$ is a Banach lattice. We write $f \geq g$ iff $f(x) \geq g(x)$ for all $x \in X$. Since the lattice operations are continuous, it is easy to verify that $A$, under the relation $\geq$, is a Banach lattice where for $f, g$ in $A$ we have

\[(f \land g)(x) = f(x) \land g(x),\]
"(f \lor g)(x) = f(x) \lor g(x), \quad \text{and} \quad |f|(x) = |f(x)|
for all \(x \in X\). For a \(\phi \in A'\) the \(\phi^+\), \(\phi^-\), \(|\phi|\) are the elements of \(A'\) which are defined on positive \(f \in A\) by
\[
\phi^+(f) = \sup \{\phi(g) : 0 \leq g \leq f\},
\phi^-(f) = -\inf \{\phi(g) : 0 \leq g \leq f\},
|\phi|(f) = \sup \{|\phi(g)| : |g| \leq f\}.
\]

**Theorem 5.1.** Each of the spaces \((A, \beta), (A, \beta_1)\) and \((A, \beta_F)\) is locally solid.

**Proof.** Let \(W\) be a convex balanced \(\beta\)-neighborhood of zero. For each \(Q \in \Omega\) there exists \(g_Q \in B_Q\) such that \(V_Q = \{f \in A : \|g_Q f\| \leq 1\} \subset W\). Each \(V_Q\) is clearly solid. Hence the set \(V = \bigcup\{V_Q : Q \in \Omega\}\) is solid. By Peressini [18, p. 161], the convex balanced hull \(V_0\) of \(V\) is solid. Since \(V_0 \subset W\), the result follows for \((A, \beta)\). The proof for \((A, \beta_1)\) is similar. For the \((A, \beta_F)\) we observe that the class of sets of the form \(\bigcap_{i=1}^\infty \{f \in A : \|f\| \leq a_i\}\), where \(0 < a_i \to \infty\) and \(G_i \in F\), consists of solid sets and is a \(\beta_F\)-base at zero.

**Definitions.** For a net \(\{f_\alpha\}\) in \(A\), we say that it decreases to zero, and write \(f_\alpha \downarrow 0\), if for each \(x \in X\) we have \(\lim f_\alpha(x) = 0\) and \(0 \leq f_\alpha(x) \leq f_\gamma(x)\) whenever \(\alpha \geq \gamma\). An element \(\phi\) of \(A'\) is called \(\tau\)-additive if \(\phi(f_\alpha) \to 0\) whenever \(f_\alpha \downarrow 0\). We will say that \(\phi\) is \(\sigma\)-additive if \(\phi(f_\alpha) \to 0\) for each sequence \(\{f_\alpha\}\) in \(A\) which decreases to zero. The set of all \(\sigma\)-additive (\(\tau\)-additive) members of \(A'\) will be denoted by \(L_\sigma(A)\) (\(L_\tau(A)\)).

**Theorem 5.2.** Each of the dual spaces \((A, \beta)'\), \((A, \beta_1)'\) and \((A, \beta_F)'\) forms a linear lattice ideal in the Riesz space \(A'\).

**Proof.** This follows easily from the fact that the spaces \((A, \beta), (A, \beta_1)\) and \((A, \beta_F)\) are locally solid.

**Theorem 5.3.** The dual space of the space \((A, \beta)\) is the space \(L_\tau(A)\).

**Proof.** Let \(\phi \in A'\) and let \(m \in M_\tau(\text{Bo}(Y), E')\) be such that \(\phi(f) = \int f dm\) for all \(f \in A\). Suppose \(\phi\) is \(\beta\)-continuous and let \(f_\alpha \downarrow 0\). We want to show that \(\phi(f_\alpha) \to 0\). Without loss of generality we may assume that \(\|f_\alpha\| \leq 1\) for all \(\alpha\). Let \(e > 0\). For each \(\alpha\), set \(Z_\alpha = \{x \in Y : \|f_\alpha(x)\| \geq e\}\). Then \(Z_\alpha \downarrow Q = \bigcap Z_\alpha\). Since \(Q \in \Omega\) we have \(|m|(Q) = 0\) by 4.9. Since \(|m|(Z_\alpha) \to |m|(Q) = 0\), there exists \(\alpha_0\) such that \(|m|(Z_\alpha) < e\) for all \(\alpha \geq \alpha_0\). Now, for \(\alpha \geq \alpha_0\), we have
\[
|\phi(f_\alpha)| \leq \left| \int_{Z_\alpha} f dm \right| + \left| \int_{Y - Z_\alpha} f dm \right|
\leq |m|(Z_\alpha) + e\|m\| \leq e(1 + \|m\|).
\]
This shows that \(\phi(f_\alpha) \to 0\) and \(\phi\) is \(\tau\)-additive.
Conversely, assume that \( \phi \) is \( \tau \)-additive. Let \( Q \in \Omega \) and \( 0 \leq s \in E, \|s\| \leq 1 \). Choose an open set \( O \) in \( Y, Q \subseteq O \), such that \( |m|(O - Q) < \epsilon \) (\( \epsilon > 0 \) arbitrary). The collection \( D = \{ hs : h \in B, 0 < h < 1, \hat{h} = 1 \text{ on } Q \text{ and } \hat{h} = 0 \text{ on } Y - O \} \) is downwards directed to zero. Hence there exists \( hs \) in \( D \) such that \( |\phi(hs)| < \epsilon \).

But

\[
|\phi(hs)| \geq \left| \int_Q s \, dm \right| - \left| \int_{O - Q} \hat{h} s \, dm \right| \geq |m(Q)s| - |m|(Q - O) \geq |m(Q)s| - \epsilon.
\]

Hence \( |m(Q)s| \leq 2\epsilon \). This proves that \( m(Q)s = 0 \) for each \( Q \in \Omega \) and each \( s \in E, \|s\| \leq 1, s > 0 \). Since \( E \) is a lattice, we have \( m(Q)s = 0 \) for all \( s \in E \).

Now the regularity of \( ms \) and 4.9 complete the proof.

**Theorem 5.4.** The dual of the space \((A, \beta_1)\) is the space \( L_o(A) \).

**Proof.** Let \( \phi \in A' \) and \( m \in M_{\tau}(B_0(Y), E') \) be such that \( \phi(f) = \int_Y f \, dm \) for all \( f \) in \( A \). Assume that \( \phi \) is \( \beta_1 \)-continuous and let \( \{f_n\} \) be a sequence in \( A \) that decreases to zero. We want to show that \( \phi(f_n) \rightarrow 0 \). We may assume without loss of generality that \( \|f_n\| \leq 1 \) for all \( n \). Let \( \epsilon > 0 \) and set \( Z_n = \{ x \in Y : \|f_n(x)\| > \epsilon \} \). Then \( Z_n \downarrow 0 \) and \( Z \in \Omega_1 \). Since \( \lim |m|(Z_n) = |m|(Z) = 0 \), there exists \( n_0 \) such that \( |m|(Z_n) < \epsilon \) if \( n \geq n_0 \). Now, if \( n \geq n_0 \), we have

\[
|\phi(f_n)| \leq \left| \int_{Z_n} \hat{f}_n \, dm \right| + \left| \int_{Y - Z_n} \hat{f}_n \, dm \right| < \epsilon + \epsilon \|m\|.
\]

This proves that \( \phi(f_n) \rightarrow 0 \) and hence \( \phi \) is o-additive. Conversely, assume \( \phi \in L_o(A) \). Let \( Z \) be in \( \Omega_1 \) and \( s \in E \) with \( s > 0 \) and \( \|s\| \leq 1 \). Given \( \epsilon > 0 \) there exists a cozero set \( V \) containing \( Z \) such that \( |m|(V - Z) < \epsilon \). Let \( g \in B \), \( 0 < g \leq 1 \), be such that \( Z = \hat{g}^{-1}(0) \). For each positive integer \( n \), let \( V_n = \{ x \in Y : \hat{g}(x) < 1/n \} \cap V \). Choose \( h_n \in B \), \( 0 < h_n \leq 1 \), \( \hat{h}_n = 1 \) on \( Z \) and \( \hat{h} = 0 \) on \( Y - V_n \). Let \( h'_n = h_1 \wedge \cdots \wedge h_n \) and set \( f_n = h'_n s \). Then \( f_n \downarrow 0 \). Hence there exists \( n \) such that \( |\phi(f_n)| < \epsilon \). But

\[
|\phi(f_n)| \geq \left| \int_Z \hat{f}_n \, dm \right| - \left| \int_{Y - Z} \hat{f}_n \, dm \right| \geq |m(Z)s| - |m|(Z - V) \geq |m(Z)s| - \epsilon.
\]

Thus \( |m(Z)s| \leq 2\epsilon \). This proves that \( m(Z)s = 0 \). From this it follows that \( m(Z)s = 0 \) for each \( s \in E \) and all \( Z \in \Omega_1 \). Now the result follows from the regularity of \( ms \) and 4.9.

**Theorem 5.5.** Let \( \tau \) be a locally convex Hausdorff topology on \( A \) for which the positive cone is normal. Then the following assertions are equivalent:

1. \((A, \tau)' \subseteq L_o(A)\).
2. If \( f_\alpha \downarrow 0 \), then \( f_\alpha \rightarrow 0 \) in the \( \tau \)-topology.
Proof. It is clear that (2) implies (1).

(1) $\Rightarrow$ (2). By Schaefer [10, p. 219], $\tau$ is the topology of uniform convergence on the $\tau$-equicontinuous subsets of $(A, \tau)^{+} = \{\phi \in (A, \tau)^{':} \phi \geq 0\}$.

Suppose now that $f_{\alpha} \downarrow 0$ and let $V$ be a $\tau$-neighborhood of zero. There exists a $\tau$-equicontinuous subset $H$ of $(A, \tau)^{+}$ such that $H^{0} \subset V$. The set $H$ is relatively weakly compact. Every $f_{\alpha}$ defines a weakly continuous linear functional on $(A, \tau)'$ by $\phi \mapsto W_{\alpha}(f) = \phi(f_{\alpha})$. If $\phi \in H$, then $W_{\alpha}(\phi) \downarrow 0$. Hence $W_{\alpha} \rightarrow 0$ uniformly on $H$ by Dini's theorem. It follows that there exists $\alpha_{0}$ such that $f_{\alpha} \in H^{0} \subset V$ for all $\alpha \geq \alpha_{0}$. This proves that $f_{\alpha} \rightarrow 0$ and the proof is complete.

We have an analogous theorem for $L_{0}(A)$ with a similar proof.

Theorem 5.6. Let $\tau$ be a locally convex Hausdorff topology on $A$ for which the positive cone is normal. The following are equivalent:

(1) The space $(A, \tau)'$ is contained in $L_{0}(A)$.
(2) $f_{n} \downarrow 0$ implies that $f_{n} \rightarrow 0$ in the $\tau$-topology.

Corollary 5.7. (1) $f_{n} \downarrow 0$ implies that $f_{n} \rightarrow 0$ in the $\beta_{1}$-topology.
(2) $f_{\alpha} \downarrow 0$ implies that $f_{\alpha} \rightarrow 0$ in the $\beta$-topology.

Theorem 5.8. Let $\tau$ be any one of the topologies $\beta, \beta_{1}, \beta_{F}$. If $W$ is a $\tau$-neighborhood of zero, then each of the sets $H_{1} = \{\phi^{+}: \phi \in W^{0}\}, H_{2} = \{\phi^{-}: \phi \in W^{0}\}$, and $H_{3} = \{\phi: \phi \in W^{0}\}$ is $\tau$-equicontinuous, where $W^{0}$ is the polar of $W$ in $(A, \tau)'$.

Proof. Since $(A, \tau)$ is locally solid, $\tau$ is the topology of uniform convergence on the $\tau$-equicontinuous subsets of $(A, \tau)^{+}$. Let $W_{1}$ be a solid $\tau$-neighborhood of zero contained in $W$ and let $H$ be a $\tau$-equicontinuous subset of $(A, \tau)^{+}$ with $H^{0} \subset W_{1}$. Let $f \in H^{0}$ and $\phi \in W_{1}^{0} \subset H^{00}$. Since $W_{1}$ is solid we have

$$|\phi^{+}(f)| \leq \phi^{+}(|f|) = \sup \{\phi(h): 0 \leq h \leq |f|\} \leq 1.$$ 

Thus $\phi^{+} \in H^{00}$. This shows that $H_{1} \subset H^{00}$. Similarly $H_{2}, H_{3} \subset H^{00}$ and the theorem is proved.

Throughout the remaining part of this paper $E$ will be assumed to be a Banach lattice with a unit element $e$ ($e$ has the property that $-e \leq s \leq e$ iff $\|s\| \leq 1$).

Theorem 5.9. (1) Every weakly compact subset of $L_{0}^{+}(A)$ is $\beta_{1}$-equicontinuous.
(2) Every weakly compact subset of $L_{1}^{+}(A)$ is $\beta$-equicontinuous.
(3) $\beta$ is the topology of uniform convergence on the weakly compact subsets of $L_{0}^{+}(A)$.
(4) $\beta_{1}$ is the topology of uniform convergence on the weakly compact subsets of $L_{0}^{+}(A)$.
Proof. (1) Let $H \subseteq L^0_\beta(A)$ be weakly compact. The set $H^0$ is convex, balanced and absorbent. Let $r > 0$. Set $\alpha = \sup \{ \| \phi \| : \phi \in H \}$ and let $Z$ be in $\Omega_1$. There exists $f \in B$, $0 \leq f \leq 1$, $Z = \hat{f}^{-1}(0)$. For each positive integer $n$, put $Z_n = \{ x \in Y : \hat{f}(x) > 1/n \}$. Choose $g_n \in B$, $0 \leq g_n \leq 1$, $\hat{g}_n = 1$ on $Z$ and $\hat{g}_n = 0$ on $Z_n$. Let $h_n = g_1 \wedge \cdots \wedge g_n$. Then $h_n \downarrow 0$ and hence $\phi(h_n e) \downarrow 0$ for each $\phi$ in $H$. By Dini's theorem, $\phi(h_n e) \rightarrow 0$ uniformly on $H$.

Hence there exists $n$ such that $\phi(h_n e) < 1/(2r)$ for all $\phi \in H$. Let $g = 1 - h_n$ and set $V = \{ h \in A : \| gh \| < 1/(2\alpha) \}$. If $h \in V \cap U_r$, then $h_n|h| \leq rh_n e$. Hence for $h \in V \cap U_r$ and $\phi \in H$ we have

$$|\phi(h)| \leq \phi(|h|g) + \phi(h_n|h|) \leq \| \phi \| \| gh \| + r\phi(h_n e) \leq a(1/2\alpha) + r(1/2r) = 1.$$  

This shows that $V \cap U_r \subseteq H^0$. Since this happens for all $r > 0$ and all $Z \in \Omega_1$ we conclude that $H^0$ is a $\beta$-neighborhood of zero.

(2) Let $Q \in \Omega$. The set $D = \{ g \in B : 0 \leq g \leq 1, \hat{g} = 1 \text{ on } Q \}$ is directed downwards to zero. From here on the proof is similar to that of (1).

(3) If $H \subseteq L^+_\beta(A)$ is weakly compact, then $H^0$ is a $\beta$-neighborhood of zero by (2). Conversely, let $W$ be a $\beta$-neighborhood of zero. Since $\beta$ is locally solid there exists a $\beta$-equicontinuous subset $H$ of $L^+_\beta(A)$ such that $H^0 \subseteq W$. If $H_1$ is the weak closure of $H$ in $L^r(A)$, then $H_1 \subseteq L^+_\beta(A)$ and $H_1$ is weakly compact. Moreover $H^0 \subseteq H^0 \subseteq W$. This proves (3).

(4) The proof is similar to that of (3).

Corollary 5.10. $\beta = \beta_1$ iff $L_\beta(A) = L_o(A)$.

Using Dini's theorem and the Alaoglu-Bourbaki theorem (see Köthe [14, p. 248]) we can easily show the following

Theorem 5.11. (1) A subset $H$ of $L^+_\beta(A)$ is $\sigma(L_\beta(A), A)$ relatively compact iff $\phi(f_\alpha) \rightarrow 0$ uniformly on $H$ for each net $\{ f_\alpha \}$ in $A$ that decreases to zero.

(2) A subset $H$ of $L^+_o(A)$ is $\sigma(L_o(A), A)$ relatively compact iff $\phi(f_\alpha) \rightarrow 0$ uniformly on $H$ whenever $f_\alpha \downarrow 0$.

Corollary 5.12. Let $H \subseteq L_\beta(A)$ ($H \subseteq L_o(A)$). The following are equivalent:

(1) $\{ \phi^+ : \phi \in H \}$ and $\{ \phi^- : \phi \in H \}$ are both weakly relatively compact in $L_\beta(A)$ (in $L_o(A)$).

(2) $\{ |\phi| : \phi \in H \}$ is weakly relatively compact in $L_\beta(A)$ (in $L_o(A)$).

Theorem 5.13. The following are equivalent:

(1) $(A, \beta)$ is a Mackey space.

(2) If $H$ is a convex, balanced, weakly compact subset of $L_\beta(A)$, then $\{ |\phi| : \phi \in H \}$ is weakly relatively compact in $L_\beta(A)$.  

Proof. (1) Let $H \subseteq L^0_\beta(A)$ be weakly compact. The set $H^0$ is convex, balanced and absorbent. Let $r > 0$. Set $\alpha = \sup \{ \| \phi \| : \phi \in H \}$ and let $Z$ be in $\Omega_1$. There exists $f \in B$, $0 \leq f \leq 1$, $Z = \hat{f}^{-1}(0)$. For each positive integer $n$, put $Z_n = \{ x \in Y : \hat{f}(x) > 1/n \}$. Choose $g_n \in B$, $0 \leq g_n \leq 1$, $\hat{g}_n = 1$ on $Z$ and $\hat{g}_n = 0$ on $Z_n$. Let $h_n = g_1 \wedge \cdots \wedge g_n$. Then $h_n \downarrow 0$ and hence $\phi(h_n e) \downarrow 0$ for each $\phi$ in $H$. By Dini's theorem, $\phi(h_n e) \rightarrow 0$ uniformly on $H$.

Hence there exists $n$ such that $\phi(h_n e) < 1/(2r)$ for all $\phi \in H$. Let $g = 1 - h_n$ and set $V = \{ h \in A : \| gh \| < 1/(2\alpha) \}$. If $h \in V \cap U_r$, then $h_n|h| \leq rh_n e$. Hence for $h \in V \cap U_r$ and $\phi \in H$ we have

$$|\phi(h)| \leq \phi(|h|g) + \phi(h_n|h|) \leq \| \phi \| \| gh \| + r\phi(h_n e) \leq a(1/2\alpha) + r(1/2r) = 1.$$  

This shows that $V \cap U_r \subseteq H^0$. Since this happens for all $r > 0$ and all $Z \in \Omega_1$ we conclude that $H^0$ is a $\beta$-neighborhood of zero.

(2) Let $Q \in \Omega$. The set $D = \{ g \in B : 0 \leq g \leq 1, \hat{g} = 1 \text{ on } Q \}$ is directed downwards to zero. From here on the proof is similar to that of (1).

(3) If $H \subseteq L^+_\beta(A)$ is weakly compact, then $H^0$ is a $\beta$-neighborhood of zero by (2). Conversely, let $W$ be a $\beta$-neighborhood of zero. Since $\beta$ is locally solid there exists a $\beta$-equicontinuous subset $H$ of $L^+_\beta(A)$ such that $H^0 \subseteq W$. If $H_1$ is the weak closure of $H$ in $L^r(A)$, then $H_1 \subseteq L^+_\beta(A)$ and $H_1$ is weakly compact. Moreover $H^0 \subseteq H^0 \subseteq W$. This proves (3).

(4) The proof is similar to that of (3).

Corollary 5.10. $\beta = \beta_1$ iff $L_\beta(A) = L_o(A)$.

Using Dini's theorem and the Alaoglu-Bourbaki theorem (see Köthe [14, p. 248]) we can easily show the following

Theorem 5.11. (1) A subset $H$ of $L^+_\beta(A)$ is $\sigma(L_\beta(A), A)$ relatively compact iff $\phi(f_\alpha) \rightarrow 0$ uniformly on $H$ for each net $\{ f_\alpha \}$ in $A$ that decreases to zero.

(2) A subset $H$ of $L^+_o(A)$ is $\sigma(L_o(A), A)$ relatively compact iff $\phi(f_\alpha) \rightarrow 0$ uniformly on $H$ whenever $f_\alpha \downarrow 0$.

Corollary 5.12. Let $H \subseteq L_\beta(A)$ ($H \subseteq L_o(A)$). The following are equivalent:

(1) $\{ \phi^+ : \phi \in H \}$ and $\{ \phi^- : \phi \in H \}$ are both weakly relatively compact in $L_\beta(A)$ (in $L_o(A)$).

(2) $\{ |\phi| : \phi \in H \}$ is weakly relatively compact in $L_\beta(A)$ (in $L_o(A)$).

Theorem 5.13. The following are equivalent:

(1) $(A, \beta)$ is a Mackey space.

(2) If $H$ is a convex, balanced, weakly compact subset of $L_\beta(A)$, then $\{ |\phi| : \phi \in H \}$ is weakly relatively compact in $L_\beta(A)$.
(3) If $H$ is a convex, balanced, weakly compact subset of $L_\tau(A)$, then \( \{ \phi^+ : \phi \in H \} \) and \( \{ \phi^- : \phi \in H \} \) are both weakly relatively compact in $L_\tau(A)$.

**Proof.** By 5.12, (2) is equivalent to (3).

(1) $\rightarrow$ (3). Let $H$ be a weakly compact, convex, balanced subset of $L_\tau(A)$. By hypothesis $H^0$ is a $\beta$-neighborhood of zero. By 5.8 the sets \( V_1 = \{ \phi^+ : \phi \in H^0 \} \) and \( V_2 = \{ \phi^- : \phi \in H^0 \} \) are weakly relatively compact in $L_\tau(A)$. Since $H \subset H^0$, (3) follows.

(3) $\rightarrow$ (1). Let $H$ be a convex balanced weakly compact subset of $L_\tau(A)$. By hypothesis and 5.9, the sets \( H_1 = \{ \phi^+ : \phi \in H \} \) and \( H_2 = \{ \phi^- : \phi \in H \} \) are $\beta$-equicontinuous. Since $H^0 \supseteq \frac{1}{2} (H_1^0 \cap H_2^0)$, it follows that $H$ is $\beta$-equicontinuous. Hence $\beta$ is finer than the Mackey topology $m(A; L_\tau(A))$. Thus $\beta = m(A, L_\tau(A))$ since $(A, \beta)' = L_\tau(A)$. This completes the proof.

We have an analogous theorem for the pair $(A, L_\sigma(A))$ and the topology $\beta_1$. The proof is similar.

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