FINITE GROUPS AS ISOMETRY GROUPS

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ABSTRACT. We show that given any finite group $G$ of cardinality $k + 1$, there is a Riemannian sphere $S^{k-1}$ (imbeddable isometrically as a hypersurface in $\mathbb{R}^k$) such that its full isometry group is isomorphic to $G$. We also show the existence of a finite metric space of cardinality $k(k + 1)$ whose full isometry group is isomorphic to $G$.

Let $G$ be a finite group of $k + 1$ elements $\{1, g_1, \ldots, g_k\}$.

THEOREM. There exists a Riemannian metric on the sphere $S^{k-1}$ such that the isometry group is isomorphic to $G$.

Proof. Label the $k + 1$ vertices of a regular $k$-simplex $\Delta_k$ by the names $1, g_1, \ldots, g_k$ of the elements of $G$. Assume $\Delta_k$ to be inscribed in a standard $S^{k-1}$ sitting in $\mathbb{R}^k$ as usual. $T_y(S^{k-1})$ denotes the tangent space at $y$.

Now in $T_1(S^{k-1})$ pick an orthonormal frame $(v_1, \ldots, v_{k-1})$. Pick $e > 0$ small and let

$$w_i = e(1 + (i - 1)/4k^2)v_i, \quad 1 \leq i \leq k - 1.$$ 

Let

$$Q = \{\exp_1(w_i) | 1 \leq i \leq k - 1\} \cup \{\exp_1(0)\} \cup \{w_1/10\}.$$ 

$\exp_1$ is the exponential map $\exp_1 : T_1(S^{k-1}) \rightarrow S^{k-1}$.

Think of $G$ as acting on $S^{k-1}$ by the isometries induced from the permutation representation on the vertices of $\Delta_k$. Let $X = \{gQ | g \in G\}$.

PROPOSITION. With the induced metric from $\mathbb{R}^k$, the metric space $X$ has its group of isometries isomorphic to $G$.

Proof. Clearly $G$ acts as a group of isometries of $X$, since $X = h[gQ | g \in G] = \{hgQ | g \in G\} = \{gQ | g \in G\} = X$.

Conversely, any isometry of $X$ must take the point 1 to some point $g$, since the points $g$ are characterized by being the only points in $X$ having their
two nearest neighbors at distance of \(\varepsilon/10\) and \(\varepsilon\) respectively. Once we know that \(1 \mapsto g\), the configuration \(gQ\) determines the image of the frame \((w_1, \ldots, w_{k-1})\) at 1, and hence determines the unique isometry of \(X\) defined by the element \(g \in G\). Of course \(\varepsilon\) must be chosen small enough so that the configurations \(gQ, g \in G\) do not "interfere" with one another.

Now we add bumps to \(S^{k-1}\) at the points of \(X\) using scalar multiplication in \(R^k\). Let

\[
\delta = (1/3)\min\{\text{dist}_{S^{k-1}}(x, y) | x, y \in X\}.
\]

Let \(f: [0, \delta] \to R\) be a smooth function satisfying

(a) \(f(s) = 100, 0 \leq s \leq \delta/2\),
(b) \(f(\delta) = 1; f^{(k)}(\delta) = 0, k = 1, 2, \ldots\),
(c) \(f^{(k)}(\delta/2) = 0, k = 1, 2, \ldots\), and
(d) \(f'(s) < 0\) if \(\delta/2 < s < \delta\).

Now for each point \(x \in X\) we remove the disk \(\exp_x(D_\delta)\) from \(S^{k-1}\) and replace it by the point set \(B_x = \{f(|v|)\exp_x(v) | v \in D_\delta\}\), where \(D_\delta\) is the \((\delta)\)-disk about the origin of \(T_x(S^{k-1})\). Clearly the set \(S^{k-1} - \bigcup_{x \in X} B_x\) is a smooth \(S^{k-1}\) imbedded in \(R^k\). We give it the induced Riemannian metric from \(R^k\) and denote it by \(M\).

**Claim:** \(\text{Isom}(M) \cong G\).

**Proof.** First we notice that the points of \(100 \cdot X \subset M\) must be taken to themselves by any isometry \(I\) of \(M\), by the choice of the function \(f\). Clearly the same arguments above for \(X\) hold for \(100 \cdot X\), hence the isometry \(I: M \to M\) restricted to \(100 \cdot X\) comes from the action of \(G\).

Let us now consider the “bump” \(B_1\) above the point 1. Let us define for \(r > 0\), \(S_r = \{f(r) \cdot \exp(v) | |v| = r, v \in T_1(S^{k-1})\}\). In other words, \(S_r\) is the \((k-2)\)-sphere of \(B_1\) lying above the \((k-2)\)-sphere about 1 of radius \(r\), for \(0 < r \leq \delta\), and for \(r = 0\) we set \(S_0 = p\), the peak point of \(B_1\).

Now it is easy to show that the orthogonal trajectories of the \(S_r\)'s are geodesics of \(M\) and as such must be preserved under any isometry taking \(p\) to \(p\).

Thus any isometry \(I\) of \(M\) which takes \(p\) to \(p\) (and which must thus leave all points of \(100 \cdot X\) fixed) must be a “rotation” on all of \(B_1\), determined by \(I|_{\partial B_1}\), carrying each \(S_r\) into itself by the “same” element of \(O(k-2)\). Similarly, this \(I\) must rotate each bump \(B_x, x \in X\).

Also this rotation must extend past the boundary of the bumps for some ways, so we can easily extend \(I|_{(M - \bigcup_x B_x)}\) to an isometry \(\tilde{I}\) of \(S^{k-1}\) to itself, by simply “coning” \(I\) over \(\exp_x(D_\delta), x \in X\). Clearly we will have \(\tilde{I}(x) = x\) for \(x \in X\), and it follows easily that \(\tilde{I}: S^{k-1} \to S^{k-1}\) is the identity. Hence \(\tilde{I}: M \to M\) must have been the identity.
Now it is clear that for each \( g \in G \) there is one isometry of \( M \) determined by the action of \( g \) on \( S^{k-1} \), extended to \( R^k \), restricted to \( M \). Now if there is another isometry \( I : M \rightarrow M \) such that \( I \mid X = g \mid X \), then \( I \circ g^{-1} : M \rightarrow M \) must leave points of \( X \) fixed, so by the above discussion must be the identity. This establishes \( \text{Isom}(M) \cong G \).

**Corollary.** Any finite group \( G \) is isomorphic to the (full) isometry group of a finite subset \( X_G \) of euclidean space. If \( \text{card}(G) = k \) then the \( X_G \) can be found with \( \text{card}(X_G) = k^2 - k \) in euclidean space of dimension \( k - 1 \).

**Proof.** Simply take \( X_G = X \) in the proof of the Theorem, and count (noting that we initially took \( \text{card}(G) = k + 1 \)).

**Remark.** Further considerations can very likely reduce the necessary cardinality for \( X_G \) to \( k(k - 3) \). The various numbers

\[
d = \min \{ \text{card}(X) \mid G \cong \text{Isom}(X) \} \quad \text{and}
\]

\[
e = \min \{ N \mid G \text{ has a faithful representation into } O(N) \}
\]

seem to be interesting invariants of a finite group \( G \).