General Position Maps for Topological Manifolds in the $2/3$-rds Range

By Jerome Dancis

Abstract. For each proper map $f$ of a topological $m$-manifold $M$ into a topological $q$-manifold $Q$, $m \leq (2/3)q - 1/3$, we build an approximating map $g$ such that the set of singularities $S$ of $g$ is a locally finite simplicial $(2m - q)$-complex locally tamely embedded in $M$, $g(S)$ is another locally finite complex $g(S)$ is a piecewise linear map and $g$ is a locally flat embedding on the complement of $S$.

Furthermore if $f|\partial M$ is a locally flat embedding then we construct $g$ so that it agrees with $f$ on $\partial M$ even when $f(\partial M)$ meets $\text{Int } Q \cap f(\text{Int } M)$.

In addition we present two other general position lemmas. Also, we show that given two codimension $\geq 3$ locally flat topological submanifolds $M$ and $V$ of a topological manifold $Q$, $\dim M + \dim V - \dim Q \leq 3$, then we can move $M$ so that $M$ and $V$ are transverse in $Q$.

1. Introduction. In this paper we shall define “general position” for the topological category and we shall establish some general position lemmas. The only tool that we shall use for this is the codim $\geq 3$ Taming Lemma 2.2 of Bryant, Seebeck, Černavskii, Homma and Miller.

We list the results and definitions:

Definition. Let $g: M \to Q$ be a continuous map of an $m$-manifold into a $q$-manifold. We say that $g$ is in general position if there exists locally finite complexes $K_m$ and $K_q$, which are closed subsets of $M$ and $Q$, respectively, such that

(i) $K_m$ is the singular set of $g$,
(ii) $\dim K_m \leq 2m - q$,
(iii) $g$ sends $K_m$ piecewise linearly onto $K_q$, and
(iv) $g|M - K_m$ is a locally flat embedding.

Furthermore, for each $x \in K_m$ there are locally flat $m$- and $q$-simplices $\Delta_m$ and $\Delta_q$ in $M$ and $Q$ respectively, $x \in \Delta_m$, such that

(v) $g$ sends $\Delta_m$ piecewise linearly into $\Delta_q$, and

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the PL structures of $K_m$ and $K_q$ are compatible with those of $\Delta^m$ and $\Delta^q$ respectively.

**Definition.** A continuous map $f: X \to Y$ is *proper* if the inverse of each compact set is a compact set. A homotopy $f: X \times I \to Y$ is *proper* if the inverse of each compact set is compact.

The main result of this paper is:

**Topological General Position Lemma 1.** Let $f: M \to Q$ be a proper map of an $m$-manifold into a $q$-manifold, $m \leq (2/3)q - 1/3$, $m \leq q - 3$ and let $e: M \to (0, 1)$ be a given continuous function. Then there is a general position map $g: M \to Q$ such that $d(f(x), g(x)) < e(x)$, for each $x \in M$, and $g$ is properly homotopic to $f$.

**Definition.** Let $g: K \to Q$ be a continuous proper map of a locally finite $k$-complex into a $q$-manifold. We say that $g$ is in *general position* if there is a locally finite complex $K_q$, which is embedded as a closed subset of $Q$ such that

(i) $g$ sends $K$ piecewise linearly onto $K_q$;
(ii) the singular set of $g$ has dimension $\leq 2k - q$;
(iii) for each point $x \in K$ there is a neighborhood, $N(g(x))$, of $g(x)$ and a homeomorphism $h_x$ sending $N(g(x))$ onto a $q$-simplex such that $h_x \circ g$ is a PL map on some neighborhood of the point $x$.

**General Position Lemma 2.** Let $f: K \to Q$ be a continuous map of a finite $k$-complex into a $q$-manifold, $k \leq q - 3$, and let $e > 0$ be given. Then there is a general position map $g: K \to Q$ such that $d(f(x), g(x)) < e$, for each $x \in K$ and $g$ is homotopic to $f$.

Furthermore if $K'$ is a subcomplex of $K$ and $f|K'$ is already a general position map then we construct $g$ such that $g|K' = f|K'$.

**Remark.** This General Position Lemma 2 is basically a corollary of the codim $\geq 3$ Taming Lemma 2.2 and as such it is part of the folklore. (We provide a proof herein because we need and use this lemma in another paper.)

**General Position Lemma 3.** Let $M^m$ and $V^v$ be topological $m$-and $v$-manifolds embedded as locally flat closed subsets of a topological $q$-manifold $Q$. Let $M$ be compact, $m, v \leq q - 3$ and $\emptyset = \partial M = \partial V = \partial Q$. Given an $e > 0$ there is an $e$-push $P$ on $M$ in $Q$ and a finite $(m + v - q)$-complex $K$ such that $P(M) \cap V = K$ and each simplex of $K$ is a locally flat subset of $M, V$ and $Q$.

Hollingsworth and Sher [HI-Sh] have shown that the triangulation techniques of Kirby and Siebenmann has the following lemma as a consequence:

**Lemma 1.1.** Let $K$ be a finite complex embedded in the interior of a manifold $M$. Let each simplex of $K$ be a locally flat subset of $M$, $\dim K \leq 3$, $\dim M \geq 5$
and $3 + \dim K \leq \dim M$. Then there is a PL manifold $N$ which is a locally flat subset of $M$, $\dim N = \dim M$, $K \subset \text{Int } N$ and $N$ is a PL regular neighborhood of $K$.

Combining General Position Lemma 3 with Lemma 1.1 and PL block transversality we will establish:

**Theorem 1.2.** Let $M$ and $V$ be topological $m$- and $v$-manifolds embedded as locally flat closed subsets of a topological $q$-manifold $Q$. Let $M$ be compact, let $w = m + v - q$; suppose

$$q - 3 \geq v \geq 5, \quad q - 3 \geq m \geq 5, \quad w \leq 3 \quad \text{and} \quad \emptyset = \partial M = \partial V = \partial Q.$$

Given an $\epsilon > 0$ there is an $\epsilon$-push $P$ on $M$ in $Q$ such that $W = P(M) \cap V$ is a PL $w$-manifold, $\partial W = \emptyset$. In addition there are PL block bundles $\eta^{q-v}|W$ and $\xi^{q-m}|W$ and an embedding of their Whitney sum $h: \eta \oplus \xi|W \hookrightarrow Q$ such that

$$h|W = 1, \quad h(\eta^{q-v}|W) \subset P(M) \quad \text{and} \quad h(\xi^{q-m}|W) \subset V.$$

**Remark.** Kirby and Siebenmann have a topological transversality theorem which is valid when the submanifolds do not have dimension 4, but their theorem requires that one of the submanifolds has a normal microbundle neighborhood in $Q$.

**Outline of proof.** Here we may apply Lemma 1.1 to the complex $K \subset P(M), V$ and $Q$ of General Position Lemma 3 obtaining PL regular neighborhoods $N^m \subset P(M), N^v \subset V$ and $N^q \subset Q$ of $K$. Let $N^v$ and $N^m$ be sufficiently small so that they are contained in $N^q$. Now $N^m \cup_K N^v$ is a polyhedron nicely embedded in $N^q$. Therefore the Taming Lemma 2.2 provides a small push $P_1$ on $N^q$ such that $P_1(N^m \cup_K N^v)$ is a subcomplex of $N^q$. Theorem 1.2 follows immediately from this and Rourke and Sanderson’s PL block transversality theory [R-S] applied to $P_1(N^m)$ and $P_1(N^v)$ in $N^q$.

We shall present our proofs in the order of increasing difficulty. General Position Lemma 2 is basically a corollary of the codim $\geq 3$ Taming Lemma 2.2. The proof of General Position Lemma 3 is moderately easy. But the proof of Topological General Position Lemma 1 is sufficiently subtle that the author does not know how to establish this lemma outside the metastable (2/3rds) range. In §6 we present two generalizations of Topological General Position Lemma 1.

2. Background. Here we shall present some definitions and theorems which will be used in the proof of the General Position Lemmas.

**Note.** “PL” is a shorthand for piecewise linear.

**Definition.** A standard representation of a neighborhood of a compact subset $X$ of an $n$-manifold, $M^n$, is a collection of compact subsets $\{N_j, I_j, M'_j, M''_j, j = 1, 2, \ldots, r\}$ of $M$ such that (see figure):

1. $N_1 = I_1 = M''_1, M'_1 = \emptyset$ and $N_r \supset X$;
2. $I_j$ is a locally flat $n$-cell in $M$;
(3) $X \subset \bigcup_j \text{Int } I_j$;

(4) $M'_j \cup M''_j = I_j$ and there is a homeomorphism $i_j$ of $I_j$ onto an $n$-simplex $\Delta^n$ such that $i_j(M'_j)$ and $i_j(M''_j)$ are combinatorial submanifolds-with-boundary of some rectilinear subdivision of $\Delta^n$;

(5) $N_j \cap N_{j-1} \cup I_j, j > i$;

(6) $\text{Int } N_j \supset X - \bigcup_{m > j} \text{Int } I_m$;

(7) $M''_j \supset I_j - N_{j-1}$ and $M'_j \subset \text{Int } N_{j-1}, j > 1$;

(8) $M''_j \cap X \subset \bigcup_{m > j} \text{Int } I_m$;

(9) $d(M''_j, N_j - I_j) > 0, j > 1$.

Furthermore, in this paper, we shall let each $I_j, 2 \leq j \leq r$, have three buffer zones or bands $B_j, B'_j, B''_j$ where $I_j, I'_j, I''_j, B'_j$ and $B''_j$ fit together as

$I_j = [-1, 1]^m, \quad I'_j = [-2, 2]^m, \quad I''_j = [-3, 3]^m,$

$B'_j = I'_j - I_j, \quad B''_j = I''_j - I'_j$ and $B_j = [-3/2, 3/2]^m - I_j$.

Additionally when $I_j$ meets $\partial M$ we want $I_j \subset I'_j \subset I''_j \supset \partial M \cap I''_j$ to be like (homeomorphic with)

$[0, 1] \times [-1, 1]^{m-1} \subset [0, 2] \times [-2, 2]^{m-1}$

$\subset [0, 3] \times [-3, 3]^{m-1} \supset (0) \times [-3, 3]^{m-1}$.

**Theorem 2.1.** For each compact subset $X$ of a manifold there is a standard representation of some neighborhood of $X$. Furthermore, if $\{I_j, j = 1, 2, \ldots, r\}$ is a particular set of locally-flat $n$-cells in $M$ such that $X \subset \bigcup_{j=1}^r \text{Int } I_j$, then some neighborhood of $X$ has a standard representation whose collection of $I'_j$'s is the given set.
Proof. Suppose that $X$ is a compact subset of an $n$-manifold with boundary $M$. The hypothesis of Theorem 2.1 or the compactness of $X$ will yield a finite set of locally flat $n$-cells \{$I_j, j = 1, 2, \ldots, r \}$ in $M$ whose interiors cover $X$.

Thus (2) and (3) are satisfied.

We begin by setting $N_1 = I_1 = M_1'$ and $M_1' = \emptyset$; thus most of (1) is satisfied.

We will construct the $N_j$'s, $M_j$'s and $M_j''$'s by induction. We assume that $N_{j-1}$ is known and satisfies (6). Therefore

\[
X \cap \partial N_{j-1} \subset X - \text{Int} N_{j-1} \subset \bigcup_{m \geq j} \text{Int} I_m,
\]

and

\[
N_{j-1} \cup \left( \bigcup_{m \geq j} \text{Int} I_m \right) \supset X \supset I_j \cap X.
\]

Thus $I_j \cap X$ is contained in the union of two sets, one of which 
($\bigcup_{m \geq j} \text{Int} I_m$) contains the boundary in $X$ of the second $(I_j \cap N_{j-1}) \cap X$. Therefore we may find a closed subset $A$ of $I_j$ such that

\[
\text{Cl}(I_j - N_{j-1}) \subset \text{Int} A \quad \text{and} \quad A \cap X \subset \bigcup_{m \geq j} \text{Int} I_m.
\]

Let $B = \text{Cl}(I_j - A)$. Thus $B \subset \text{Int} N_{j-1}$.

One may now use some basic theory of regular neighborhoods (e.g. see
Theorem 2.11 of [Hd]) in order to "expand" $A$ and $B$ into "combinatorial sub-
manifolds" $M_j'$ and $M_j$ respectively of $I_j$ such that (4), (7) and (8) are satisfied.

Condition (8) implies that there is an open set $O_j$ such that

\[
X \cap M_j'' \subset O_j \subset \text{Cl} O_j \subset \bigcup_{m \geq j} \text{Int} I_m.
\]

We are now ready to define $N_j$, namely

\[
N_j = \text{Cl}(N_{j-1} - O_j) \cup I_j.
\]

A brief checking of (10) and (11) will show that (5), (6) and (7) are satisfied.

We have shown that all conditions of a standard representation are satisfied.

Therefore Theorem 2.1 is established.

Zeeman's Unknotting Ball Theorem [Z]. Let $B^m \subset B^q$ be PL $m$-
and-$q$-balls, $q \geq m + 3$ and $\partial B^m = B^m \cap \partial B^q$. Then there is an onto PL homeo-
morphism

\[
h: \left( I^q = I^m \times I^{q-m}, I^m \right) \rightarrow (B^q, B^m).
\]

Definition. A map $f: K \rightarrow Q$ of a polyhedron $K$ into a topological man-
ifold $Q$ is a nice map if there is a triangulation of $f(K)$ such that: $f: K \rightarrow f(K)$
is piecewise linear and for each simplex $\sigma \in K$, $f(\sigma)$ is a locally flat subset of $Q$.

We shall be using the next lemma in order to “straighten” nice maps.

**Taming Lemma 2.2** (Bryant, Seebeck, Černavský, Homma and Miller). Let $K$ be a $k$-complex in the interior of a combinatorial $q$-manifold $Q$, $k < q - 3$. Let the interior of each simplex of $K$ be locally flat in $Q$ and let $\varepsilon > 0$. Then there is an ambient $\varepsilon$-isotopy

$$\{H_t : Q \rightarrow Q, t \in [0, 1] \text{ and } H_0 = 1\}$$

such that

(i) $H_1|K$ is piecewise linear, and

(ii) $H_t(x) = (x), d(x, K) > \varepsilon$ and $t \in [0, 1]$.

Furthermore, if $L$ is a subcomplex of both $K$ and $Q$, then we may insist that $H_1|L = 1$.

**Remark.** This Taming Lemma 2.2 is a corollary of the theorems of [Br-Sb], [MI], [Cr-2] and [Br]. Of course [Br-Sb] uses the ideas of [Hm]. The case $k \leq (2/3)q - 1$ of Taming Lemma 2.2 is established in [Cr-1].

**Definitions.** An ambient isotopy of a space $Q$ is a continuous map $H : Q \times I \rightarrow Q$, such that if we set $h_t(x) = H(x, t)$, then $h_t$ is a homeomorphism of $Q$ onto itself, for each $t \in I$.

We say that $h_a$ is “ambient isotopic” to $h_b$, for $a, b \in I$.

An ambient $\varepsilon$-isotopy of $Q$ is an ambient isotopy which satisfies the additional condition:

$$d(h_t(x), x) < \varepsilon, \text{ for each } x \in Q, t \in I.$$  

A push is an ambient isotopy of a space $Q$ such that for some compact proper subset $A$ of $Q$,

$$h_t(x) = x, \text{ for all } x \in Q - A \text{ and } t \in I,$$

and $h_0 = 1$. An $\varepsilon$-push $P$ of $A$ is an ambient $\varepsilon$-isotopy of $Q$ such that

$$h_t(x) = x, \text{ when } d(x, A) \geq \varepsilon, t \in I$$

and $h_0 = 1$. Also $P(x) = h_1(x)$.

**3. Proof of General Position Lemma 2.** Let $f, K$ and $Q$ satisfy the hypothesis of this lemma. Let $\{I_i, M_i, M''_i, N_i, i = 1, 2, \ldots, r\}$ be a standard representation for $Q$ and let each $I_i$ be contained in a Euclidean neighborhood $E_i^q$. Let $K_1, \ldots, K_r$ be subcomplexes of $K$ such that

$$\bigcup_{i=1}^r K_i = K \quad \text{and} \quad f(K_i) \subset \text{Interior } I_i, \quad i = 1, 2, \ldots, r.$$
We shall proceed inductively building general position maps \( g_i : \bigcup_{j=1}^{i} K_j \to Q \), such that \( g_i(K_j) \subset E_j, j = 1, 2, \ldots, i \).

First we observe that since \( g|K' \) is already a general position map that for each simplex \( \Delta \) of \( K' \) (possibly after subdivision of \( K' \)) that \( g \) sends \( \Delta \) piecewise linearly into some "patch", interior \( I^q \), in \( Q \). As a consequence of Zeeman's Unknotting Ball Theorem (in \( \S 2 \)) we see that \( f(\text{Int} \, \Delta) \) is a locally flat subset of \( Q \). Thus we shall be able to use the Taming Lemma (in \( \S 2 \)) on \( f|K_i \cap K' : K_i \cap K' \to E^q \).

**Step 1.** As just noted we may use Taming Lemma 2.2 in order to obtain a push \( P_1 \) on \( f|K_1 \cap K' \) such that
\[
P_1 \circ f|K_1 \cap K' : K_1 \cap K' \to I^q_1
\]
is piecewise linear. Let \( g_1 : K_1 \to I^q_1 \) be a general position piecewise linear map which extends \( P_1 \circ f|K_1 \cap K' \) and approximates \( f \).

**Induction Assumption.** Assume that we have constructed \( g_{i-1} : K' \cup \bigcup_{j=1}^{i-1} K_j \to Q \) as a general position map such that
\[
g_{i-1}|K' = f|K' \quad \text{and} \quad g_{i-1}(K_j) \subset E^q_j, \quad j = 1, 2, \ldots, i - 1.
\]

**Step i.** Let \( K'_i = K_i \cap (K' \cup \bigcup_{j=1}^{i-1} K_j) \). As before we may use the Taming Lemma 2.2 in order to obtain a push \( P_i \) on \( g_{i-1}|K'_i \) in \( E^q_i \) so that \( P_i \circ g_{i-1}|K'_i : K'_i \to E^q_i \) is piecewise linear. Let \( g'_i : K'_i \to E^q_i \) be a general position piecewise linear map which extends \( P_i \circ g_{i-1}|K'_i \) and approximates \( f \).

Let \( g_i : K' \cup \bigcup_{j=1}^{i} K_j \to Q \) be defined by
\[
g_i(x) = \begin{cases} P_i^{-1} \circ g'_i(x), & x \in K, \\ g_{i-1}(x), & x \in K' \cup \bigcup_{j=1}^{i-1} K_j. \end{cases}
\]
Clearly \( g_i \) is a well-defined general position map of the type needed for our induction. So the induction works,
\[
g_r : K = K' \cup \bigcup_{i=1}^{r} K_i \to Q, \quad g_r|K' = f|K'
\]
and \( g_r \) is a general position map.

Since the euclidean patches \( E^q_i \) are convex \( g_r \) will be homotopic to \( f \). This completes the proof of General Position Lemma 2.

**Remark.** In a manner analogous to the one we shall use at the end of the proof of the Topological General Position Lemma 1, it is easily shown that General Position Lemma 2, is also valid when \( f : K \to Q \) is a proper map and \( K \) is a locally finite complex.

4. **Proof of General Position Lemma 3.** Let \( M, V \) and \( Q \) be as in the statement of this lemma. Choose a standard representation \( \{N_j^o, I_j^o, V_j', V_j^r, j = 1, 2, \}

. . . , s) of a neighborhood of the compact subset $M \cap V$ of the manifold $V$ such that each $I^v_j$ has a euclidean neighborhood $E^q_j = E^v \times E^{q-v}$ where $1: I^v_j \hookrightarrow E^v$ and $\text{diam } E^q_j < \varepsilon$ (courtesy of Theorem 2.1). That we can find such a collection \{I^v_j\} follows from the local flatness of $V$ in $Q$.

We shall perform an induction on these $N^v_j$'s of this standard representation as follows:

**INDUCTIVE STATEMENT $A_n$.** There is a set of $n$ pushes $\{P_1, \ldots, P_n\}$ where each $P_j$ is an $(\varepsilon/s)$-push on $V^v_j$ in $E^q_j$ such that $\text{Int } N^u_v$ and $P_n \circ P_{n-1} \circ \cdots \circ P_1(M)$ are in general position and

$$V \cap P_n \circ P_{n-1} \circ \cdots \circ P_1(M) \subset N^u_v.$$ 

Thus statement $A_s$ will establish the lemma.

**Preliminaries for proof that statement $A_{n-1}$ implies statement $A_n$, $n = 2, \ldots, s$.** Let

$$K_{n-1} = P_{n-1} \circ \cdots \circ P_1(M) \cap \text{Int}(N^u_{n-1}).$$

Statement $A_{n-1}$ implies that $K_{n-1}$ is a locally finite complex which has a nice proper embedding in $\text{Int } N^u_{n-1}$.

Now $K_{n-1}$ has a subcomplex $K^*_{n-1}$ which “approximates” $K_{n-1} \cap V^v_n$ in the sense that $K_{n-1} \cap V^v_n \subset K^*_{n-1} \subset K_{n-1} \cap E^q_n$. Therefore Taming Lemma 2.2 provides a push on $K^*_{n-1}$ in $E^q_n$ which sends $K^*_{n-1}$ onto a polyhedron in $E^q_n$. Therefore without loss of generality we may assume that $K^*_{n-1}$ is a subcomplex of $E^q_n$ and hence $K_{n-1} \cap V^v_n$ is a subcomplex of subdivisions of both $K_{n-1}$ and $V^v_n$.

This completes the preliminaries for showing that statement $A_{n-1}$ implies statement $A_n$.

Now for each integer $n = 1, 2, \ldots, s$, there is a standard representation

$$\{N^m_{n,i}, I^m_{n,i}, M^m_{n,i}, M^m_n, i = 1, 2, \ldots, k_n\}$$

of $P_{n-1} \circ \cdots \circ P_1(M) \cap V^v_n$ in $M$. We shall perform an inner induction on these $N^m_{n,i}$'s as follows:

**INDUCTIVE STATEMENT $B_{n,i}$.** There is a collection

$$\{P^i_{n,j}\}$$

of an $(\varepsilon/s\kappa_n)$-push on the image of $M^m_j$ in $E^q_n$, $j = 1, 2, \ldots, k_n$ and, if we set

$$g_{n,i} = P^i_{n,i} \circ \cdots \circ P^i_{n,1} \circ P_{n-1} \circ \cdots \circ P_1,$$

then $g_{n,i}(\text{Int } N^m_{n,i})$ and $E^u_v$ are in general position in $E^q_n$. In addition

$$I^u_n \cap g_{n,i}(M) \subset (K^*_{n-1} \cap V^v_n) \cup g_{n,i}(N^m_{n,k_n})$$

and $g_{n,i}$ agrees with $P_{n-1}$ on $N^u_v - V^v_n$. 

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Thus for each \( n \) the statement \( B_{n,k_n} \) will establish the statement \( A_n \) so we now show that statement \( B_{n,i-1} \) implies statement \( B_{n,i} \). For this we work entirely in \( E_n^q \) where \( V \cap E_n^q = E_n^u \), a \( u \)-dimensional vector subspace.

Statement \( B_{n,i-1} \) implies that there is a finite subcomplex \( L_{n,i-1}^* \) of \( E_n^u \cap g_{n,i-1}(\text{Int } N_{n,i-1}^m) \) which "approximates" \( E_n^u \cap g_{n,i-1}(M'_{n,i}) \) in the sense that

\[
E_n^u \cap g_{n,i-1}(M'_{n,i}) \subset L_{n,i-1}^* \subset E_n^u \cap g_{n,i-1}(\text{Int } N_{n,i-1}^m)
\]

and \( L_{n,i-1}^* \) is nicely embedded in \( V, Q \) and the image of \( M \). Therefore the Taming Lemma 2.2 provides a small push on \( g_{n,i-1}^{-1}(L_{n,i-1}^*) \cap M'_{n,i} \) in \( M \) which pushes a small neighborhood of \( g_{n,i-1}^{-1}(L_{n,i-1}^*) \cap M'_{n,i} \) onto a subpolyhedron of a euclidean neighborhood of \( f_{n,i}^m \). Therefore without loss of generality we may assume that the set defined as

\[
(4.1) \quad L_{n,i-1} = L_{n,i-1}^* \cap g_{n,i-1}(M'_{n,i}) = E_n^u \cap g_{n,i-1}(M'_{n,i})
\]

is a subcomplex of both \( L_{n,i-1}^* \) and \( g_{n,i-1}(M'_{n,i}) \).

Therefore \( L_{n,i-1} \) is nicely embedded in \( E_n^u \) and, by the Taming Lemma 2.2 again, we may assume, without loss of generality, that \( L_{n,i-1} \) is also a subcomplex of \( E_n^u \).

The clever idea of this proof is that we have not arranged things (with the aid of (4.1)) so that the set

\[
R_{n,i-1} \overset{\text{def}}{=} E_n^u \cup g_{n,i-1}(M'_{n,i})
\]

is a complex nicely embedded in \( E_n^q \).

The Taming Lemma 2.2 provides an \((e/2s_n)\)-push \( P_{n,i}^m \) of \( g_{n,i-1}(M'_{n,i}) \) in \( E_n^q \) such that

(a) \( P_{n,i}^m \) moves \( R_{n,i-1} \) onto a subcomplex of \( E_n^q \);

(b) \( P_{n,i}^m \) moves no point of \( E_n^u \); and

(c) \( P_{n,i}^m \circ g_{n,i-1}^{-1} : M'_n \rightarrow E_n^q \) is PL.

Now the Taming Lemma 2.2 provides an \((e/2s_n)\)-push \( P_{n,i}^m \) of \( P_{n,i}^m g_{n,i-1}(M'_{n,i}) \) in \( E_n^q \) such that

\[
P_{n,i}^m P_{n,i}^m g_{n,i-1} : I_{n,i} \rightarrow E_n^q
\]

is PL and in PL-general-position with respect to \( E_n^u \) and such that

\[
P_{n,i}^m((P_{n,i}^m g_{n,i-1}(N_{n,i}^m - M'_{n,i}))) \cap E_n^u = 1.
\]

Now \( P_{n,i}^m = P_{n,i}^m \circ P_{n,i}^m \) is the push demanded by statement \( B_{n,i} \) since both \( P_{n,i}^m \) and \( P_{n,i}^m \) preserve the intersection of \( E_n^u \) with the image of \( M'_{n,i} \) and \( P_{n,i}^m \) places the image of \( M'_{n,i} \) in general position with respect to \( E_n^u \).

This establishes statement \( B_{n,i}, i = 1, 2, \ldots, i_n \). We set \( P_n = P_{n,k_n} \circ \ldots \circ P_{n,2} \circ P_{n,1} \) and then statements \( A_n, n = 1, 2, \ldots, s \), are established.
Finally statement $A_s$ establishes General Position Lemma 3.

5. Proof of Topological General Position Lemma 1. In this section we shall first establish the Topological General Position Lemma 1 for the case when $M$ is compact and $e(x)$ is a constant function, i.e. $e(x) = \epsilon > 0$. Following this we shall indicate how the noncompact case follows from the compact case.

The epsilontics will be omitted from the proof. The reader may easily fill them in.

Proof of Topological General Position Lemma 1. Case (i) $M$ is compact.

Let $f$, $M$, and $Q$ satisfy the hypotheses of the lemma and let $M$ be compact. Let $\{I_i, M_i', M_i'', N_i, i = 1, \ldots, r\}$ be a standard representation of $M$ such that each $f(I_i')$ is contained in some Euclidean open set $E^q_i$ of $Q$ (see Theorem 2.1).

The proof is a double induction, namely;

**Big Induction.** We shall construct approximations $g_i: M \to Q$ of $f_i$, $i = 1, 2, \ldots, r$, such that $g_i|\text{Int } N_i$ is a general position map without triple points.

**Little Induction.** We shall construct approximations $g_{ij}: M \to Q$ of $g_{i-1}$, $j = 1, 2, \ldots, i - 1$, and small pushes $P_{ij}$ of $(Q, g_{i-1}(I_j))$, $j = 1, \ldots, i - 1$ such that

(i) $g_{ij}|\text{Int } N_j \cup I_i - B^q_i$ is a general position map; and
(ii) $g_{ij}|I_i = g_{i-1}|I_i$ is a piecewise linear general position map of $I_i$ into $E^q_i \subset Q$;
(iii) $g_{ij}|N_i - I_i = P_{ij}g_{i-1}|N_i - I_i$;
(iv) $g_{ij}|\text{Int } N_i$ has no triple points;
(v) $g_{ij}(x) = g_{i-1}(x), x \in M - \bigcup_{k<i} I_k$ and $j > 1$.

All that we have to do in order to prove General Position Lemma 1 is to indicate how to do the steps of this double induction. Here goes: (note: The subscripts $i$ and $j$ will refer to the Big Induction and the Little Induction, respectively).

*Step $i = 1$.* Approximate $f|I_1: I_1 \to E^q_1 \subset Q$ by a piecewise linear general position map. Extend this map to all of $M$ obtaining $g_1: M \to Q$.

*Step $i = 2$. Beginning.* (i) Let $S_1$ be the singularities of $g_1|N_1$. Then there is a subpolyhedron $K$ of $I_1$ such that $S_1 \cap I_2 \subset K \subseteq S_1 \cap I_2''$. One may use Taming Lemma 2.2 in order to find a small push $P$ of $(I_2'', K)$ such that $P(K)$ is a subpolyhedron of $I_2''$. Therefore without loss of generality we may assume that $S_1 \cap I_2'$ is a subpolyhedron of $I_2'$ and that the triangulations of $I_1$ and $I_2'$ are compatible on $S_1 \cap I_2'$.
(ii) We can construct a small push $P$ of $(Q, g_1(M_2'))$ such that $Pg_1|M_2': M_2' \to E^q_2 \subset Q$ is a piecewise linear map. This follows from Taming Lemma 2.2, Steps $i = 1$ and (i) above. Thus, without loss of generality we may assume
that $g_1|M'_2: M'_2 \to E'^2 \subset Q$ is a piecewise linear map.

(iii) Extend $g_1|M'_2$ to a piecewise linear general position map $g_2: I_2 \to E'^2$, such that

$$g_2(I_2 - M'_2) \cap g_1((S_1 - I_2) \cap N_2) = \emptyset$$

using the dimension restriction $m \leq (2/3)q - (1/3)$ and Lemma 5 of [D], namely.

**Lemma 5.1.** Let $f$ be a map of an $r$-complex $K$ into a combinatorial $n$-manifold $M$. Let $X$ be the finite union of tame $s$-complexes in $M$, $r + s < n$. If $\epsilon > 0$ is given, then there is a piecewise linear map $g: K \to M$ such that $g(K) \cap X = \emptyset$ and $d(g, f) < \epsilon$.

Furthermore, if $L$ is a subcomplex of $K$ such that $f|L$ is already piecewise linear and $f(L) \cap X = \emptyset$, then $g|L = f|L$.

We shall finish Step $i = 2$ after we complete:

**Step $i = 2$ and $j = 1$.** Let $K'_2$ be a subpolyhedron of $K_2$ such that

$$\text{Closure}[I_1 - I'_1] \cap f^{-1}(E'^2) \subset K'_2 \subset [I_1 - I_2 - B_2] \cap f^{-1}(E'^2).$$

Let $K_2 = K'_2 \cup M'_2$. Thus $K_2$ is a polyhedron with $K'_2$ and $M'_2$ as disjoint subpolyhedra.

A little checking shows that $g_1|K_2$ is a nice map. Therefore, we may use Taming Lemma 2.2 in order to obtain a small push $P_{2,1}$ of $(Q, g_1(K'_2))$ such that:

(i) $P_{2,1}g_1|K_2: K_2 \to E'^2$ is piecewise linear,

(ii) $P_{2,1}g_1|K_2$ is in general position with respect to $g_2|I_2$, and

(iii) $P_{2,1}g_1(M'_2) = 1$.

Thus we may easily construct $g_{2,1}: M \to Q$ as an extension of $P_{2,1} \circ g_1|K_2$ and $g_2|I_2$. This $g_{2,1}$ together with $P_{2,1}$ will satisfy the Little Induction hypothesis for $i = 2$ and $j = 1$.

**Step $i = 2$. Conclusion.** All that remains is to "correct" the intersection of $g_{2,1}(B''_2)$ with $g_{2,1}(M''_2)$.

One may use an argument which is similar to the beginning of Step $i = 2$ in order to show that we may assume without loss of generality that:

(i) The singularities of $g_{2,1}|K_2 \cup I_2$ intersect $B''_2$ (as well as $I_2$) as a subpolyhedron of $I''_2$.

(ii) Let $K''_2 = (B''_2 \cap \text{Int} N_1) \cup I_2$. Then $g_{2,1}|K''_2: K''_2 \to E''_2 \subset Q$ is a piecewise linear map.

As a consequence of equation (5.1) there exists a closed neighborhood $U_2$ of $[g^{-1}_{2,1}g_{2,1}(M''_2)] \cap B''_2 \cap N_2$ such that $U_2 \cap (S_1 \cup M''_2 \cup (M - N_1)) = \emptyset$ and $K''_2 \cup U_2$ is a combinatorial submanifold of $I''_2$. (Remember that $S_1$ is the singularities of $g_1|N_1$.)
Remark. We only need the dimension restriction \( m \leq (2/3)q - (1/3) \) in order to obtain this set \( U_2 \) via equation (5.1).

Let \( K''_2 \) be a subpolyhedron of \( I''_2 \) such that

\[
(U_2 \cup M'_2 \cup B''_2) \cap N_2 \subset K''_2 \subset (U_2 \cup M'_2 \cup B''_2) \cap N_1.
\]

Since \( g_{2,1}|K''_2 = P_2 g_{1,1}|K''_2 \), we see that \( g_{2,1} \) is a nice map of \( K''_2 \) into \( E_2^q \).

Therefore there is a push \( P_2 \) of \( (E_2^q, g_{2,1}(U_2)) \) such that

\[
P_2 g_{2,1}|U_2 : U_2 \rightarrow E_2^q \subset Q \text{ is piecewise linear},
\]

\[
P_2 g_{2,1}(K''_2 - U_2) \cup (N_1 - I''_2) = 1,
\]

\( P_2 g_{2,1}|U_2 \) is in general position with respect to \( g_{2,1} M'_2 \).

A little checking will show that the desired map is

\[
g_2(x) = \begin{cases} 
  g_{2,1}(x), & x \in (M - N_1) \cup I_2, \\
  P_2 g_{2,1}(x), & x \in N_1.
\end{cases}
\]

This completes Step \( i = 2 \).

Step \( i = 3 \). Beginning. This is essentially the same as the beginning of Step \( i = 2 \). Thus we obtain a piecewise linear map \( g_3 : I_3 \rightarrow E_3^q \) such that

(i) \( g_3(I_3 - M'_3) \cap g_2(S_2 - M'_2) \cap N_3 = \emptyset \), and without loss of generality that

(ii) \( S_2 \cap I'_2 \) is a subpolyhedron of \( I'_3 \),

(iii) \( g_3|M'_3 = g_2|M'_2 \),

where \( S_2 \) is the set of singularities of \( g_2|N_2 \).

Step \( i = 3 \) and \( j = 1 \). This is essentially the same as Step \( i = 2 \) and \( j = 1 \) for \( I_1 \cap N_2 \) and \( I_3 \) instead of \( I_1 \) and \( I_2 \), respectively. Some minor problems appear at \( I_1 \cap \partial N_2 \). These problems should be ignored since \( I_1 \cap \partial N_2 \subset \bigcup_{k>1} I_k \), and hence these bad points will automatically disappear later.

Step \( i = 3 \) and \( j = 2 \). Let \( K_{3,2} \) be a subpolyhedron of \( I_3 \cup (M'_2 - (I_3 \cup B_3)) \) such that \( I_3 \cup M'_2 - B'_3 \subset K_{3,2} \). We observe that \( g_{3,1}|K_{3,2} \) is a nice map of \( K_{3,2} \) into \( E_3^q \). Hence, as before without loss of generality, we temporarily assume that \( g_{3,1}|K_{3,2} \) is piecewise linear.

Let \( K'_{3,2} \) be a polyhedron which “approximates” \( (I_2 - I_3) \cup M'_3 \) namely \( K'_{3,2} \) is a subpolyhedron of \( [I_2 - (I_3 \cup B_3)] \cup M'_3 \) which contains \( (I_2 - I'_3) \cup M'_3 \).

We see that \( g_{3,1}|K'_{3,2} \) is a nice map. Hence the Taming Lemma 2.2 applies. Therefore there is a push \( P_{3,2} \) on \( (Q, g_{3,1}(I_2 - M'_3)) \) such that

(i) \( P_{3,2} g_{3,1}|K'_{3,2} \) is piecewise linear;

(ii) \( P_{3,2} g_{3,1}(M'_3) = 1 \);

(iii) \( P_{3,2} g_{3,1}|K'_{3,2} \) is in general position with respect to \( g_{3,1}|I_3 \).

The reader may now easily finish the construction of \( g_{3,2} : M \rightarrow Q \) as an extension of \( P_{3,2} \circ g_{3,1}|K'_{3,2} \) and \( g_{3,1}|I_3 \); this \( g_{3,2} \) together with \( P_{3,2} \) above
will satisfy the conditions of the Little Induction for \( i = 3 \) and \( j = 2 \).

**Step \( i = 3 \). Conclusion.** This is essentially the same as the conclusion of Step \( i = 2 \).

This completes the proof of Step \( i = 3 \) of the Big Induction.

By continuing in this manner the proof of the General Position Lemma will be completed. All the steps in each of the following collections are essentially the same:

(i) Beginning of Step \( i, i \geq 2 \),

(ii) Steps \( j = 1 \) for \( i \geq 2 \),

(iii) Steps \( j \geq 2 \) for \( i \geq 3 \),

(iv) Conclusion of Step \( i, i \geq 2 \).

**Remark on Steps \( j \geq 2 \) and \( i \geq 4 \).** We note that

\[
K_{ij} \quad \text{will approximate} \quad I_i \cup (M_{ij} - I_i - N_{i,j-1});
\]

\[
K'_{ij} \quad \text{will approximate} \quad M_{ij}' \cup (I_j - I_i - N_{i,j-1}).
\]

There are some difficulties at the boundary of \( N_{i-1} \). These should be ignored since \( \partial N_{i-1} \subset \bigcup_{k \geq i} I_k \) and hence will automatically be corrected later.

The final map \( g: M \to Q \) is the map demanded by General Position Lemma 1.

Finally we observe that, since every manifold is locally contractible, if \( g \) is sufficiently close to \( f \) then \( g \) is homotopic to \( f \). This completes the proof for the case \( M \) is compact.

**Case (ii).** \( M \) is not compact. Here \( M = \bigcup_{i=1}^{\infty} X_i \) where each \( X_i \) is compact and \( X_i \subset \text{Int} \, X_{i+1}, i = 1, 2, \ldots \). Let \( \epsilon_i = \min \{ \epsilon(x) \mid x \in X_i \} \).

So we construct \( g \) on \( M \) by constructing a sequence of \( g_i \)'s on neighborhoods of the \( X_i \)'s with the aid of standard representations for neighborhoods of the \( X_i \)'s in \( M \). These \( g_i \)'s will be general position maps on some small neighborhoods of the \( X_i \)'s and it is easily arranged that \( d(g_i(x), f(x)) < \epsilon_i, x \in X_i - X_{i-1} \).

Furthermore, since \( f \) is a proper map for each integer \( i_0 \) there is an integer \( i_1 \) such that \( f(X_{i_0}) \cap f(M - X_{i_1}) = \emptyset \).

Hence we may arrange the \( g_i \)'s so that \( g_{i_1} \mid X_{i_0} = g_n \mid X_{i_0} \) for each \( n > i_1 \). Therefore the \( g_i \)'s will converge to a continuous function \( g \) which satisfies all the conditions of a general position map.

Again we observe that, since every manifold is locally contractible, if we keep \( g \) sufficiently close to \( f \), then \( g \) will be properly homotopic to \( f \).

This completes the proof of the Topological General Position Lemma 1.

6. **Corollaries of Topological General Position Lemma 1.** In this section we shall establish two corollaries of (the proof of) the Topological General Position Lemma 1. We need and use these two corollaries in papers on topological embeddings which we are currently writing.
Corollary 6.1. Let $f: (M, \partial M) \to (Q, \partial Q)$ be a continuous proper map of an $m$-manifold into a $q$-manifold, $m \leq (2/3)q - (1/3)$, $m \leq q - 3$ and let $\varepsilon: M \to (0, 1)$ be a given continuous function. Then there is a general position map $g: M \to Q$ such that $g|\partial M \to \partial Q$ is also a general position map,

$$d(f(x), g(x)) < \varepsilon(x), \quad \text{for each } x \in M,$$

and $g$ is properly homotopic to $f$.

Proof. The Topological General Position Lemma 1 provides a general position map $g_1|\partial M \to \partial Q$ which approximates $f|\partial M$. The collars of $\partial M$ in $M$ and of $\partial Q$ in $Q$ enables one to extend this map $g_1$ to a general position map $g_1 \times 1$ defined on an open collar of $\partial M$ into a small collar of $\partial Q$ so that $g_1 \times 1$ approximates $f$ on the open collar of $\partial M$. Let $f_1$ be a close approximation of $f$ which agrees with $g_1 \times 1$ on the domain of $g_1 \times 1$. A general position map $g: (M, \partial M) \to (Q, \partial Q)$ which approximates $f_1$ and which agrees with $f_1$ and $g_1 \times 1$ on some small closed collar of $\partial M$ is obtained by applying the proof of the Topological General Position Lemma 1 to $f_1$ on $M$ minus the domain of $g_1 \times 1$. Thus Corollary 6.1 is established.

Corollary 6.2. Let $f: M \to Q$ be a proper map of a topological $m$-manifold into a topological $q$-manifold, $m \leq (2/3)q - 1/3$, and let $\varepsilon: M \to (0, \infty)$ be a continuous function. Then there is a proper general position map $g: M \to Q$ such that $d(f(x), g(x)) < \varepsilon(x)$.

Furthermore if $f|\partial M$ is already a locally flat embedding then $g$ may be constructed so that $g|\partial M = f|\partial M$.

Note. Here $f(M)$ is still permitted to intersect both the interior of $Q$ and $f(\text{Int } M)$.

The proof of Corollary 6.2 is a slightly complicated variation of the proof of the Topological General Position Lemma 1.

Outline of the Proof of Corollary 6.2. Let $f: M \to Q$ satisfy the hypothesis of this Corollary 6.2. We use here the same standard representation and the same big and little induction statements as used in the proof of the Topological General Position Lemma 1. These inductions here have an analogous set of "Steps" to the previous set of steps.

Step $i = 1$. First push $f(I_1 \cap \partial M)$ onto a subpolyhedron of $E_1^q$ by some small push $P_1^*$; this is possible because $f|\partial M$ is a locally flat embedding, locally flat embeddings are locally tame and because of the Taming Lemma 2.2. Now approximate $P_1^* f|I_1 \cap \partial M$. Extend $(P_1^*)^{-1} f|I_1 \to E_1^q$ which extends $P_1^* f|I_1 \cap \partial M$. Extend $(P_1^*)^{-1} f|I_1$ to all of $M$ obtaining $g_1: M \to Q$.

Note that on $I_1 \cap \partial M$, $g_1$ agrees with $(P_1^*)^{-1} P_1^* f = f$.
Step i = 2. Beginning. Parts (i) and (ii) are the same as in the proof of the Topological General Position Lemma 1.

(iii) First use the Taming Lemma 2.2 in order to push \( g_1(\partial M \cap M''_2) \) onto a subpolyhedron of \( E_2^n \) by a push \( P_2^* \) which moves no point of \( g_1(\partial M \cap M'_2) \) such that

\[
[P_2^*g_1(\partial M \cap M''_2)] \cap g_1((S_1 - I_2) \cap N_2) = \emptyset
\]

and \( P_2^*g_1|\partial M \cap I_2 \) is in PL general position with respect to \( g_1|M'_2 \) using the dimension restriction \( m \leq (2/3)q - (1/3) \) and

\[
\text{Lemma 6.3. If the } f \text{ of Lemma 5.1 is a locally tame embedding and } r \leq n - 3, \text{ then the } g \text{ of Lemma 5.1 may be constructed so that there is an } \epsilon \text{-push } P^* \text{ such that } g = P^*f \text{ and } P^* \text{ moves no point of } L.
\]

Remark. Lemma 6.3 is a corollary of the proof of Lemma 19 of [D] and the Taming Lemma 2.2.

Now extend \( P_2^*g_1|\partial M \times I_2 \) and \( g_1|M'_2 \) to a PL general position map \( g_2: I_2 \rightarrow E_2^n \) such that

\[
g_2(I_2 - M'_2) \cap g_1((S_1 - I_2) \cap N_2) = \emptyset
\]

using Lemma 5.1 and the hypothesis \( m \leq (2/3)q - (1/3). \)

Note. \( g_2|\partial M \times I_2 = P_2^*f|\partial M \times I_2. \)

This completes Step i = 2. Beginning.

We would like to use Step i = 2 and \( j = 1 \) of the proof of the Topological General Position Lemma here while being careful that

\[
P_{2,1}g_1(\partial M \cap (I_1 - I_2)) \cap P_2^*g_1(\partial M \cap M''_2) = \emptyset
\]

Unfortunately, we do not know how to guarantee that \( g_{2,1}|\partial M \) will have no self intersection at \( N_1 - N_2. \) To overcome this problem we shall run through the previous Step \( i = 2 \) and \( j = 1 \) twice; the first time we will adjust just the image of \( \partial M \cap K'_2 \) with respect to the image of \( I_2, \) without destroying the fact that \( \partial M \) is embedded in \( Q, \) and the second time to adjust the image of \( K'_2 \) while fixing the image of \( \partial M \cap K'_2. \)

Step i = 2 and \( j = 1. \) Let \( K'_2 \) be as in Step \( i = 2 \) and \( j = 1 \) of the proof of Topological General Position Lemma. Let \( (K'_2)^* = K'_2 \cap \partial M \) and \( K_2^* = (K'_2)^* \cup M'_2. \) As in the previous Step \( i = 2 \) and \( j = 1, \) the Taming Lemma 2.2 provides a small push \( P_{2,1}^* \) of \( (Q, g_1((K'_2)^*)) \) such that \( P_{2,1}^*g_1|K_2^* \) is PL and is in general position with respect to \( g_2|I_2 \) and

\[
P_{2,1}^*P_2^*g_1(M'_2) \cup [\partial M \cap (B'_1 \cup B_2 \cup I_2)] = 1.
\]

Here \( P_{2,1}^* \) is a \( \delta \)-push where

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Therefore, if \( x, y \in \partial M \) and \( P^*_{2,1}g_1(x) = g_1(y) \) then both \( x \) and \( y \) must be in \((K'_2)^* \cap (\partial M \cap B'_2)\).

Now let \((K'_2)^{**}\) be a subpolyhedron of \( K'_2 \) which approximates \( K'_2 - g_1^{-1}[g_1(S_1 \cap (N_1 - N_2 - B'_2))] \).

Using \((K'_2)^{**}\) instead of \( K'_2 \) in the procedure of Step \( i = 2 \) and \( j = 1 \) of the Topological General Position Lemma yields a small push \( P^{**}_{2,1} \) such that

\[
P^{**}_{2,1}P^*_{2,1}P^*_2g_1|K_2 = [K'_2 - (K'_2)^{**}] \text{ is PL}
\]

and is in general position with respect to \( g_2|I_2 \) and such that

\[
P^{**}_{2,1}|P^*_{2,1}P^*_2g_1(M'_2 \cup (\partial M \cap (I_1 - I_2 - B'_2))) = 1.
\]

By using \((K'_2)^{**}\) instead of \( K'_2 \) above we were able to avoid moving the image of \( S_1 \cap (N_1 - N_2 - B'_2) \cap \partial M \).

Therefore we may easily construct \( g_{2,1} : M \to Q \) as an extension of \( P^*_2, P^*_{2,1}g_1|K_2, g_2|I_2 \) and \( P^*_{2,1}P^*_2g_1|\partial M \).

\textbf{Note.} Since \( g_{2,1}|\partial M = P^*_2|g_2|\partial M \), it is a locally flat embedding.

\textit{Step} \( i = 2. \ Conclusion. \) This is the same as Step \( i = 2 \) (conclusion) of the proof of the Topological General Position Lemma 1 with the addition that here \( P_2 \) is a \( \delta \)-push where

\[
2\delta > d(g_2(\partial M \cap (N_2 - I_2)), g_2(\partial M - N_1)).
\]

This choice of \( \delta \) yields that \( g_2|\partial M \) is a locally flat embedding. A little checking will show that \( g_2|\partial M \) was obtained from \( g_1|\partial M \) by three small pushes.

\textbf{Step} \( i = 3 \) and \( j = 2. \) This step begins the same as Step \( i = 3 \) and \( j = 2 \) of the proof of the Topological General Position Lemma 1 with the same \( K_{3,2} \) and also having \( g_{3,1}|K_{3,2} \) being PL. In the proof of Topological General Position Lemma 1, the remainder of Step \( i = 3 \) and \( j = 2 \) is just like Step \( i = 2 \) and \( j = 1 \). So we complete this step here by retracing our path through Step \( i = 2 \) and \( j = 1 \).

As before, the other steps in the inductions are analogous to the ones we just discussed. The result of going through all the steps is a general position map \( g_r : M \to Q \) such that \( g_r|\partial M \) was obtained from \( f|\partial M \) by a series of small pushes. If \( P \) is the composition of these pushes then \( g_r|\partial M = Pf|\partial M \) and hence \( f|\partial M = P^{-1}g_r|\partial M \). Therefore \( P^{-1}g_r : M \to Q \) is the desired general position approximation to \( f \). Thus Corollary 6.2 is established.

\textbf{Added in proof.} We mention here two more corollaries of the techniques of this paper. We shall use both of these corollaries as lemmas in another paper that we are currently writing.
Corollary 6.3. Let $f : M \rightarrow Q$ be a proper map of a topological $m$-manifold into a topological $q$-manifold, $m \leq (2/3)q - 1/3$ and let $\varepsilon : M \rightarrow (0, \infty)$ be a continuous function. Suppose $\partial M = M_1 \cup M_2$, where $M_1$ and $M_2$ are two disjoint $(n-1)$-submanifolds of $\partial M$, $\partial M_1 = \emptyset = \partial M_2$. Suppose $f|M_1$ is a locally flat embedding. Then there is a proper general position map $g : M \rightarrow Q$ such that $d(f(x), g(x)) < \varepsilon(x)$, for each $x \in M$, and $g|M_1 = f|M_1$.

Remark. Corollary 6.3 is an immediate consequence of our proof of Corollary 6.2 (just treat $M_2$ as if it is part of the interior of $M$).

Corollary 6.4. Let $V$ be a proper locally flat topological $v$-submanifold in the interior of a topological $q$-manifold $Q$, $q \geq v + 3$. Let $K$ be a locally finite simplicial $k$-complex, $k \leq q - 3$, which is properly locally-tamely embedded in $Q$. Then there is an ambient isotopy

$$\{H_t : Q \rightarrow Q, H_0 = 1\}$$

such that $H_1(K) \cap V$ is a $(v + k - q)$-subcomplex $K_2$ of $H_1(K)$ and $K_2$ is locally-tamely embedded in $V$.

Remark. Corollary 6.4 is a corollary of the proof of General Position Lemma 3. The proof of Corollary 6.4 is actually simpler because of the triangulation of $K$. Therefore, here the standard representation of $M$, in the proof of General Position Lemma 3 (when $M$ is compact), is replaced by a set of subcomplexes $\{K_1, \ldots, K_n\}$ of $K$ such that $K = \bigcup_{j=1}^n K_j \subset E_j^q, j = 1, 2, \ldots, n$.

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DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742