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ESTIMATES FOR THE $\bar{\partial}$-NEUMANN OPERATOR IN WEIGHTED HILBERT SPACES

BY

SIDNEY L. HANTLER

ABSTRACT. Estimates for the $\bar{\partial}$ operator are used to derive estimates for the Neumann operator in weighted Hilbert spaces. The technique is similar to that used to prove regularity of solutions of elliptic partial differential equations. A priori estimates are first obtained for smooth compactly supported forms and these estimates are then extended by suitable approximation results. These estimates are applied to give new bounds for the reproducing kernels in the subspaces of entire functions.

1. **Introduction.** L. Hörmander gave fundamental estimates for the $\bar{\partial}$ operator in weighted Hilbert spaces in [5]. Based on this work, we give here improved estimates for the associated Neumann operator. These estimates are applied to give new bounds for the reproducing kernels of the subspaces of analytic functions. In [3], this technique is applied to problems of weighted polynomial approximation.

To introduce our results, we present some notation. The Hilbert space of (classes of) functions on C^n which are square integrable with respect to the measure $e^{-\omega}\,d\lambda$, where ω is measurable with respect to the Lebesgue measure $d\lambda$, is denoted by $L^2(\omega)$. Similarly $L^2_{(p,q)}(\omega)$ is the space of differential forms of type (p, q) whose coefficients belong to $L^2(\omega)$.

To illustrate our results, suppose $n = 1$ and there is a positive constant c so that $\partial^2\omega/\partial z\partial\bar{z} \geqslant c$. Then the Neumann operator, N, maps $L^2_{(0,1)}(\omega)$ into $L^2_{(0,1)}(\omega)$. It is shown, in Theorem 2, that if

$$\alpha < \sqrt{2c} < \beta \quad \text{and} \quad \theta \in L^2_{(0,1)}(\omega - \beta|z|),$$

then $N(\theta) \in L^2_{(0,1)}(\omega - \alpha|z|)$. This result is applied, in Theorem 6, to show that the reproducing kernels of $A^2(\omega)$, the subspace of analytic functions in $L^2(\omega)$, belong to $A^2(\omega - \alpha|z|)$. Estimates for the Neumann operator in a more general setting are given in Theorem 4 and general estimates for reproducing kernels are presented in Theorems 7 and 8.

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Our method for proving estimates for the Neumann operator is similar to that used in proving regularity of solutions of elliptic partial differential equations. A priori estimates are obtained for smooth forms with compact support. Approximation results are then proved in order to extend these estimates to forms which can be suitably approximated by smooth, compactly supported forms.

The technique for obtaining estimates for reproducing kernels is based on the idea of N. Kerzman [6] to represent reproducing kernels in terms of the Neumann operator on the space of (0, 1) forms. Kerzman used estimates derived by J. J. Kohn [7] to prove that the Bergman kernel functions of strongly pseudoconvex domains are smooth up to the boundary. Our estimates for the reproducing kernels of $A^2(\omega)$ are derived in a similar fashion, using estimates for the Neumann operator on $L^2_{(0,1)}(\omega)$.

Our estimates for the Neumann operator in weighted Hilbert spaces were motivated by the study of polynomial approximation in weighted Hilbert spaces of entire functions in [8] and [9]. These estimates can be used to prove approximation theorems in $A^2(\omega)$. In [3], this technique is used to show that the polynomials are dense in $A^2(\omega)$ when ω is nearly radial.

The Neumann operator is defined and some estimates of Hörmander are given in §2. §§3 and 4 are devoted to a discussion of a special case of our results. In §3, a priori estimates for the Neumann operator are derived, and in §4 these estimates are extended by means of an approximation result. In §§5 and 6 we consider the most general case of our results. We give an indication, in §5, of the new techniques required to generalize the results of §§3 and 4. §6 is devoted to an exposition of general estimates for the Neumann operator. Finally, in §7, we apply these results to derive new bounds for reproducing kernels.

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2. Notation and preliminaries. We recall some notation and results of Hörmander from [4]. The space of infinitely differentiable functions with compact support in C^n is denoted by $\mathcal{D}$ and the space of (p, q) forms with coefficients in $\mathcal{D}$ is denoted by $\mathcal{D}_{(p,q)}$. The $\bar\partial$ operator, mapping $\mathcal{D}_{(p,q)}$ into $\mathcal{D}_{(p,q+1)}$, is defined by

$$\bar\partial\left(\sum_{I,J}{}' f_{I,J}\, dz^I \wedge d\bar z^J\right) = \sum_{I,J}{}' \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial \bar z_k}\, d\bar z^k \wedge dz^I \wedge d\bar z^J,$$

where the prime indicates that the summation extends only over increasing multi-indices.

In order to apply the estimates proved in [4], we shall assume that ϕ and ψ are in $C^2(C^n)$ and that there are positive constants δ and c so that

$$\text{(a)}\quad e^{-\psi} + e^{-2\psi}|\bar{\partial}\psi|^2 \leqslant \delta(1 + |z|^2),$$

$$\text{(2.1)}\quad \text{(b)}\quad \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z)w_j\bar{w}_k \geqslant (2|\partial\psi(z)|^2 + ce^{\psi(z)})|w|^2,$$

$$\text{(c)}\quad \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2(\phi+\psi)}{\partial z_j \partial \bar{z}_k}(z)w_j\bar{w}_k \geqslant (2|\partial\psi(z)|^2 + ce^{\psi(z)})|w|^2,$$

hold for all $z, w \in C^n$. We write $\phi_1 = \phi - 2\psi$, $\phi_2 = \phi - \psi$, $\phi_3 = \phi$, and $\phi_4 = \phi + \psi$.

The weak extensions of the $\bar{\partial}$ operator defined on $L^2_{(p,q)}(\phi_1) \xrightarrow{\bar{\partial}} L^2_{(p,q+1)}(\phi_2)$ and $L^2_{(p,q+1)}(\phi_2) \xrightarrow{\bar{\partial}} L^2_{(p,q+2)}(\phi_3)$ are denoted by T and S respectively. Then T and S are closed densely defined operators with Hilbert space adjoints T^* and S^*. The domain and range of an operator A are denoted by D_A and R_A respectively.

From Lemma 4.2.1 of [4] it follows that

$$\text{(2.2)}\qquad c^2\|f\|^2_{\phi_2} + c\int {\sum_{I,J}}' \sum_{k=1}^{n} \left|\frac{\partial f_{I,J}}{\partial \bar{z}_k}\right|^2 e^{-\phi}\,d\lambda \leqslant 2\|(TT^* + S^*S)f\|^2_{\phi_2}$$

for all $f \in D_{TT^*+S^*S}$.

From (2.2) we conclude that $TT^* + S^*S$ has a bounded inverse, N, mapping $L^2_{(p,q+1)}(\phi_2)$ into $L^2_{(p,q+1)}(\phi_2)$. The operator N is the Neumann operator on $L^2_{(p,q+1)}(\phi_2)$ and $\|N\| \leqslant 2/c$. Our object is to prove that the Neumann operator maps certain proper subspaces of $L^2_{(p,q+1)}(\phi_2)$ into other proper subspaces of $L^2_{(p,q+1)}(\phi_2)$.

3. **A priori estimates in a special case.** In this section we obtain a priori estimates for the Neumann operator when $N(\theta)$ has smooth, compactly supported coefficients. Strong a priori estimates cannot be extended directly due to difficulties encountered in proving suitable approximation results. In §4 an approximation result is proved which enables us to prove weak estimates for a more general class of forms. These estimates are then used to prove stronger estimates for this larger class of forms.

Certain technical difficulties arise in proving estimates for the Neumann operator when $n > 1$ and $\psi \neq 0$. These details obscure the general argument and, therefore, in this section and in §4 we assume that $n = 1$ and $\psi = 0$. We also assume that $p = q = 0$. In §5, we indicate how these restrictions can be removed. General results with references to complete proofs are presented in §6.

Recall that $\phi \in C^2(C)$ and that $\partial^2\phi/\partial z\partial\bar{z} \geqslant c > 0$ from (2.1). Because $n = 1$, we have $S = 0$ and, thus, N is the inverse of TT^*. Where there is no danger of confusion we write $f = f d\bar{z}$. It follows from (4.1.9) in [4] that

$$T^*(f) = (\partial\phi/\partial z)f - \partial f/\partial z \quad \text{when } f \in D_{T^*} \tag{3.1}$$

and thus that

$$TT^*(f) = (\partial^2\phi/\partial z\partial\bar{z})f + T^*(\partial f/\partial\bar{z})d\bar{z} \quad \text{when } f \in D_{TT^*}. \tag{3.2}$$

We conclude that

$$T^*(Gf) = GT^*(f) - (\partial G/\partial z)f \quad \text{when } f \text{ and } Gf \text{ are in } D_{TT^*} \tag{3.3}$$

and that

$$TT^*(Gf) = GTT^*(f) + \frac{\partial G}{\partial\bar{z}}T^*(f)d\bar{z} - \frac{\partial^2 G}{\partial z\partial\bar{z}}f - \frac{\partial G}{\partial z}\frac{\partial f}{\partial\bar{z}}d\bar{z} \quad \text{when } f \text{ and } Gf \text{ are in } D_{TT^*}. \tag{3.4}$$

Finally, observe that if $f \in D_{TT^*}$, we have

$$\|f\|_\phi \leqslant \|N\|\|TT^*(f)\|_\phi \tag{3.5}$$

and

$$\|T^*(f)\|_\phi \leqslant \|N\|^{1/2}\|TT^*(f)\|_\phi. \tag{3.6}$$

We can now obtain a priori estimates for the Neumann operator by considering the commutator of TT^* and multiplication by a smooth function. Let $\theta \in L^2_{(0,1)}(\phi)$ with $N(\theta) \in \mathcal{D}_{(0,1)}$ and $G \in C^\infty(C^n)$. Notice that

$$GN(\theta) = N(G\theta) + N[TT^*, G]N(\theta) \tag{3.7}$$

where $[A, B]$ denotes the commutator $AB - BA$. This yields

$$\|GN(\theta)\|_\phi \leqslant \|N\|\|G\theta\|_\phi + \|N\|\|[TT^*, G]N(\theta)\|_\phi.$$

Our goal is to choose G and a subspace H of $L^2_{(0,1)}(\phi)$ containing $\mathcal{D}_{(0,1)}$ so that the right-hand side of (3.7) is finite when $\theta \in H$. We then prove an approximation result on H and apply a weak compactness argument to conclude that $GN(\theta) \in L^2_{(0,1)}(\phi)$.

As an example, suppose that $N(\theta) \in \mathcal{D}_{(0,1)}$, that $0 \leqslant a < \|N\|^{-1/2}$ and that $G(z) = e^{a\bar{z}}$. Applying TT^* to both sides of (3.7) yields

$$TT^*(e^{a\bar{z}}N(\theta)) = e^{a\bar{z}}\theta + [TT^*, e^{a\bar{z}}]N(\theta).$$

By (3.4) and the triangle inequality it follows that

$$\|TT^*(e^{a\bar{z}}N(\theta))\|_\phi \leqslant \|e^{a\bar{z}}\theta\|_\phi + a\|e^{a\bar{z}}T^*N(\theta)\|_\phi.$$

From (3.3) and (3.6) we conclude that

$$\|TT^*(e^{a\bar{z}}N(\theta))\|_\phi \leq \|e^{a\bar{z}}\theta\|_\phi + a\|N\|^{1/2}\|TT^*(e^{a\bar{z}}N(\theta))\|_\phi$$

and thus that

$$\|TT^*(e^{a\bar{z}}N(\theta))\| \leq (1 - a\|N\|^{1/2})^{-1}\|e^{a\bar{z}}\theta\|_\phi.$$

As an immediate consequence of (3.5) and (3.6) we conclude that

$$\|e^{a\bar{z}}N(\theta)\|_\phi \leq \|N\|(1 - a\|N\|^{1/2})^{-1}\|e^{a\bar{z}}\theta\|_\phi \tag{3.9}$$

and that

$$\|e^{a\bar{z}}T^*N(\theta)\|_\phi \leq \|N\|^{1/2}(1 - a\|N\|^{1/2})^{-1}\|e^{a\bar{z}}\theta\|_\phi. \tag{3.10}$$

A natural choice of $H = \{f \in L^2_{(0,1)}(\phi)\colon e^{a\bar{z}}f \in L^2_{(0,1)}(\phi)\}$ arises from (3.9) and (3.10). If $\mathcal{D}_{(0,1)}$ were dense in H in the norm $\|f\|_H = \|e^{a\bar{z}}f\|_\phi$ it would follow from (3.9) and (3.10) by a weak compactness argument that $e^{a\bar{z}}N(\theta) \in L^2_{(0,1)}(\phi)$ and that $e^{a\bar{z}}T^*N(\theta) \in L^2(\phi)$ when $\theta \in H$. We have been unable to prove this density result directly.

We now consider a different choice of G and H. Suppose that $N(\theta) \in \mathcal{D}_{(0,1)}$ and that $G(z) = \bar{z}^m$ where m is a nonnegative integer. Proceeding as above we obtain

$$TT^*(\bar{z}^m N(\theta)) = \bar{z}^m\theta + m\bar{z}^{m-1}T^*N(\theta)\,d\bar{z}$$

and

$$\|TT^*(\bar{z}^m N(\theta))\|_\phi \leq \|\bar{z}^m\theta\|_\phi + m\|N\|^{1/2}\|TT^*(\bar{z}^{m-1}N(\theta))\|_\phi.$$

An inductive argument shows that

$$\|TT^*(\bar{z}^m N(\theta))\|_\phi \leq m!\|N\|^{m/2}\sum_{k=0}^{m}\frac{\|N\|^{-k/2}}{k!}\|\bar{z}^k\theta\|_\phi. \tag{3.11}$$

From (3.5) and (3.6) it follows that

$$\|\bar{z}^m N(\theta)\|_\phi \leq m!\|N\|^{(m+2)/2}\sum_{k=0}^{m}\frac{\|N\|^{-k/2}}{k!}\|\bar{z}^k\theta\|_\phi \tag{3.12}$$

and that

$$\|\bar{z}^m T^*N(\theta)\|_\phi \leq m!\|N\|^{(m+1)/2}\sum_{k=0}^{m}\frac{\|N\|^{-k/2}}{k!}\|\bar{z}^k\theta\|_\phi. \tag{3.13}$$

We denote by H_m the space of those $f \in L^2_{(0,1)}(\phi)$ such that $\bar{z}^k f \in L^2_{(0,1)}(\phi)$ for $k = 0, 1, \ldots, m$. The approximation result needed to prove (3.12) and (3.13) for all $\theta \in H_m$ is proved in §4. We remark that (3.13) is the critical estimate used in §6 to prove estimates for the reproducing kernels in weighted Hilbert spaces of entire functions.

4. Estimates for the Neumann operator in a special case. We continue to assume that $n = 1$, $\psi = 0$, and $p = q = 0$ in this section. We now prove the approximation lemma needed to prove (3.12) and (3.13) for $\theta \in H_m$.

LEMMA. *Suppose that $\theta \in H_l$ and that $\bar{z}^{j-1}N(\theta) \in D_{T^*}$ for $j = 1, 2, \ldots, l$. Then there exists a sequence f_p, $p = 1, 2, \ldots$, of elements of $\mathcal{D}_{(0,1)}$ such that*

(a) $\bar{z}^{j-1}f_p \to \bar{z}^{j-1}N(\theta)$ *in* $L^2_{(0,1)}(\phi)$ *for* $j = 1, 2, \ldots, l$, *and*
(b) $\bar{z}^{j}TT^*(f_p) \to \bar{z}^{j}\theta$ *in* $L^2_{(0,1)}(\phi)$ *for* $j = 0, 1, \ldots, l$.

PROOF. The proof proceeds in two steps. First compactly supported $\tilde{f}_p$ are found satisfying (a) and (b). Then the sequence f_p is defined by mollifying the forms $\tilde{f}_p$.

Let $f = N(\theta)$ and choose a sequence η_p, $p = 1, 2, \ldots$, of elements of $\mathcal{D}$ such that

$$(4.1)\quad \begin{cases} (1) & 0 \leqslant \eta_p \leqslant 1, \\ (2) & \eta_p = 1 \text{ on the disc of radius } p/2, \\ (3) & \text{there exists a constant } s \text{ so that} \\ & |\partial\eta_p/\partial z|^2 + |\partial\eta_p/\partial\bar{z}|^2 + |\partial^2\eta_p/\partial z\partial\bar{z}|^2 \leqslant s(1 + |z|^2)^{-1}. \end{cases}$$

Put $\tilde{f}_p = \eta_p f$ and notice that $\tilde{f}_p$ has compact support and is in D_{TT^*} for each p.

Because $|\bar{z}^{j-1}\tilde{f}_p - \bar{z}^{j-1}N(\theta)|^2 \leqslant |\bar{z}^{j-1}N(\theta)|^2e^{-\phi}$, (a) follows by the dominated convergence theorem.

Further, it follows from (3.4) and (4.1)(1) that

$$|TT^*(\tilde{f}_p) - \theta|^2e^{-\phi} = |TT^*(\eta_p f) - TT^*(f)|^2e^{-\phi}$$

$$\leqslant 2|[TT^*, \eta_p]f|^2e^{-\phi} + 2|(\eta_p - 1)TT^*(f)|^2e^{-\phi}$$

$$\leqslant 6\left|\frac{\partial\eta_p}{\partial\bar{z}}T^*(f)\right|^2e^{-\phi} + 6\left|\frac{\partial^2\eta_p}{\partial z\partial\bar{z}}f\right|^2e^{-\phi} + 6\left|\frac{\partial\eta_p}{\partial z}\frac{\partial f}{\partial\bar{z}}\right|^2e^{-\phi} + 2|TT^*(f)|^2e^{-\phi}.$$

From this inequality and (4.1)(3) we conclude that

$$(4.2)\quad \begin{aligned} |TT^*(\tilde{f}_p) - \theta|^2e^{-\phi} &\leqslant 6s|T^*N(\theta)|^2e^{-\phi} + 6s|N(\theta)|^2e^{-\phi} \\ &\quad + 6s|\partial f/\partial\bar{z}|^2e^{-\phi} + 2|\theta|^2e^{-\phi} \end{aligned}$$

and that for $j = 1, 2, \ldots, l$,

$$(4.3)\quad \begin{aligned} |\bar{z}^jTT^*(\tilde{f}_p) - \bar{z}^j\theta|^2e^{-\phi} &\leqslant 6s|T^*(\bar{z}^{j-1}N(\theta))|^2e^{-\phi} + 6s|\bar{z}^{j-1}N(\theta)|^2e^{-\phi} \\ &\quad + 12s|(\partial/\partial\bar{z})(\bar{z}^{j-1}f)|^2e^{-\phi} \\ &\quad + 12s(j-1)^2|\bar{z}^{j-2}N(\theta)|^2e^{-\phi} + 2|\bar{z}^j\theta|^2e^{-\phi}. \end{aligned}$$

By (2.2) and the hypotheses of this lemma, the right-hand sides of (4.2) and (4.3) are integrable and (b) follows by the dominated convergence theorem.

It now suffices to prove (a) and (b) when $f \in D_{TT^*}$ has compact support. Choose $\chi \in \mathcal{D}$ with support in the unit disk and $\int \chi \, d\lambda = 1$. Writing $\chi_p(z) = p^2\chi(pz)$, put $f_p(\zeta) = \int f(\zeta - z)\chi_p(z)\, d\lambda$ and $f_p = f_p\, d\bar{z}$. Then $f_p \in \mathcal{D}_{(0,1)}$ and straightforward arguments show that (a) and (b) are satisfied.

We can now show that the Neumann operator maps H_m into H_m.

THEOREM 1. *Suppose $\theta \in H_m$. Then*

(a) $$\bar{z}^k N(\theta) \in D_{TT^*},$$

(b) $$\|\bar{z}^k N(\theta)\|_\phi \leq k!\, \|N\|^{(k+2)/2} \sum_{j=0}^{k} \frac{\|N\|^{-j/2}}{j!} \|\bar{z}^j\theta\|_\phi,$$

(c) $$\|\bar{z}^k T^* N(\theta)\|_\phi \leq k!\, \|N\|^{(k+1)/2} \sum_{j=0}^{k} \frac{\|N\|^{-j/2}}{j!} \|\bar{z}^j\theta\|_\phi$$

for $k = 0, 1, 2, \ldots, m$.

PROOF. The proof proceeds by induction on k. When $k = 0$, there is nothing to prove. Suppose that (a), (b) and (c) are true for nonnegative integers k, strictly less than l, where $l \leq m$.

By the preceding lemma, there exists a sequence, f_p, of elements of $\mathcal{D}_{(0,1)}$ such that

(a) $\bar{z}^{j-1} f_p \to \bar{z}^{j-1} N(\theta)$ in $L^2_{(0,1)}(\phi)$ for $j = 1, 2, \ldots, l$, and

(b) $\bar{z}^j TT^*(f_p) \to \bar{z}^j\theta$ in $L^2_{(0,1)}(\phi)$ for $j = 0, 1, \ldots, l$.

By (3.11), we have

$$\|TT^*(\bar{z}^l f_p) - TT^*(\bar{z}^l f_q)\|_\phi \leq l!\, \|N\|^{l/2} \sum_{j=0}^{l} \frac{\|N\|^{-j/2}}{j!} \|\bar{z}^j TT^*(f_p) - \bar{z}^j TT^*(f_q)\|_\phi$$

so that $TT^*(\bar{z}^l f_p)$ is a convergent sequence in $L^2_{(0,1)}(\phi)$. Now $\bar{z}^l f_p = NTT^*(\bar{z}^l f_p)$ and N is continuous, so that $\bar{z}^l f_p$ is also convergent in $L^2_{(0,1)}(\phi)$. Since $\bar{z}^l f_p \to \bar{z}^l f$ in the sense of distributions we conclude that $\bar{z}^l f \in L^2_{(0,1)}(\phi)$. Moreover, TT^* is a closed operator and thus $\bar{z}^l f \in D_{TT^*}$. This proves (a) when $k = l$. By passing to the limit in (3.12) and (3.13), we see that (b) and (c) are true for $k = l$. This completes the proof.

As an application of Theorem 1, we obtain a result similar to the strong a priori estimates (3.9) and (3.10).

THEOREM 2. *Suppose that $\alpha < \sqrt{2c} < \beta$ and $\theta \in L^2_{(0,1)}(\phi - \beta|z|)$. Then $N(\theta) \in L^2_{(0,1)}(\phi - \alpha|z|)$ and $T^*N(\theta) \in L^2(\phi - \alpha|z|)$.*

PROOF. Notice that for all nonnegative integers k, $|z|^k e^{-\beta|z|/2} \leq k!(\beta/2)^{-k}$. Thus, we have

$$\int |z|^{2k}|\theta|^2 e^{-\phi}\,d\lambda = \int (|z|^k e^{-\beta|z|/2})^2 |\theta|^2 e^{-\phi+\beta|z|}\,d\lambda \leq (k!)^2(\beta/2)^{-2k}\|\theta\|^2_{\phi-\beta|z|}, \tag{4.4}$$

so that $\theta \in H_m$ for all m. By Theorem 1 it follows that $\bar{z}^j N(\theta) \in D_{TT^*}$ for $j = 0, 1, 2, \ldots$, and furthermore that

(a) $$\|\bar{z}^k N(\theta)\|_\phi \leq k!\,\|N\|^{(k+2)/2} \sum_{j=0}^{k} \frac{\|N\|^{-j/2}}{j!}\|\bar{z}^j\theta\|_\phi$$

and

(b) $$\|\bar{z}^k T^* N(\theta)\|_\phi \leq k!\,\|N\|^{(k+1)/2} \sum_{j=0}^{k} \frac{\|N\|^{-j/2}}{j!}\|\bar{z}^j\theta\|_\phi.$$

From these inequalities, (4.4), and the fact that $\|N\| \leq 2/c$, we conclude that

(a) $$\|\bar{z}^k N(\theta)\|_\phi \leq k!\left(\frac{2}{c}\right)^{(k+2)/2} \sum_{j=0}^{k} \left(\frac{\sqrt{2c}}{\beta}\right)^j \|\theta\|_{\phi-\beta|z|} \leq \frac{\beta k!\,(2/c)^{(k+2)/2}\|\theta\|_{\phi-\beta|z|}}{\beta-\sqrt{2c}}$$

and

$$\|\bar{z}^k T^* N(\theta)\|_\phi \leq \frac{\beta k!\,(2/c)^{(k+1)/2}\|\theta\|_{\phi-\beta|z|}}{\beta-\sqrt{2c}}.$$

The proof is completed by observing that these inequalities insure that the power series for $e^{\alpha|z|/2}N(\theta)$ converges in $L^2_{(0,1)}(\phi)$ and the power series for $e^{\alpha|z|/2}T^*N(\theta)$ converges in $L^2(\phi)$.

As an example of Theorem 2, let $\phi(z) = 2x^2$. If $\theta \in \mathcal{D}_{(0,1)}$ then $N(\theta) \in L^2_{(0,1)}(2x^2 - |z|)$ and $T^*N(\theta) \in L^2(2x^2 - |z|)$.

5. The general method. In §§3 and 4 we used an a priori estimate and an approximation result to obtain estimates for the Neumann operator when $n = 1$, $\psi = 0$ and $p = q = 0$. The fact that $\psi = 0$ governed the choice of $G(z) = \bar{z}$ in the a priori estimate (3.8). In this section we derive a generalization of (3.8) and discuss the associated approximation result needed to prove estimates in the absence of these restrictions.

Suppose now that $\theta \in L^2_{(p,q+1)}(\phi_2)$, $N(\theta) \in \mathcal{D}_{(p,q+1)}$, and $G \in C^\infty(C^n)$. A natural generalization of (3.8) is

$$\|GN(\theta)\|_{\phi_2} \leq \|N\|\,\|G\theta\|_{\phi_2} + \|N\|\,\|[TT^* + S^*S, G]N(\theta)\|_{\phi_2}.$$

Unfortunately, we cannot derive the estimates for the commutator $[TT^* + S^*S, G]$ required to obtain a useful a priori estimate.

Observe that (2.1) insures that the operator $SS^* + L^*L$ has a bounded inverse, M, where L is the weak extension of the $\bar{\partial}$ operator defined on $L^2_{(p,q+2)}(\phi_3) \xrightarrow{\bar{\partial}} L^2_{(p,q+3)}(\phi_4)$. Then M is the Neumann operator on $L^2_{(p,q+2)}(\phi_3)$ and $\|M\| \leqslant c/2$.

In order to derive a useful generalization of (3.8), observe that the projection of $GN(\theta)$ onto R_T is $NTT^*(GN(\theta))$ and the projection of $GN(\theta)$ onto $R_T^\perp$ is $S^*MS(GN(\theta))$. We then have

$$GN(\theta) = NTT^*(GN(\theta)) + S^*MS(GN(\theta)).$$

Applying the triangle inequality we obtain

$$\|GN(\theta)\|_{\phi_2} \leqslant \|N\|\,\|TT^*(GN(\theta))\|_{\phi_2} + \|S^*MS(GN(\theta))\|_{\phi_2}.$$

Since $\|S^*MS(GN(\theta))\|_{\phi_2} \leqslant \|M\|^{1/2}\|S(GN(\theta))\|_{\phi_2}$ we have

$$\|GN(\theta)\|_{\phi_2} \leqslant \|N\|\,\|TT^*(GN(\theta))\|_{\phi_2} + \|M\|^{1/2}\|S(GN(\theta))\|_{\phi_2}. \tag{5.1}$$

Assume now that $\theta \in \ker S$. It follows that $N(\theta) \in \ker(S)$. From (5.1) we conclude that

$$\begin{aligned}\|GN(\theta)\|_{\phi_2} \leqslant{}& \|N\|\,\|G\theta\|_{\phi_2}\\ &+ \|N\|\,\|[TT^*, G]N(\theta)\|_{\phi_2} + \|M\|^{1/2}\|[S, G]N(\theta)\|_{\phi_3}.\end{aligned} \tag{5.2}$$

When $n = 1$, (5.2) is identical to (3.8), and (5.2) is the a priori estimate used to prove the general results in §6.

It remains to choose G and H so that the right-hand side of (5.2) is finite when $\theta \in H$, and to prove an approximation result to extend (5.2) to H. Appropriate conditions on G are given in (6.1). Since (5.2) differs from (3.8) only in that the term $\|M\|^{1/2}\|[S, G]N(\theta)\|_{\phi_3}$ is explicitly present in (5.2), only this term presents new difficulties in the general case. Otherwise, the approximation result in the general case is identical to the lemma of §4.

6. General estimates for the Neumann operator. General results are stated without proof in this section. Proofs of the general results differ in that they are technically more complicated than those given in §4 for a special case. However, the main ideas remain the same. Complete proofs are given in [2].

We continue to assume that ϕ and ψ are in $C^2(C^n)$ and satisfy (2.1). In order that the right-hand side of (5.2) be finite, we shall be interested in $G \in C^\infty(C^n)$ such that there exist positive constants c_0 and c_1 so that

$$(6.1)\qquad \begin{aligned} &(1)\quad |\partial G|^2 e^{-\psi} \leqslant c_0^2,\\ &(2)\quad |\bar\partial G|^2 e^{-\psi} \leqslant c_0^2,\\ &(3)\quad |\partial G|^2 |\bar\partial \psi|^2 e^{-2\psi} \leqslant c_0^2,\\ &(4)\quad |\partial\bar\partial G|^2 e^{-2\psi} \leqslant c_0^2,\\ &(5)\quad |G|^2 e^{-\psi} \leqslant c_1^2(1+|z|^2),\\ &(6)\quad |G|^2 |\bar\partial\psi|^2 e^{-2\psi} \leqslant c_1^2(1+|z|^2). \end{aligned}$$

For example, when $n = 1$ and $\psi = 0$ the function $G(z) = \bar z$ satisfies (6.1). More generally, when $\psi = 0$, the function $G(z) = \bar z_k$ for $k = 1, 2, \ldots, n$ satisfies (6.1). Similarly, if $\psi(z) = (a - 1)\log(1 + |z|^2)$, then $G(z) = (1 + |z|^2)^{a/2}$ satisfies (6.1) when $a \geqslant 0$.

We now state a generalization of Theorem 1.

THEOREM 3. *Suppose that G satisfies* (6.1), *that* $\theta \in \ker(S)$ *and that* $G^k\theta \in L^2_{(p,q+1)}(\phi_2)$ *for* $k = 0, 1, 2, \ldots, m$. *Then there exists a positive constant* ϵ, *depending only on c and* c_0, *such that*

(a) $$G^k N(\theta) \in D_{TT^*} \cap D_S,$$

(b) $$\|G^k N(\theta)\|_{\phi_2} \leqslant 2\epsilon^k \|N\| k! \sum_{j=0}^{k} \frac{\epsilon^{-j}}{j!} \|G^j\theta\|_{\phi_2},$$

(c) $$\|G^k T^* N(\theta)\|_{\phi_1} \leqslant (\|N\|^{1/2} + 2)\epsilon^k k! \sum_{j=0}^{k} \frac{\epsilon^{-j}}{j!} \|G^j\theta\|_{\phi_2}.$$

The following analogue of Theorem 2 follows from Theorem 3 in the same way that Theorem 2 follows from Theorem 1. The constant ϵ is the same constant as in Theorem 3.

THEOREM 4. *Suppose that G satisfies* (6.1) *and that* $\alpha < 2/\epsilon < \beta$. *If* $\theta \in \ker(S) \cap L^2_{(p,q+1)}(\phi_2 - \beta|G|)$, *then* $N(\theta) \in L^2_{(p,q+1)}(\phi_2 - \alpha|G|)$ *and* $T^*N(\theta) \in L^2_{(p,q)}(\phi_1 - \alpha|G|)$.

As an example of Theorem 4, suppose that $c > 0$ and $a \geqslant 1$ are constants and that $\phi \in C^2(\mathcal{C}^n)$ satisfies

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2\phi}{\partial z_j \partial \bar z_k} w_j \bar w_k \geqslant (c + 2(a-1)^2)|w|^2$$

for each $w \in \mathcal{C}^n$. Then ϕ and $\psi(z) = (a - 1)\log(1 + |z|^2)$ satisfy (2.1), and $G(z) = (1 + |z|^2)^{a/2}$ satisfies (6.1). We conclude from Theorem 4 that if $\theta \in \ker(S) \cap \mathcal{D}_{(p,q+1)}$ then there is a positive constant α so that

$$N(\theta) \in L^2_{(p,q+1)}(\phi_2 - \alpha|z|^a)$$

and $T^*N(\theta) \in L^2_{(p,q)}(\phi_1 - \alpha|z|^a)$.

In particular, if $\sum_{j=1}^n \sum_{k=1}^n (\partial^2\phi/\partial z_j\partial\bar z_k) w_j \bar w_k \geqslant c|w|^2$ for each $w \in \mathcal{C}^n$ and if $\theta \in \ker(S) \cap \mathcal{D}_{(p,q+1)}$, then $N(\theta) \in L^2_{(p,q+1)}(\phi - \alpha|z|)$ and $T^*N(\theta) \in$

$L^2_{(p,q)}(\phi - \alpha|z|)$. Thus, if $\phi(z) = \Sigma_{j=1}^n x_j^2$ and $\theta \in \mathcal{D}_{(p,q+1)}$ we may conclude that

$$N(\theta) \in L^2_{(p,q+1)}\left(\sum_{j=1}^n x_j^2 - \alpha|z|\right) \quad \text{and} \quad T^*N(\theta) \in L^2_{(p,q)}\left(\sum_{j=1}^n x_j^2 - \alpha|z|\right).$$

7. **Estimates for reproducing kernels.** Suppose that ω is measurable with respect to the Lebesgue measure in C^n and is bounded from above on compact subsets of C^n. Then the space of analytic functions in $L^2(\omega)$ is a closed subspace of $L^2(\omega)$ denoted $A^2(\omega)$. To each ζ in C^n there corresponds an element K_ζ of $A^2(\omega)$, with the property that $f(\zeta) = (f, K_\zeta)_\omega$ for all $f \in A^2(\omega)$. This analytic function K_ζ is called the *reproducing kernel at* ζ of $A^2(\omega)$. (For the general theory of reproducing kernels see [1].)

For example, the reproducing kernel at ζ in $A^2(|z|^2)$ is $K_\zeta(z) = \pi^{-n}e^{z\cdot\bar{\zeta}}$. Notice that for each $\epsilon < 1$, the reproducing kernels of $A^2(|z|^2)$ belong to $A^2(|z|^2 - \epsilon|z|^2)$. In this section we apply the results derived above to prove similar estimates in $A^2(\phi_1)$.

We begin by stating a representation theorem for reproducing kernels. This theorem is a based on a representation result used by N. Kerzman in [6] to prove that the Bergman kernel function in a strongly pseudoconvex domain is smooth up to the boundary.

THEOREM 5. *Suppose that ϕ and ψ satisfy* (2.1) *and K_ζ is the reproducing kernel at ζ in $A^2(\phi_1)$. Then there exists $g_\zeta \in C^1(C^n)$ with compact support such that $K_\zeta = g_\zeta - T^*NT(g_\zeta)$.*

We can use this representation theorem to derive estimates for reproducing kernels from those estimates proved above.

THEOREM 6. *Suppose that $\phi \in C^2(C^n)$ and that there is a positive constant c so that $\partial^2\phi/\partial z\partial\bar{z} \geqslant c$. If $\alpha < \sqrt{2c}$, then the reproducing kernels of $A^2(\phi)$ belong to $A^2(\phi - \alpha|z|)$.*

PROOF. We observe from Theorem 5 that $K_\zeta = g_\zeta - T^*NT(g_\zeta)$ where $T(g_\zeta)$ is continuous and has compact support. Since ϕ is continuous, we have

$$T(g_\zeta) \in L^2_{(0,1)}(\phi - \beta|z|)$$

for all β. We conclude from Theorem 2 that $T^*NT(g_\zeta) \in L^2(\phi - \alpha|z|)$. The proof is completed by observing that $g_\zeta \in L^2(\phi_1 - \alpha|z|)$, since g_ζ is continuous and compactly supported.

Notice that Theorem 6 guarantees that the reproducing kernels of $A^2(|z|^2)$ belong to $A^2(|z|^2 - \alpha|z|)$ for $\alpha < \sqrt{2}$ but it does not give the sharp result stated above.

THEOREM 7. *Suppose that $\phi \in C^2(C^n)$, that ϕ and ψ satisfy* (2.1) *and* *that G satisfies* (6.1). *Then there is a positive constant* α, *depending only on c and* c_0, *so that the reproducing kernels of* $A^2(\phi_1)$ *belong to* $A^2(\phi_1 - \alpha|G|)$.

PROOF. Observe that $R_T \subset \ker(S)$ and use Theorem 4 in place of Theorem 2 in the proof of Theorem 6.

It is of interest to note that when $\psi = 0$, the assumption that $\phi \in C^2(\mathbf{C}^n)$ can be eliminated. We omit the proof and refer the reader to [2, Theorem 6.7] for details.

THEOREM 8. *Suppose that* ϕ *is a plurisubharmonic function on* $\mathbf{C}^n$ *which is bounded from below on compact sets and that the distribution*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k - c|w|^2 \, d\lambda$$

is nonnegative for all $w \in \mathbf{C}^n$. *Suppose also that* $\alpha < 2\sqrt{c}/n(\sqrt{c} + \sqrt{2})$. *Then the reproducing kernels of* $A^2(\phi)$ *belong to* $A^2(\phi - \alpha|z|)$.

8. **Conclusion.** We have proved estimates for the Neumann operator on $L^2_{(p,q+1)}(\phi - \psi)$ when ϕ and ψ satisfy (2.1). These estimates have been applied to prove new estimates for the reproducing kernels in $A^2(\phi - 2\psi)$.

Finally, we remark that the estimates of this paper are not sharp in any of the spaces in which we can calculate the Neumann operator explicitly. Such spaces are extremely regular, however (e.g. $\phi \in C^\infty(\mathbf{C}^n)$ and $\phi(z) = \phi(|z|)$), and it is possible that they do not exhibit the extreme behavior of the general case.

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IBM T. J. WATSON RESEARCH CENTER, YORKTOWN HEIGHTS, NEW YORK 10598