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THE MULTIPLICATIVE BEHAVIOR OF $\mathcal{H}$

BY

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ABSTRACT. Various results are given describing the product of two $\mathcal{H}$-classes in an arbitrary semigroup in terms of groups and homomorphisms.

1. It is known that in an arbitrary semigroup S the relation $\mathcal{H}$ of Green need not be a congruence: the product HK of two $\mathcal{H}$-classes H and K is usually spread out between several $\mathcal{H}$-classes. Our main results give a fairly precise description of the set HK and the multiplication $H \times K \longrightarrow HK$.

This is achieved by the first four of our five main theorems. Let $(R_i)_{i\in I}$, $(L_\lambda)_{\lambda\in\Lambda}$ be the families of all $\mathcal{R}$- and $\mathcal{L}$-classes which intersect HK. The first main theorem gives a partition of H into sets $(B_i)_{i\in I}$ (and a partition of K into sets $(Y_\lambda)_{\lambda\in\Lambda}$) indexed so that a product ax $(a \in H, x \in K)$ lies in R_i (in L_λ) if and only if $a \in B_i$ $(x \in Y_\lambda)$; and these are the orbits in H (in K) under the action of a common subgroup G_1^* of the right Schützenberger group $G^*(H)$ of H (a common subgroup G_1 of $G(K)$). In particular each $H_{i\lambda} = R_i \cap L_\lambda$ intersects AX, with $AX \cap H_{i\lambda} = B_iY_\lambda$.

The second main theorem goes further in describing B_iY_λ and the multiplication $B_i \times Y_\lambda \longrightarrow B_iY_\lambda$, by giving finer partitions $(C_j)_{j\in J}$, $(Z_\mu)_{\mu\in M}$ of H, K with the following properties: the sets C_jZ_μ with $C_j \subseteq B_i$, $Z_\mu \subseteq Y_\lambda$ form a partition of B_iY_λ; $C_jx = C_jZ_\mu = aZ_\mu$ whenever $a \in C_j$, $x \in Z_\mu$; the sets C_j are the orbits in H under the action of a common subgroup $G_2^* \subseteq G_1^*$ of $G^*(H)$ (which does not depend on i, λ), and dually for the sets Z_μ.

The third main theorem investigates the relationship between the groups $G_2^* \subseteq G_1^* \subseteq G^*(H)$, $G_2 \subseteq G_1 \subseteq G(K)$ and the common Schützenberger group G of all $H_{i\lambda}$ (as obtained in [8]), by providing homomorphisms Γ: $G_1^* \longrightarrow G$, Δ: $G_1 \longrightarrow G$ such that $\operatorname{Ker}\Gamma \subseteq G_2^*$, $\operatorname{Ker}\Delta \subseteq G_2$ and $\Gamma G_2^* = \Delta G_2 = \Gamma G_1^* \cap \Delta G_1$. This is in turn used in the fourth main theorem to show that, up to bijections, the multiplication of B_i and Y_λ into B_iY_λ can be described by the multiplication of G_1^* and G_1 into (ΓG_1^*) (ΔG_1), and to describe the equivalence relation induced by $H \times K \longrightarrow HK$ as follows: if $a, b \in H$ and $x, y \in K$, so that $b = a \cdot h$,

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$x = g \cdot y$ with $g \in G(K)$, $h \in G^*(H)$, then $ax = by$ if and only if $g \in G_2$, $h \in G_2^*$ [which is clear by the second main theorem] and $\Gamma h = \Delta g$.

These four theorems are supported by a number of other results. In particular, the fifth main theorem states that groups and homomorphisms with all the properties in the third main theorem can be realized from a product of $\mathcal{H}$-classes in some semigroup; this is shown by a rather complicated example, and implies that no additional properties can be obtained for these groups and homomorphisms in general.

2. These results go somewhat further than the previously published studies of multiplicative behavior of $\mathcal{H}$, a subject whose importance for semigroup theory is apparent from the repeated use throughout the theory of the Miller-Clifford theorem, or from the arisal out of one key multiplicative lemma of the Petrich representation [12] and the Schützenberger representation (see [6]). Although the general behavior is significantly more complex, the author feels his results should have been obtained some years ago. The literature contains two attempts in this direction. Of particular significance to the author are the results of Bastida [1], [2], [3], whose reading inspired this paper and its original proofs. These give a fairly detailed study of the product Hx of an $\mathcal{H}$-class by one element; [1] studies one intersection $Hx \cap K$ (with K an $\mathcal{H}$-class), its preimage in H, and the left Schützenberger groups of these two sets; [2] shows how these depend on K and contains part of our first two main theorems; [3] has examples and a small part of our last main theorem. The concept of a coset is implicit in [1], but its two-sidedness is necessary to go further. The sets considered by Bastida are also used in our proofs, in particular for the last two main theorems. The other paper of general interest is [5], where Butler uses yet another method to study Hx (essentially, the third decomposition in §3 below); the editor's note appended to [5] is a clear forerunner of the methods used here. Otherwise, good behavior for $\mathcal{H}$ was noted in full transformation semigroups by Kim [10], but Butler [4] showed it does not happen in general (so, of course, does our last main theorem).

3. The paper is organized as follows. The necessary basic concepts for what follows are given in §1. This includes basic properties of cosets (recalled from [7]); some general facts about coset decompositions, including a remark due to Professor Clifford which led to considerable simplification in the proofs of the main theorems; and a general treatment of the Bastida sets and homomorphism, completed by a right-sided analogue of the Bastida homomorphism, which later gives rise to the maps Γ, Δ in the third main theorem. §2 then proves the first two main theorems, which are fairly straightforward, while the next two are proved in §3, together with a weaker version which does not depend in [8] and is of some interest in itself. The sets which appear in these theorems occur naturally as left cosets (as defined in [7]), and it is of great importance in this ap-

proach that they also have right Schützenberger groups.

The rest of the paper supports the main theory with a number of other results. It would seem more natural at the start to use $G(H)$, $G^*(K)$ rather than $G^*(H)$, $G(K)$; this approach is used in §4, but is necessarily of limited interest since the main theorems cannot be easily expressed in terms of the left action of the Schützenberger group of H and dually for K. The fifth main theorem is proved in §5 by the construction of an example which consists of two maximal subgroups that multiply into a rectangular group, plus whatever additional elements are necessary to obtain a semigroup. The last section deals with the finite case: Lagrange's theorem implies that in this case there are a number of divisibility relationships between the orders of the various sets which appear in the main theorems (already a concern in [5]); this raises the fascinating possibility that the same arguments from elementary number theory, that are so useful in finite group theory, can be used for semigroups too, provided one is willing to disregard the order of S, which is not of high significance in general (but see e.g. [13]), and instead begin with properties of $\mathcal{H}$-classes, which in many ways have a much greater similarity with the group case.

4. The notation is generally as in [6], to which the reader is referred for all unexplained terminology, except that we have left Schützenberger groups act on the left rather than on the right. All families under consideration are implicitly assumed to be faithfully indexed (i.e. when we consider $(P_\alpha)_{\alpha\in A}$ then $P_\alpha = P_\beta$ implies $\alpha = \beta$). The two $\mathcal{H}$-classes denoted by H and K in this introduction are thereafter called A and X.

The results of this paper have been announced under the same title in a short paper (Semigroup Forum 8 (1974), 74–81).

1. Cosets and Schützenberger groups.

1. A *coset* of a semigroup S is a nonempty subset C of S which is contained in a single $\mathcal{H}$-class and satisfies:

(C) $u \in S^1$, $uC \cap C \neq \varnothing$ implies $uC \subseteq C$.

It is shown in [7] that (C) is then equivalent to its dual:

(C*) $u \in S^1$, $Cu \cap C \neq \varnothing$ implies $Cu \subseteq C$,

so that the definition is self-dual. (More general one-sided definitions are given in [7].) Green's Lemma easily implies that every $\mathcal{H}$-class H of S is a coset; if conversely H is a maximal subgroup of S (for instance, if $S = H$ is a group), the cosets of S contained in H are precisely the cosets of the group H in the usual sense, whence the terminology.

Any coset C of S gives rise to a simply transitive group $G(C)$ of bijections of C, which can be defined in two equivalent ways as follows. First one may consider the submonoid $T(C) = \{u \in S^1; uC \subseteq C\}$ of S^1, noting that, by (C),

$u \in T(C) \Longleftrightarrow uC \cap C \neq \varnothing$; this acts on the set C by left multiplication, with $u, v \in T(C)$ equiacting if and only if $ua = va$ for all $a \in C$, equivalently (since all elements of C are $\mathcal{R}$ equivalent) if and only if $ua = va$ for one $a \in C$. We write this relation as $u \equiv v$; it is a congruence on $T(C)$. In what follows it is convenient to avoid selecting one particular group $G(C)$, and hence we define $G(C)$ as any group together with a surjective homomorphism π: $T(C) \longrightarrow G(C)$ which induces on $T(C)$ precisely the congruence $\equiv$ [that $G(C)$ is a group actually follows from this condition]; then a left action of $G(C)$ on C is well defined by: $\pi(u) \cdot c = uc$ for all $u \in T(C)$, $c \in C$; and it is a simply transitive action. This defines $G(C)$ up to action-preserving isomorphism. [Some parts of what follows would become hopelessly confusing if we followed the very elegant presentation in [9], namely $G(C) = T(C)/\equiv$; defining $G(C)$ as a group of bijections of C, as in [6], is not particularly advantageous.]

When applied to an $\mathcal{H}$-class H, the above evidently yields the left Schützenberger group of H; hence we call $G(C)$ the *left Schützenberger group* of C. If H is the $\mathcal{H}$-class containing C, then $G(C)$ can be recovered from $G(H)$ through the action-preserving injection $G(C) \longrightarrow G(H)$ induced by the inclusion $T(C) \longrightarrow T(H)$; or, one may always select for $G(C)$ the subgroup $\{g \in G(H); g \cdot C \subseteq C\} = \{g \in G(H); g \cdot C \cap C \neq \varnothing\}$ of $G(H)$. This is shown in [7]; the two definitions are also given and compared in [11].

This alternate definition of $G(C)$ yields an equivalent definition of a coset, as a nonempty subset C of an $\mathcal{H}$-class H such that:

(C′) $C = K \cdot a$ for some $a \in H$ and some subgroup K of $G(H)$. In this condition, a can be any element of C, but K is unique; namely $K = G(C)$ $[\subseteq G(H)] = \{g \in G(H); g \cdot C \subseteq C\}$.

Dually, every coset C gives rise to a *right Schützenberger group* $G^*(C)$; this is any group together with a surjective homomorphism π^* of $T^*(C) = \{u \in S^1; Cu \subseteq C\}$ to $G^*(C)$ which induces on $T^*(C)$ the congruence $\equiv$: $u \equiv v \Longleftrightarrow au = av$ for all $a \in C$. A simply transitive right action of $G^*(C)$ on C is then well defined by: $c \cdot \pi^*(u) = cu$ for all $c \in C$ and $u \in T^*(C)$. The actions of $G(C)$ and $G^*(C)$ commute, i.e. $g \cdot (c \cdot g') = (g \cdot c) \cdot g'$ for all $c \in C$, $g \in G(C)$, $g' \in G^*(C)$; hence $g \cdot c \cdot g'$ may be written without parentheses.

This in turn implies that the groups $G(C)$, $G^*(C)$ are always isomorphic: an isomorphism ϵ_c: $G(C) \longrightarrow G^*(C)$, $g \longmapsto g^c$ is defined for each $c \in C$ by: $g \cdot c = c \cdot g^c$. [Had the two groups been made to act on the same side, as in [6], this would be an anti-isomorphism.] Unlike the other isomorphisms discussed below, ϵ_c is not canonical and in general depends on the choice of c.

2. A *left coset decomposition* of an $\mathcal{H}$-class H is a partition of H into cosets which all have the same left Schützenberger group in $G(H)$; equivalently, it is a partition of H into all subsets of the form $K \cdot a$ $(a \in H)$, with K a given

subgroup of $G(H)$ [= the partition of H into orbits under the action of some $K < G(H)$]. *Right coset decompositions* of H are defined dually. Although in each case H is partitioned into cosets the case of a group shows that the two concepts are not equivalent; the reader will easily verify that a left coset decomposition of H is also a right coset decomposition if and only if the corresponding subgroup K of $G(H)$ is a normal subgroup of $G(H)$.

Left (or right) coset decompositions are a basic tool in what follows, and so is the following characterization, observed by Professor Clifford (see introduction):

PROPOSITION 1.1 (CLIFFORD). *Let $\sim$ be an equivalence relation on an $\mathcal{H}$-class H. Then $\sim$ partitions H to a left coset decomposition if and only if $a \sim b$ implies $au \sim bu$ for all $u \in T^*(H)$.*

PROOF. First assume that the $\sim$-classes do form a left coset decomposition of H, so that there is a subgroup K of $G(H)$ such that the $\sim$-class of $a \in H$ is $K \cdot a$ for all a. If $a \sim b$, then $b \in K \cdot a$, hence $b \cdot g \in K \cdot a \cdot g$ for all $g \in G^*(H)$; with $g = \pi^*(u)$ this yields $au \sim bu$.

Conversely, assume that $a \sim b$ implies $au \sim bu$ for all $u \in T^*(H)$. Let C be a class modulo $\sim$. If $u \in S^1$ and $Cu \cap C \neq \varnothing$, then $au \in C$ for some $a \in C$, and also $u \in T^*(H)$; for any $c \in C$, $c \sim a$ then implies $cu \sim au \in C$ and $cu \in C$; thus, $Cu \subseteq C$. Thus every $\sim$-class satisfies (C*) and thus is a coset. Now let C, C' be any two $\sim$-classes. By (C′) we can write $C = K \cdot a$, $C' = K' \cdot b$ for some $a, b \in H$ (actually, $a \in C$, $b \in C'$) and subgroups K, K' of $G(H)$. If $k \in K$, then $k \cdot a \sim a$; by the hypothesis, this implies $k \cdot a \cdot g \sim a \cdot g$ for all $g \in G^*(H)$ and hence $k \cdot c \sim c$ for all $c \in C'$; thus $k \cdot C' \subseteq C'$ and therefore $k \in K'$. Thus $K \subseteq K'$ and by symmetry $K = K'$. □

3. We now investigate homomorphisms that naturally arise between Schützenberger groups of different cosets.

First let C, D be cosets and assume $Cr \subseteq D$ for some $r \in S^1$. It is easy to show (see [7]) that this implies $T(C) \subseteq T(D)$, and that this inclusion induces a homomorphism $\varphi = \varphi_D^C$: $G(C) \longrightarrow G(D)$. Thus $\varphi(\pi_C(u)) = \pi_D(u)$ for all $u \in T(C) \subseteq T(D)$; φ depends only on C, D (and not on r). We call φ an *associativity homomorphism* because of its further property $\varphi g \cdot cr = (g \cdot c)r$ whenever $g \in G(C)$, $c \in C$, $Cr \subseteq D$.

These associativity homomorphisms compose: φ_C^C is the identity on $G(C)$, and $\varphi_E^D \varphi_D^C = \varphi_E^C$ whenever $Cr \subseteq D$, $Ds \subseteq E$ for some $r, s \in S^1$. In particular, if C, D are cosets and $Cr \subseteq D$, $Ds \subseteq C$ for some $r, s \in S^1$, then φ_D^C and φ_C^D are mutually inverse isomorphisms (so that $G(C) \cong G(D)$). For instance, this is often used to identify the left Schützenberger groups of two $\mathcal{R}$-equivalent $\mathcal{H}$-classes; it also has an indirect bearing on 1.1 since the inclusion $K \subseteq K'$ in that proof is

readily seen to be an associativity homomorphism.

Dually, if C, D are cosets and $rC \subseteq D$ for some $r \in S^1$, then $T^*(C) \subseteq T^*(D)$ and there is a canonical homomorphism $\psi = \psi_D^C\colon G^*(C) \longrightarrow G^*(D)$ such that $\psi(\pi_C^*(u)) = \pi_D^*(u)$ for all $u \in T^*(C) \subseteq T^*(D)$ and $rc \cdot \psi g = r(c \cdot g)$ whenever $g \in G^*(C)$, $c \in C$, $Cr \subseteq D$. These homomorphisms also compose. Hence if C, D are cosets such that $rC \subseteq D$, $sD \subseteq C$ for some $r, s \in S^1$, then $G^*(C) \cong G^*(D)$ canonically (via ψ_D^C, ψ_C^D).

The remaining definitions and results are taken from [1] (except for the immediate generalization to cosets). Let $r \in S^1$ and C, D be cosets such that $Cr \cap D \neq \emptyset$. The *Bastida sets* of C, D, r are:

$$A = A_{C,r;D} = D \cdot\!\cdot_C r = \{c \in C;\, cr \in D\}$$

$$B = B_{C,r;D} = Cr \cap D \quad (= Ar).$$

It is immediate that $Ar = B$ and that $A, B \neq \emptyset$. The following result is not stated in that form in [1], but its proof is a basic argument in [1] used in constructing the groups $G(A)$, $G(B)$.

PROPOSITION 1.2. *Let C, D be cosets and $r \in S^1$ be such that $Cr \cap D \neq \emptyset$. Then $A_{C,r;D}$ and $B_{C,r;D}$ are cosets.*

PROOF. It suffices to show that $A = A_{C,r;D} \subseteq C$ and $B = B_{C,r;D} \subseteq D$ satisfy (C). Assume $uA \cap A \neq \emptyset$, say $ua \in A$ for some $a \in A$. Then $a, ua \in C$, $ar, uar \in D$, whence $u \in T(C)$, $u \in T(D)$. For any $a' \in A$, we have $a' \in C$, $a'r \in D$, so that $ua' \in C$, $ua'r \in D$ and hence $ua' \in A$. Therefore $uA \subseteq A$.

Assume $uB \cap B \neq \emptyset$, say $ub \in B$ for some $b \in B$. Then $b, ub \in D$ and there exist $a, a' \in A$ such that $b = ar$, $ub = a'r$. Since A is a coset, there exists $v \in T(A)$ with $a' = va$; as in the first part of the proof, we also have $v \in T(C)$, $v \in T(D)$; furthermore $vb = var = a'r = ub$. Now take any $b' \in B$. Since D is a coset and $b, b' \in D$, we have $b' = bt$ for some $t \in S^1$ and hence $vb = ub$ implies $vb' = ub'$. But $vb' \in vCr \subseteq Cr$ and $vb' \in vD \subseteq D$ since $v \in T(C)$, $T(D)$. Therefore $ub' = vb' \in B$. Thus $uB \subseteq B$. □ Incidentally we have shown that for each $u \in T(B)$ there is some $v \in T(A)$ such that $vb = ub$ for all $b \in B$.

The *Bastida homomorphism* $\gamma = \gamma_{C,r;D}\colon G(A) \longrightarrow G(B)$ is simply the associativity homomorphism φ_B^A whose existence is insured by $Ar = B$. The remark which follows the proof of 1.2 shows that:

PROPOSITION 1.3. *$\gamma_{C,r;D}$ is surjective whenever defined.* □

In what follows we will also need the groups $G^*(A)$, $G^*(B)$ and a right-sided analogue of γ. We note that in the proof of 1.2, which is essentially the original proof in [1], we established (C) for A and B; we then know from [7] that (C*) also holds, but there seems to be no immediate way to prove it directly.

This undoubtedly explains why $G^*(A)$, $G^*(B)$ are not considered in [1] (and our main results were not obtained there).

Compositions of γ and ϵ's yield many surjective homomorphisms $G^*(A) \cong G(A) \longrightarrow G(B) \cong G^*(B)$. That one of these is canonical is rather more surprising:

PROPOSITION 1.4. *Let C, D, r, A, B, γ be as above. There is a canonical surjective homomorphism γ^*: $G^*(A) \longrightarrow G^*(B)$ such that $ar \cdot \gamma^* h = \gamma(h^a) \cdot ar = (a \cdot h)r$ for all $a \in A$, $h \in G^*(A)$.*

PROOF. First note that $(a \cdot h)r = (h^a \cdot a)r = \gamma(h^a) \cdot ar = ar \cdot (\gamma(h^a))^{ar}$ for all $a \in A$, $h \in G^*(A)$. We show that $(\gamma(h^a))^{ar}$ does not depend on a. For any $c \in A$, we have $c = g \cdot a$ for some $g \in G(A)$ and

$$\begin{aligned} cr \cdot (\gamma(h^c))^{cr} &= (c \cdot h)r = (g \cdot a \cdot h)r = \gamma g \cdot (a \cdot h)r \\ &= \gamma g \cdot ar \cdot (\gamma(h^a))^{ar} = (g \cdot a)r \cdot (\gamma(h^a))^{ar} = cr \cdot (\gamma(h^a))^{ar}, \end{aligned}$$

which implies $(\gamma(h^c))^{cr} = (\gamma(h^a))^{ar}$. We define a mapping γ^*: $G^*(A) \longrightarrow G^*(B)$ by $\gamma^* h = (\gamma(h^a))^{ar}$; this satisfies the last condition in the statement. Since the definition is independent on a, we can for all g, $h \in G^*(A)$ calculate

$$ar \cdot \gamma^*(gh) = (a \cdot gh)r = ((a \cdot g) \cdot h)r = (a \cdot g)r \cdot \gamma^* h = ar \cdot \gamma^* g \cdot \gamma^* h,$$

which implies $\gamma^*(gh) = (\gamma^* g)(\gamma^* h)$, i.e. γ^* is a homomorphism. That γ^* is surjective follows from 1.3 and $ar \cdot \gamma^* h = \gamma(h^a) \cdot ar$. □

The map γ^* is basically one of the commutation mappings ω in [7]. In general these mappings fail to be canonical and may also fail to be homomorphisms.

The dual Bastida sets and homomorphisms arise from cosets C, D and $r \in S^1$ such that $rC \cap D \neq \emptyset$. They will be denoted by $A^*_{r,C;D} = \{c \in C; rc \in D\}$, $B^*_{r,C;D} = rC \cap D = rA^*$, $\delta^*_{r,C;D}$: $G^*(A^*) \longrightarrow G^*(B^*)$ and $\delta_{r,C;D}$: $G(A^*) \longrightarrow G(B^*)$; there δ^* is the associativity homomorphism and δ the commutation map ($\delta g \cdot ra = r(g \cdot a)$ for all $a \in A^*$, $g \in G(A^*)$).

2. The first two main theorems.

1. In this and the following sections A and X denote two given $\mathcal{H}$-classes of a semigroup S. We also denote by $(R_i)_{i \in I}$ and $(L_\lambda)_{\lambda \in \Lambda}$ the family of all $\mathcal{R}$- and $\mathcal{L}$-classes, respectively, which intersect the product AX. Since $\mathcal{R}$ is a left congruence and $\mathcal{L}$ a right congruence, AX is contained in a single $\mathcal{D}$-class, and hence $H_{i\lambda} = R_i \cap L_\lambda$ is an $\mathcal{H}$-class for all i, λ. [There may be other $\mathcal{H}$-classes in our $\mathcal{D}$-class, but they cannot intersect AX. On the other hand, the first main theorem will show that every $H_{i\lambda}$ does meet AX.] In this section we study how AX is spread between the various $H_{i\lambda}$ and investigate the sets $AX \cap H_{i\lambda}$.

2. For each $x \in S$ (e.g. $x \in X$) we let $\sim_x$ be the equivalence relation on A defined by: $a \sim_x b \Longleftrightarrow ax \mathcal{R} bx$.

LEMMA 2.1. *When $x \mathcal{R} y$, $\sim_x = \sim_y$. If $x \in X$ and C is a class modulo $\sim_x$, then CX is contained in a single $\mathcal{R}$-class, and this yields a bijection between $\sim_x$-classes and the $\mathcal{R}$-classes which intersect AX.*

PROOF. Assume $y = xr$, $x = ys$ (where $r, s \in S^1$). Since $\mathcal{L}$ is a right congruence, Ax and Ay are contained in single $\mathcal{L}$-classes L, L', respectively. By Green's Lemma, right multiplication by r is an $\mathcal{R}$-class preserving bijection of L onto L'. Hence $a \sim_x b$, i.e. $ax \mathcal{R} bx$ implies $ay = axr \mathcal{R} ax \mathcal{R} bx \mathcal{R} bxr = by$, i.e. $a \sim_y b$; by symmetry, $\sim_x = \sim_y$.

Now assume $x \in X$ and let C be a class modulo $\sim_x$; pick $c \in C$. Since $\mathcal{R}$ is a left congruence, cX is contained in a single $\mathcal{R}$-class R. When $y \in X$, $cy \in R$; since C is also a $\sim_y$-class, we see that $Cy \subseteq R$. This proves $CX \subseteq R$. To each $\sim_x$-class C we may then assign the $\mathcal{R}$-class which contains CX (and hence intersects AX). If two classes C and C' go to the same $\mathcal{R}$-class R, then $a \in C$ and $b \in C'$ are such that $ax \mathcal{R} bx$; therefore $C = C'$. If conversely R is an $\mathcal{R}$-class which meets AX, then it contains some product ax and hence corresponds to the class of a. □

Since $\mathcal{R}$ is a left congruence, it is evident that $a \sim_x b$ implies $ua \sim_x ub$ for all $u \in T(A)$. Hence $\sim_x$ induces on A a right coset decomposition, by 1.1. This decomposition does not depend on the choice of x in X (in fact, 2.1 shows that it depends solely on A and the $\mathcal{R}$-class of X); we call it the *first decomposition* of A. It also follows from 2.1 that the cosets in the first decomposition of A correspond bijectively to the $\mathcal{R}$-classes $(R_i)_{i \in I}$ which meet AX; hence they can be written as a family $(B_i)_{i \in I}$ indexed by I in such a way that $B_i X \subseteq R_i$ for all $i \in I$. Note that in fact $B_i X = AX \cap R_i$ since the R_i's are pairwise disjoint. Because we have a right coset decomposition, the subgroup $G^*(B_i)$ of $G^*(A)$ does not depend on i; it will be hereafter denoted by G_1^*; it depends only on A and the $\mathcal{R}$-class of X.

Dually, there is a left coset decomposition of X, the *first decomposition* of X, into cosets which can be written as a family $(Y_\lambda)_{\lambda \in \Lambda}$ in such a way that $AY_\lambda \subseteq L_\lambda$ for all $\lambda \in \Lambda$ (actually, $AY_\lambda = AX \cap L_\lambda$). The subgroup $G(Y_\lambda)$ of $G(X)$ does not depend on λ; it depends only on X and the $\mathcal{L}$-class of A; it is hereafter denoted by G_1.

We note that $B_i Y_\lambda \subseteq H_{i\lambda}$; actually, $B_i Y_\lambda = AX \cap H_{i\lambda}$ since the $H_{i\lambda}$'s are pairwise disjoint. We state the results obtained so far as

THEOREM 2.2 (FIRST MAIN THEOREM). *There exists a subgroup G_1^* of $G^*(A)$, which depends only on A and the $\mathcal{R}$-class of X, and a subgroup G_1 of $G(X)$, which depends only on X and the $\mathcal{L}$-class of A, such that the orbits under G_1^* (under G_1) correspond bijectively to the $\mathcal{R}$-classes ($\mathcal{L}$-classes) which intersect*

AX. If B_i, Y_λ are the orbits corresponding to R_i, L_λ, then $B_iX = AX \cap R_i$, $AY_\lambda = AX \cap L_\lambda$, $B_iY_\lambda = AX \cap H_{i\lambda}$. □

Some corollaries are:

COROLLARY 2.3. *Every $H_{i\lambda}$ intersects AX.* □

COROLLARY 2.4. *Each R_i intersects every Ax $(x \in X)$, with $Ax \cap R_i = B_ix$; each L_λ intersects every aX $(a \in A)$, with $aX \cap L_\lambda = aY_\lambda$; $B_ix = Ax \cap H_{i\lambda}$, $aY_\lambda = aX \cap H_{i\lambda}$ when $a \in B_i$, $x \in B_i$, $x \in Y_\lambda$.*

PROOF. When $x \in X$, $a \in A$, then $ax \in R_i \iff a \in B_i$, $ax \in L_\lambda \iff x \in Y_\lambda$. □

3. This last corollary suggests that we investigate the sets B_ix, aY_λ more closely, and this will lead us to the second main theorem.

LEMMA 2.5. *The sets B_ix $(x \in Y_\lambda)$ form a partition of B_iY_λ and are part of a left coset decomposition of $H_{i\lambda}$. The sets aY_λ $(a \in B_i)$ form a partition of B_iY_λ and are part of a right coset decomposition of $H_{i\lambda}$. Furthermore each B_ix $(x \in Y_\lambda)$ intersects every aY_λ $(a \in B_i)$.*

PROOF. First note that $B_ix = Ax \cap H_{i\lambda}$ is a Bastida set $B_{A,x;H_{i\lambda}}$ and hence a coset, by 1.2. Now take $x, y \in Y_\lambda$; in particular, $y = xr$ for some $r \in S^1$. When $g = \pi(u) \in G(B_ix) \subseteq G(X)$, i.e. $uB_ix = g \cdot B_ix \subseteq B_ix$, then $uB_iy = uB_ixr \subseteq B_ixr = B_iy$ and hence $g \in G(B_iy)$; thus $G(B_ix) \subseteq G(B_iy)$ and, by symmetry, $G(B_ix) = G(B_iy)$. Therefore the sets B_ix $(x \in Y_\lambda)$ are part of a left coset decomposition of $H_{i\lambda}$; in particular they are pairwise disjoint, i.e. $B_ix \cap B_iy \neq \varnothing$ implies $B_ix = B_iy$ (though not $x = y$), and hence constitute a partition of B_iY_λ. The second part of the statement is dual, and the last part is immediate since $a \in B_i$, $x \in Y_\lambda$ implies $ax \in B_ix \cap aY_\lambda$. □

We now pull back to A and X the partitions of B_iY_λ obtained in 2.5. Define an equivalence relation $\sim$ on A by $a \sim b \iff aX = bX$.

LEMMA 2.6. *For each $\lambda \in \Lambda$, $aX = bX \iff aY_\lambda = bY_\lambda$. Furthermore $\sim$ depends only on A and the $\mathcal{R}$-class of X.*

PROOF. If $aX = bX$ then aX, bX lie in the same $\mathcal{R}$-class and hence a, b belong to the same set B_i; furthermore $aY_\lambda = aX \cap H_{i\lambda} = bX \cap H_{i\lambda} = bY_\lambda$ for all λ. Conversely, assume $aY_\lambda = bY_\lambda$ for some λ. For any $\mu \in \Lambda$, we can pick $x \in Y_\lambda$, $y \in Y_\mu$ and have $y = xr$ for some $r \in S^1$; noting $r \in T^*(X)$, let $g = \pi_X^*(r)$. Then $Y_\mu = G_1 \cdot y = G_1 \cdot x \cdot g = Y_\lambda \cdot g = Y_\lambda r$; therefore $aY_\mu = aY_\lambda r = bY_\lambda r = bY_\mu$. Thus $aY_\mu = bY_\mu$ for all μ and hence $aX = bX$. Finally, $aX = bX$ implies $aXr = bXr$ for all $r \in S^1$; in particular, $aY = bY$ for every $\mathcal{H}$-class Y which is $\mathcal{R}$-equivalent to X. □

It is clear that $a \sim b$ i.e. $aX = bX$, implies $ua \sim ub$ for all $u \in T(A)$. Hence 1.1 implies that $\sim$ induces a right coset decomposition of A. We call it the *second decomposition* of A; by 2.6, it depends only on A and the $\mathcal{R}$-class of X. In view of 2.6 it can also be defined by $aY_\lambda = bY_\lambda$ and is then independent of λ. We just saw that when $aX = bX$ then a, b must lie in the same set B_i; hence the second decomposition is finer than the first decomposition. The cosets in the second decomposition of A will be denoted by $(C_j)_{j\in J}$; each is contained in a single set B_i. The subgroup $G^*(C_j)$ of $G^*(A)$ does not depend on j, since we have a right coset decomposition, and will be denoted by G_2^*. When $C_j \subseteq B_i$, $g \in G_2^*$ implies $C_j \cdot g \subseteq C_j$ and $B_i \cdot g \cap B_i \neq \emptyset$, so that $G_2^* \subseteq G_1^*$.

Dually, the equivalence relation $Ax = Ay$ (equivalently, $B_i x = B_i y$ for any $i \in I$) induces on X a left coset decomposition, the second decomposition of X. The cosets in that decomposition will be denoted by $(Z_\mu)_{\mu\in M}$; each Z_μ is contained in some Y_λ. The common subgroup $G(Z_\mu)$ of $G(X)$ will be denoted by G_2; it depends only on X and the $\mathcal{L}$-class of A; furthermore $G_2 \subseteq G_1$.

We state these results and some additional properties as

THEOREM 2.7 (SECOND MAIN THEOREM). *There exists a subgroup $G_2^* \subseteq G_1^*$ of $G^*(A)$, which depends only on A and the $\mathcal{R}$-class of X, and a subgroup $G_2 \subseteq G_1$ of $G(X)$, which depends only on X and the $\mathcal{L}$-class of A, such that the families $(C_j)_{j\in J}$, $(Z_\mu)_{\mu\in M}$ of orbits under G_2^*, G_2 have the following properties: for each $i \in I$, $\lambda \in \Lambda$, the sets C_jZ_μ with $C_j \subseteq B_i$, $Z_\mu \subseteq Y_\lambda$ form a partition of $B_iY_\lambda = AX \cap H_{i\lambda}$ into cosets; furthermore $C_jZ_\mu = aZ_\mu = C_jx = B_ix \cap aY_\lambda$ whenever $a \in C_j \subseteq B_i$, $x \in Z_\mu \subseteq Y_\lambda$.*

PROOF. It is immediate that the intersection of two cosets is again a coset; hence it follows from 2.5 that the sets $B_ix \cap aY_\lambda$ $(a \in B_i,\ x \in Y_\lambda)$ form a partition of $B_iY_\lambda = AX \cap H_{i\lambda}$ into cosets. Hence it suffices to prove the last part of the statement.

Assume $a \in C_j \subseteq B_i$, $x \in Z_\mu \subseteq Y_\lambda$. We know that $bY_\lambda = aY_\lambda$ for all $b \in C_j$, so that $C_jY_\lambda = aY_\lambda$ and $C_jx \subseteq B_ix \cap aY_\lambda$. Conversely, any $d \in B_ix \cap aY_\lambda$ can be written as $d = bx$ with $b \in B_i$; then $d \in bY_\lambda \cap aY_\lambda$ and, by 2.5, $bY_\lambda = aY_\lambda$; therefore $b \in C_j$ and $d \in C_jx$. Thus $C_jx = B_ix \cap aY_\lambda$. Dually, $aZ_\mu = B_ix \cap aY_\lambda$. If finally $y \in Z_\mu$, then $C_jx = aZ_\mu = C_jy$ (by the above applied to y), and hence $C_jZ_\mu = aZ_\mu$. $\square$

The significance of this theorem is that it locates the contraction which occurs when B_i and Y_λ are multiplied, i.e. the extent to which the multiplication $B_i \times Y_\lambda \longrightarrow B_iY_\lambda$ fails to be injective. Indeed $aZ_\mu = bZ_\mu$ whenever a, b lie in the same C_j; and, since this is in turn equivalent to $aX = bX$, each C_j is evidently maximal with this property.

3. The third and fourth main theorems.

1. The next two main theorems aim at a description of the equivalence relation induced by the multiplication $B_i \times Y_\lambda \longrightarrow B_iY_\lambda$ (equivalently, $A \times X \longrightarrow AX$). This is accomplished by the fourth main theorem, which in this sense refines the second main theorem. The accomplishment of this requires a careful study of the relationship between the groups already introduced and the Schützenberger group of the $\mathcal{D}$-class D that contains AX, the results of which are expressed in the third main theorem.

Two more groups are necessary for this; they arise from yet another decomposition of A and X, which is obtained as follows. Define an equivalence relation $\equiv$ on A by $a \equiv b \Longleftrightarrow ax = bx$ (where $x \in X$); it is clear that $\equiv$ does not depend on the choice of x in X (in fact it does not depend on the choice of x in its $\mathcal{R}$-class). [Note that $\equiv$ provides a lower bound for the contraction of $A \times X$ into AX.] We also see that $a \equiv b$ implies $ua \equiv ub$ for all $u \in T(A)$; hence $\equiv$ induces a right coset decomposition of A, the *third decomposition* of A. This third decomposition depends only on A and the $\mathcal{R}$-class of X; we denote the cosets in the third decomposition of A by $(D_k)_{k\in K}$ and the common subgroup $G^*(D_k)$ of $G^*(A)$ by G_3^*. Since $ax = bx$ implies $aY_\lambda = bY_\lambda$ (where $x \in Y_\lambda$) by 2.5, we see that $G_3^* \subseteq G_2^*$.

Dually, there exists a subgroup $G_3 \subseteq G_2$ of $G(X)$, which depends only on X and the $\mathcal{L}$-class of A, whose corresponding left coset decomposition of X (the *third decomposition* of X) is the family $(W_\nu)_{\nu\in N}$ of classes modulo the equivalence relation $\equiv$ on X defined by $x \equiv y \Longleftrightarrow ax = ay$ (where a is any element of A). Evidently, $a \in D_k$, $x \in W_\nu$ implies $D_kW_\nu = aW_\nu = D_kx = \{ax\}$.

2. We now investigate the relationship between the groups $G_3^* \subseteq G_2^* \subseteq G_1^* \subseteq G^*(A)$, $G_3 \subseteq G_2 \subseteq G_1 \subseteq G(X)$ and the groups $G(H_{i\lambda})$, $G^*(H_{i\lambda})$. This is based on the recognition of B_i, C_j, D_k, B_ix, C_jx and the dual sets as Bastida sets, which yields Bastida homomorphisms. Assume $x \in W_\nu \subseteq Z_\mu \subseteq Y_\lambda$. By the first main theorem, $ax \in H_{i\lambda} \Longleftrightarrow a \in B_i$, so that $B_i = A_{A,x;H_{i\lambda}}$, $B_ix = B_{A,x;H_{i\lambda}}$ (the latter was used in the proof of 2.5). When $a \in B_i$, the second main theorem implies that $a \in C_j \Longleftrightarrow ax \in C_jx = C_jZ_\mu$, so that $C_j = A_{B_i,x;C_jZ_\mu}$, $C_jx = B_{B_i,x;C_jZ_\mu}$, equivalently $C_j = A_{B_i,x;aY_\lambda}$, $C_jx = B_{B_i,x;aY_\lambda}$ whenever $a \in C_j \subseteq B_i$. The similar result for D_k is trivial since D_kx is a one-element coset, but still $D_k = A_{B_i,x;D_kx}$, $D_kx = B_{B_ix;D_kx}$. In each case we obtain a Bastida homomorphism γ with the associativity property $\gamma g \cdot cx = (g \cdot c)x$, and a homomorphism γ^* with the commutation property $cx \cdot \gamma^*h = (c \cdot h)x$; we denote these homomorphisms by $\gamma_{i,x}\colon G(B_i) \longrightarrow G(B_ix)$, $\gamma_{i,x}^*\colon G^*(B_i) \longrightarrow G^*(B_ix)$ and similarly $\gamma_{j,x}$, $\gamma_{j,x}^*$, $\gamma_{k,x}$, $\gamma_{k,x}^*$. The last two maps are trivial; by 1.3, 1.4, they are all surjective.

The following result implies that only the first two maps $\gamma_{i,x}$, $\gamma_{i,x}^*$ need be considered:

LEMMA 3.1. *For each $x \in X$ and $a \in D_k \subseteq C_j \subseteq B_i$, the following diagrams (where the vertical maps are inclusions) are commutative*:

$$\begin{tikzcd}
G(B_i) \arrow[r, "\gamma_{i,x}"] & G(B_i x) \\
G(C_j) \arrow[r, "\gamma_{j,x}"] \arrow[u] & G(C_j x) \arrow[u] \\
G(D_k) \arrow[r, "\gamma_{k,x}"] \arrow[u] & G(D_k x) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1^* = G^*(B_i) \arrow[r, "\gamma^*_{i,x}"] & G^*(B_i x) \\
G_2^* = G^*(C_j) \arrow[r, "\gamma^*_{j,x}"] \arrow[u] & G^*(C_j x) \arrow[u] \\
G_3^* = G^*(D_k) \arrow[r, "\gamma^*_{k,x}"] \arrow[u] & G^*(D_k x) \arrow[u]
\end{tikzcd}$$

PROOF. The inclusion $G(C_j) \subseteq G(B_i)$ arises when we consider $G(C_j)$, $G(B_i)$ as subgroups of $G(A)$: for $g \in G(C_j)$ implies $g \cdot C_j \subseteq C_j$, $g \cdot B_i \cap B_i \neq \emptyset$ and $g \in G(B_i)$. The other inclusions arise in a similar manner.

Consider first the first diagram, and let $c \in C_j$, $g \in G(C_j)$. Then also $c \in B_i$, $g \in G(B_i)$ and $\gamma_{i,x} g \cdot cx = (g \cdot c)x = \gamma_{j,x} g \cdot cx$, which implies $\gamma_{i,x} g = \gamma_{j,x} g$. Commutativity of the bottom square is proved similarly.

If now $c \in C_j$, $h \in G^*(C_j)$ then similarly $cx \cdot \gamma^*_{i,x} h = (c \cdot h)x = cx \cdot \gamma^*_{j,x} h$ and hence $\gamma^*_{i,x} h = \gamma^*_{j,x} h$. The last square is treated similarly. □

Since all γ, γ^* maps are surjective, it follows from the lemma that $\gamma_{i,x}(G(C_j)) = G(C_j x)$, $\gamma_{i,x}(G(D_k)) = \{1\}$, $\gamma^*_{i,x}(G_2^*) = G^*(C_j x)$ and $\gamma^*_{i,x}(G_3^*) = \{1\}$. In particular $G_3^* \subseteq \operatorname{Ker} \gamma^*_{i,x}$. In fact:

LEMMA 3.2. $G_3^* = \operatorname{Ker} \gamma^*_{i,x}$; *hence* $G_3^* \lhd G_1^*$.

PROOF. Assume $\gamma^*_{i,x} h = 1$, where $h \in G_1^*$; pick $a \in A$. Then $(a \cdot h)x = ax \cdot 1 = ax$; by definition of the third decomposition, $a \cdot h \equiv a$ and hence $h \in G_3^*$. □

The next two diagram lemmas investigate how $\gamma_{i,x}$, $\gamma^*_{i,x}$ depend on i and x.

LEMMA 3.3. *When $x \in Y_\lambda$, $y \in Y_\mu$, the following diagram commutes*:

$$\begin{tikzcd}
 & G(B_i x) \arrow[r, "\subseteq"] \arrow[dd, "\varphi"] & G(H_{i\lambda}) \arrow[dd, "\varphi^{i\lambda}_{i\mu}"] \\
G(B_i) \arrow[ur, "\gamma_{i,x}"] \arrow[dr, "\gamma_{i,y}"'] & & \\
 & G(B_i y) \arrow[r, "\subseteq"'] & G(H_{i\mu})
\end{tikzcd}$$

PROOF. First $B_i x \subseteq H_{i\lambda}$, $B_i y \subseteq H_{i\mu}$ by the first main theorem, which provides the inclusion maps in the diagram; also, $x, y \in X$ implies $y = xr$, $x = ys$

for some $r, s \in S^1$, whence $B_i y = (B_i x)r$ and (by Green's Lemma) $H_{i\mu} = H_{i\lambda} r$; this provides the associativity homomorphisms in the diagram (which we see are actually isomorphisms). When $m \in B_i x$, $g \in G(B_i x)$, then also $m \in H_{i\lambda}$, $g \in G(H_{i\lambda})$, and $\varphi_{i\mu}^{i\lambda} g \cdot mr = (g \cdot m)r = \varphi g \cdot mr$, so that $\varphi_{i\mu}^{i\lambda} g = \varphi g$ and the square in our diagram commutes. [Since the inclusions in the diagram are associativity homomorphisms, this also follows from the composition property of these homomorphisms.] Also we remember that the Bastida maps γ are associativity homomorphisms, and hence the triangle commutes too. □

LEMMA 3.4. *When $x \in Y_\lambda$ the following diagram commutes for all $i, j \in I$:*

$$\begin{tikzcd}
 & G^*(B_i x) \arrow[r, "\subseteq"] \arrow[dd, "\psi"] & G^*(H_{i\lambda}) \arrow[dd, "\psi_{j\lambda}^{i\lambda}"] \\
G_1^* \arrow[ur, "\gamma_{i,x}^*"] \arrow[dr, "\gamma_{j,x}^*"'] & & \\
 & G^*(B_j x) \arrow[r, "\subseteq"'] & G^*(H_{j\lambda})
\end{tikzcd}$$

PROOF. Take $a \in B_i$, $b \in B_j$. Since $a, b \in A$, we have $b = ra$, $a = sb$ for some $r, s \in S^1$. Since B_i, B_j are part of a right coset decomposition, it follows from 1.1 (more precisely, from the dual of 1.1) that $rB_i \subseteq B_j$; hence $r(B_i x) \subseteq B_j x$, and there is an associativity homomorphism ψ: $G^*(B_i x) \longrightarrow G^*(B_{jx})$. Applying Green's Lemma to ax, bx we obtain the other associativity homomorphism $\psi_{j\lambda}^{i\lambda}$. That these commute with the inclusion maps is proved as for 3.3.

Now let $h \in G_1^* = G^*(B_i) = G^*(B_j)$. Since $h = \pi^* v$ for some v we have $ra \cdot h = r(a \cdot h)$. Hence

$$rax \cdot \gamma_{j,x}^* h = (ra \cdot h)x = r(a \cdot h)x = r(ax \cdot \gamma_{i,x}^* h) = rax \cdot \psi\gamma_{i,x}^* h,$$

which implies $\gamma_{j,x}^* h = \psi\gamma_{i,x}^*$, and the triangle commutes too. □

Dually, there are similar commutative diagrams involving the commutation maps $\delta_{a,\lambda}$: $G_1 = G(Y_\lambda) \longrightarrow G(aY_\lambda)$ and associativity (Bastida) maps $\delta_{a,\lambda}^*$: $G^*(Y_\lambda) \longrightarrow G^*(aY_\lambda)$.

3. With these diagram lemmas we can now prove a weaker version of the third and fourth main theorems. First we may observe that, when $a \in B_i \subseteq C_j$, $x \in Y_\lambda \subseteq Z_\mu$, then 3.1, 3.2 imply $\gamma_{i,x}^* G_2^* = G^*(C_j x)$ and, since $\gamma_{i,x}^*$ is surjective, $G^*(C_j x) \cong G_2^*/G_3^*$. Dually, $G(aZ_\mu) \cong G_2/G_3$. Since $C_j x = aZ_\mu$ by the second main theorem, it follows that $G_2/G_3 \cong G(aZ_\mu) \cong G^*(C_j x) \cong G_2^*/G_3^*$. However, the middle isomorphism (an ϵ map) is not canonical. That a canonical isomorphism exists is part of the following theorem.

THEOREM 3.5. *There exists a canonical isomorphism* Θ: $G_2/G_3 \longrightarrow G_2^*/G_3^*$ *such that, for all* $g \in G_2$, $h \in G_2^*$, $a \in A$, $x \in X$: $\Theta(gG_3) = hG_3^*$ *if and only if* $a(g \cdot x) = (a \cdot h)x$.

PROOF. First we let $a \in A$, $x \in X$ be fixed for the time being, say $a \in C_j \subseteq B_i$, $x \in Z_\mu \subseteq Y_\lambda$ and calculate

$$ax \cdot \gamma_{i,x}^* h = (a \cdot h)x, \quad a(g \cdot x) = a(x \cdot g^x) = ax \cdot \delta_{a,\lambda}^* g^x,$$

so that $(a \cdot h)x = a(g \cdot x)$ is equivalent to $\gamma_{i,x}^* h = \delta_{a,\lambda}^* g^x$, for all $g \in G_2$, $h \in G_2^*$. Since $\gamma_{i,x}^* h = \gamma_{i,x}^* h'$ implies $h' \in hG_3^*$ by 3.2, there is a partial mapping $\theta_{a,x}$: $G_2 \longrightarrow G_2^*/G_3^*$ such that $\theta_{a,x} g = hG_3^*$ if and only if $\gamma_{i,x}^* h = \delta_{a,\lambda}^* g^x$ (i.e. $(a \cdot h)x = a(g \cdot x)$). Since $a(G_2 \cdot x) = aZ_\mu = C_j x = (a \cdot G_2^*)x$ by the second main theorem, we see that $\theta_{a,x}$ is in fact defined on all of G_2 (= is a mapping) and is surjective. Furthermore it is a homomorphism, since $\gamma_{i,x}^*$, $\delta_{a,\lambda}^*$ and ϵ_x are homomorphisms. Now $\theta_{a,x} g = 1 \in G_2^*/G_3^*$ if and only if $a(g \cdot x) = (a \cdot 1)x = ax$, i.e. $g \cdot x \equiv x$, i.e. $g \in G_3$; thus Ker $\theta_{a,x} = G_3$ and $\theta_{a,x}$ induces an isomorphism $\Theta_{a,x}$: $G_2/G_3 \longrightarrow G_2^*/G_3^*$. We see that

$$\begin{aligned}\Theta_{a,x}(gG_3) = hG_3^* &\Longleftrightarrow \theta_{a,x} g = hG_3^* \Longleftrightarrow \gamma_{i,x}^* h \\ &= \delta_{a,\lambda}^* g^x \Longleftrightarrow (a \cdot h)x = a(g \cdot x).\end{aligned}$$

It remains to show that $\Theta_{a,x}$ does not depend on the choice of a, x in A, X (but only on A and X). Take another $b \in A$, say $b \in C \subseteq B_k$ (where $k \in I$). It follows from 3.4 that $\gamma_{k,x}^* h = \psi_{k\lambda}^{i\lambda} \gamma_{i,x}^* h$ for all $h \in G_2^*$, and from the dual of 3.3 that $\delta_{b,\lambda}^* g^* = \psi_{k\lambda}^{i\lambda} \delta_{a,\lambda}^* g^*$ for all $g^* \in G^*(Z_\mu)$. Therefore $\Theta_{a,x}(gG_3) = hG_3^*$, i.e. $\gamma_{i,x}^* h = \delta_{a,\lambda}^* g^x$, implies $\gamma_{k,x}^* h = \delta_{b,\lambda}^* g^x$, i.e. $\Theta_{b,x}(gG_3) = hG_3^*$. Thus $\Theta_{a,x}$ does not depend on a. The alternate definition of $\Theta_{a,x}$ by $(a \cdot h)x = a(g \cdot x)$ is self-dual; hence duality implies that $\Theta_{a,x}$ does not depend on x. □

Note that we have essentially reached our goal of describing when $ax = by$ (for $a, b \in A$, $x, y \in X$); we can always write $b = a \cdot h$, $x = g \cdot y$ for some $h \in G^*(A)$, $g \in G(X)$; then $ax = by$ reads $a(g \cdot y) = (a \cdot h)y$, and therefore happens if and only if $g \in G_2$, $h \in G_2^*$ (by the second main theorem) and $\Theta(gG_3) = hG_3^*$ (by the above).

At the same time it is clear that the diagram lemmas above contain information that does not appear in 3.5, regarding the relationship of G_1, G_1^* and their subgroups to $G(H_{i\lambda})$ and $G^*(H_{i\lambda})$. Taking this into account will yield the third main theorem, and the definitive form of the equality criterion above as fourth main theorem.

4. In order to express these results in as simple and natural a way as possible, it is severely inconvenient to deal with the many groups $G(H_{i\lambda})$, $G^*(H_{i\lambda})$,

and wasteful as well since they all are isomorphic. However, the lack of canonical isomorphisms $G(H_{i\lambda}) \cong G^*(H_{i\lambda})$ renders the identification of these groups dangerous since we might then lose the commutativity properties expressed by our diagram lemmas. That it can be done without any loss of commutativity properties in the diagrams is the main result of [8]. This can be stated as follows. Let D be any $\mathcal{D}$-class of S (where S is any semigroup), with $\mathcal{R}$-classes $(R_i)_{i\in I'}$ and $\mathcal{L}$-classes $(L_\lambda)_{\lambda\in\Lambda'}$; there exists a cross-section $p = (p_{i\lambda})$ (with $p_{i\lambda} \in H_{i\lambda} = R_i \cap L_\lambda$) of $\mathcal{H}$ in D such that every diagram formed by the associativity homomorphisms $\varphi^{i\lambda}_{i\mu}$, $\psi^{i\lambda}_{j\lambda}$ and the commutation isomorphisms $\epsilon_{i\lambda} = \epsilon_{p_{i\lambda}}$ is commutative. Equivalently one may select one group G as left Schützenberger group and right Schützenberger group of every $H_{i\lambda}$ in such a way that $(g \cdot a)r = g \cdot ar$ whenever $a, ar \in D$ and $a \mathcal{R} ar$, $r(a \cdot g) = ra \cdot g$ whenever $a, ra \in D$ and $a \mathcal{L} ra$, and $g \cdot p_{i\lambda} = p_{i\lambda} \cdot g$ for all i, λ [i.e. so that all $\varphi^{i\lambda}_{i\mu}$, $\psi^{i\lambda}_{i\lambda}$ and $\epsilon_{i\lambda}$ (relative to the given cross-section) are the identity on G]. The cross-section p is a *coherence basis* and can always be selected as follows: pick any $1 \in I$, $1 \in \Lambda$; pick all p_{i1}, $p_{1\lambda}$ arbitrarily (in their $\mathcal{H}$-classes); then $p_{i1} = u_i p_{11}$, $p_{1\lambda} = p_{11} v_\lambda$ for some $u_i, v_\lambda \in S^1$ and for $i, \lambda \neq 1$ one may choose $p_{i\lambda} = u_i p_{11} v_\lambda$.

We shall apply this result to the $\mathcal{D}$-class D of S which contains AX and assume then that $G(H_{i\lambda}) = G = G^*(H_{i\lambda})$ as above for all $i \in I'$, $\lambda \in \Lambda'$, in particular for all $i \in I$, $\lambda \in \Lambda$ (we may assume $I \subseteq I'$, $\Lambda \subseteq \Lambda'$). The choice of a coherence basis must be done carefully, and in what follows we choose p according to the following recipe:

LEMMA 3.6. *Let $(a_i)_{i\in I}$, $(x_\lambda)_{\lambda\in\Lambda}$ be cross-sections of the first decompositions of A and X. There exists a coherence basis of D such that $p_{i\lambda} = a_i x_\lambda$ for all $i \in I$, $\lambda \in \Lambda$.*

PROOF. Pick $1 \in I$, $1 \in \Lambda$ and select $p_{11} = a_1 x_1$, $p_{i1} = a_i x_1$ for all $i \in I\backslash 1$, $p_{i1} \in H_{i1}$ for all $i \in I'\backslash I$, $p_{1\lambda} = a_1 x_\lambda$ for all $\lambda \in \Lambda\backslash 1$, $p_{1\lambda} \in H_{1\lambda}$ for all $\lambda \in \Lambda'\backslash\Lambda$. Since $a_i x_\lambda \in H_{i\lambda}$ for all $i \in I$, $\lambda \in \Lambda$, this selects a cross-section of $\mathcal{H}$ in $R_1 \cup L_1$. When $i \neq 1$, let $u_i \in S^1$ be such that $u_i a_1 = a_i$ if $i \in I$, $u_i p_{11} = p_{i1}$ if $i \in I'\backslash I$; when $\lambda \neq 1$ let $v_\lambda \in S^1$ be such that $x_1 v_\lambda = x_\lambda$ if $\lambda \in \Lambda$, $p_{11} v_\lambda = p_{1\lambda}$ if $\lambda \in \Lambda'\backslash\Lambda$. We see that in fact $p_{i1} = u_i p_{11}$, $p_{1\lambda} = p_{11} v_\lambda$ for all $i, \lambda \neq 1$; hence we obtain a coherence basis if we complete p by $p_{i\lambda} = u_i p_{11} v_\lambda$ for all $i, \lambda \neq 1$. When $i \in I$, $\lambda \in \Lambda$, this yields $p_{i\lambda} = a_i x_\lambda$. □

We note that this choice of a coherence basis depends on A and Y and not just on D; if two other $\mathcal{H}$-classes A', X' multiply into D, study of their product may require a different coherence basis for D (and hence a different common group $G(H_{i\lambda}) = G^*(H_{i\lambda})$, which e.g. may differ from G by a nontrivial action-preserving automorphism).

5. Once our group G (with his actions) has been selected, we obtain homomorphisms $\gamma^*_{i\lambda} = \gamma^*_{i,x_\lambda}\colon G^*_1 \longrightarrow G$, $\delta_{i\lambda} = \delta_{a_i,\lambda}\colon G_1 \longrightarrow G$. The next result shows that these are canonical homomorphisms (i.e. depend only on A, X and the cross-sections in A, X which determine the coherence basis necessary for the choice of G).

LEMMA 3.7. *$\gamma^*_{i\lambda}$ and $\delta_{i\lambda}$ do not depend on i nor λ.*

PROOF. Lemma 3.4 shows that $\gamma^*_{i\lambda}$ does not depend on i (as $\psi^{i\lambda}_{j\lambda}$ is now the identity on G). To show that it does not depend on λ we use the calculation properties of G and its actions, and the following remarks. First,

$$p_{i\lambda} \cdot \gamma^*_{i\lambda} h = a_i x_\lambda \cdot \gamma^*_{i\lambda} h = (a_i \cdot h) x_\lambda = (h^{a_i} \cdot a_i) x_\lambda = \gamma_{i\lambda} h^{a_i} \cdot a_i x_\lambda,$$

where $\gamma_{i\lambda} = \gamma_{i,x_\lambda}$. Second, it follows from 3.3 that $\gamma_{i\lambda} = \gamma_{i\mu}$ for all $\lambda, \mu \in \Lambda$. Taking any $\lambda, \mu \in \Lambda$ we observe that $x_\mu = x_\lambda u$ for some $u \in S^1$ so that $p_{i\mu} = p_{i\lambda} u \mathcal{R} p_{i\lambda}$ and hence

$$p_{i\mu} \cdot g = g \cdot p_{i\mu} = g \cdot p_{i\lambda} u = (g \cdot p_{i\lambda}) u = (p_{i\lambda} \cdot g) u$$

for all $g \in G$; therefore

$$\begin{aligned} p_{i\mu} \cdot \gamma^*_{i\lambda} h &= (p_{i\lambda} \cdot \gamma^*_{i\lambda} h) u = (\gamma_{i\lambda} h^{a_i} \cdot a_i x_\lambda) u \\ &= \gamma_{i\lambda} h^{a_i} \cdot a_i x_\lambda u \quad (\text{since } a_i x_\lambda u \mathcal{R} a_i x_\lambda) \\ &= \gamma_{i\mu} h^{a_i} \cdot a_i x_\mu = p_{i\mu} \cdot \gamma^*_{i\mu} h \end{aligned}$$

for all $h \in G^*_1$. Therefore $\gamma^*_{i\lambda} = \gamma^*_{i\mu}$. Dually, $\delta_{i\lambda}$ does not depend on i nor on λ. □

The canonical homomorphisms $\Gamma\colon G^*_1 \longrightarrow G$, $\Delta\colon G_1 \longrightarrow G$ are defined by $\Gamma = \gamma^*_{i\lambda}$, $\Delta = \delta_{i\lambda}$ for any $i \in I$, $\lambda \in \Lambda$. Their basic properties are expressed by the third main theorem:

THEOREM 3.8 (THIRD MAIN THEOREM). *There exist canonical homomorphisms $\Gamma\colon G^*_1 \longrightarrow G$, $\Delta\colon G_1 \longrightarrow G$, such that* $\operatorname{Ker} \Gamma \subseteq G^*_2$, $\operatorname{Ker} \Delta \subseteq G_2$ *and* $\Gamma G^*_2 = \Delta G_2 = \Gamma G^*_1 \cap \Delta G_1$.

PROOF. By 3.2, $\operatorname{Ker} \Gamma = G^*_3 \subseteq G^*_2$ and dually $\operatorname{Ker} \Delta = G_3 \subseteq G_2$. Picking any $i \in I$, $\lambda \in \Lambda$, we have $\Gamma G^*_2 \cdot a_i x_\lambda = a_i x_\lambda \cdot \Gamma G^*_2 = (a_i \cdot G^*_2) x_\lambda = C_j x_\lambda$, where $a_i \in C_j$, and dually $\Delta G_2 \cdot a_i x_\lambda = a_i Z_\mu$, where $x_\lambda \in Z_\mu$; hence the second main theorem implies $C_j x_\lambda = a_i Z_\mu$ and $\Gamma G^*_2 = \Delta G_2$. Finally, assume $\Gamma h = \Delta g$, where $h \in G^*_1$, $g \in G_1$. Then $(a_i \cdot h) x_\lambda = a_i x_\lambda \cdot \Gamma h = \Gamma h \cdot a_i x_\lambda = \Delta g \cdot a_i x_\lambda = a_i (g \cdot x_\lambda)$; by the remarks following 3.5 we conclude $h \in G^*_2$, $g \in G_2$; thus $\Gamma G^*_1 \cap \Delta G_1 \subseteq \Gamma G^*_2 = \Delta G_2$; the converse inclusion is trivial. □

The last part of the proof shows that $\Gamma h = \Delta g$ if and only if $(a_i \cdot h)x_\lambda = a_i(g \cdot x_\lambda)$ (for any i, λ); hence, by 3.5, if and only if $\Theta(gG_3) = hG_3^*$, if and only if $(a \cdot h)x = a(g \cdot x)$ for any a, x. A more general calculation shows

$$(a_i \cdot h)(g \cdot x_\lambda) = (a_i \cdot h)(x_\lambda u) = ((a_i \cdot h)x_\lambda)u = (\Gamma h \cdot a_i x_\lambda)u$$
$$= \Gamma h \cdot a_i x_\lambda u = \Gamma h \cdot a_i(g \cdot x_\lambda) = \Gamma h \cdot \Delta g \cdot a_i x_\lambda$$

for all $h \in G_1^*$, $g \in G_1$. We incorporate this to part of 3.5 to obtain

THEOREM 3.9 (FOURTH MAIN THEOREM). *Let $a, b \in A$, $x, y \in X$, so that $b = a \cdot h$, $x = g \cdot y$ for some unique $h \in G^*(A)$, $g \in G(X)$. Then $ax = by$ if and only if $h \in G_2^*$, $g \in G_2$ and $\Gamma h = \Delta g$. Furthermore $(a_i \cdot h)(g \cdot x_\lambda) = \Gamma h \cdot \Delta g \cdot a_i x_\lambda$ for all $i \in I$, $\lambda \in \Lambda$, $h \in G_1^*$, $g \in G_1$.* □

The mathematical object consisting of the groups G, $G_2^* \subseteq G_1^* \subseteq G^*(A)$, $G_2 \subseteq G_1 \subseteq G(X)$ and homomorphisms Γ: $G_1^* \longrightarrow G$, Δ: $G_1 \longrightarrow G$ is the *group model* of the product of the $\mathcal{H}$-classes A and X. The basic group-theoretical properties of the group model are given by the third main theorem; the fifth main theorem will show that no additional properties can be found in general.

The main theorems show that much of the information obtained on the product AX can be expressed solely in terms of the group model. The first main theorem readily yields a bijection of I to the set of right cosets of G_1^* in $G^*(A)$; thus the index of G_1^* in $G^*(A)$ yields the number of $\mathcal{R}$-classes among which AX is spread. The fourth main theorem shows that the multiplication $B_i \times Y_\lambda \longrightarrow H_{i\lambda}$ can be described by the composite $G_1^* \times G_1 \longrightarrow G$ of Γ, Δ and the multiplication in G, up to the bijections $h \longmapsto a_i \cdot h$, $g \longmapsto g \cdot x_\lambda$, $g \longmapsto g \cdot a_i x_\lambda$ of G_1^*, G_1, G upon B_i, Y_λ, $H_{i\lambda}$ respectively.

6. The first two main theorems included statements that G_1^*, G_2^* (G_1, G_2) depend only on A and the $\mathcal{R}$-class of X (on X and the $\mathcal{L}$-class of A). It is not easy to complete the other two main theorems with similar statements. Basically, a change of A or X inside its $\mathcal{L}$- or $\mathcal{R}$-class necessarily modifies the coherence basis that was used in the selection of G and the description of Γ, Δ; it is conceivable that if two choices are made for A, X, the coherence bases used will not agree on the $\mathcal{H}$-classes that meet both A_1X_1 and A_2X_2, either by lack of further coherence or simply because $A_1X_1 \cap A_2X_2 = \varnothing$. Either phenomenon is still something of a mystery to the author, and does not seem to have been explored by anybody else.

A much simpler difficulty is that the groups $G^*(A_1)$, $G^*(A_2)$ are technically different. This is much easier to surmount, since the two groups can be identified through the associativity isomorphism ψ or by using the Hunter-Anderson presentation for both. It is not too difficult to see that all our groups G_1^*

etc. then depend only on the $\mathcal{L}$-class of A and the $\mathcal{R}$-class of X; the same is true of the isomorphism Θ in 3.5, essentially because we only used $x \mathcal{R} y$ in the proof of 3.3 and $a \mathcal{L} b$ in that of 3.4. This achieves for the weaker version 3.5 what could not be obtained for the third and fourth main theorems.

4. Group actions.

1. In this section we give an alternate description of AX, achieved by means of actions of $G(A)$, $G^*(X)$ on the union $A \square X$ of all $\mathcal{H}$-classes which intersect AX (all $H_{i\lambda}$).

First we construct a left action of $G(A)$ on AS^1. When $u \in T(A)$, $uA \subseteq A$ implies $uAS^1 \subseteq AS^1$; furthermore $\pi(u) = \pi(v)$ implies $ua = va$ for all $a \in A$ and hence $um = vm$ for all $m \in AS^1$. Therefore a left action $\circ$ of $G(A)$ on AS^1 is well defined by: $g \circ as = uas = (g \cdot a)s$ whenever $g = \pi(u) \in G(A)$, $u \in T(A)$, $s \in S^1$. Dually, a right action $\circ$ of $G^*(X)$ on S^1X is well defined by: $sx \circ h = sxu = s(x \cdot h)$ for all $h = \pi^*(u) \in G^*(X)$, $x \in X$, $s \in S^1$. Each is clearly a group action, though not in general simple nor transitive.

PROPOSITION 4.1. *The actions of $G(A)$, $G^*(X)$ on AS^1, S^1X induce commuting actions on AX. Furthermore, in AX, each orbit under $G(A)$ intersects every orbit under $G^*(X)$.*

PROOF. Take $a \in A$, $x \in X$, $g = \pi(u) \in G(A)$, $h = \pi^*(v) \in G^*(X)$. Then $g \circ ax = (g \cdot a)x \in AX$. Dually, $ax \circ h \in AX$. Also, $(g \circ ax) \circ h = (u(ax)) \circ h = (u(ax))v = u((ax)v) = g \circ (ax \circ h)$, so the induced actions commute. We see that $G(A) \cdot ax = (G(A) \cdot a)x = Ax$, $ax \cdot G^*(X) = aX$; it is clear that each Ax intersects every aX (both contain ax). $\square$

PROPOSITION 4.2. *The actions of $G(A)$ and $G^*(X)$ on AX induce a transitive left action of $G(A)$ on I and a transitive right action of $G^*(X)$ on Λ, such that $g \circ (AX \cap R_i) = AX \cap R_{g \circ i}$, $(AX \cap L_\lambda) \circ h = AX \cap L_{\lambda \circ h}$ and*

$$g \circ (AX \cap H_{i\lambda}) \circ h = AX \cap H_{g \circ i, \lambda \circ h}$$

for all $i \in I$, $\lambda \in \Lambda$, $g \in G(A)$, $h \in G^*(X)$.

PROOF. Take any $m, n \in AX$, $g \in G(A)$. Since $\mathcal{R}$ is a left congruence, $m \mathcal{R} n$ implies $g \circ m \mathcal{R} g \circ n$. Hence, for each i, $g \circ (AX \cap R_i) \subseteq AX$ is contained into one $\mathcal{R}$-class which also intersects AX. Thus $g \circ i \in I$ is well defined by $g \circ (AX \cap R_i) \subseteq AX \cap R_{g \circ i}$. This clearly defines a group action. To show transitivity, take any $i, j \in I$. There exist $a, b \in A$, $x, y \in X$ with $ax \in R_i$, $by \in R_j$; also, $b = g \cdot a$ for some $g \in G(A)$. Then $ay \in R_i$ since $\mathcal{R}$ is a left congruence, and $g \circ ay = by$, so that $g \circ (AX \cap R_i)$ intersects R_j; this implies $j = g \circ i$. Finally, if $n \in AX \cap R_{g \circ i}$, then $m = g^{-1} \circ n \in AX \cap R_i$, and $n = g \circ m$, which shows $g \circ (AX \cap R_i) = AX \cap R_{g \circ i}$.

Dually, there is a transitive right action of $G^*(X)$ on Λ such that $(AX \cap L_\lambda) \circ h = AX \cap L_{\lambda \circ h}$. It is immediate that $g \circ (AX \cap H_{i\lambda}) \circ h \subseteq AX \cap H_{g \circ i, \lambda \circ h}$; equality is proved as above. □

This result provides another proof of the fact that every $H_{i\lambda}$ intersects AX. Indeed, for any i, λ, we know that L_λ intersects AX and hence some $H_{j\lambda}$ intersects AX; since $i = g \circ j$ for some $g \in G(A)$ it follows that $AX \cap H_{i\lambda} = g \circ (AX \cap H_{j\lambda}) \neq \varnothing$.

2. We now let $A \square X$ be the union of all $H_{i\lambda}$ (all $\mathcal{H}$-classes which intersect AX).

PROPOSITION 4.3. *The actions of $G(A)$, $G^*(X)$ on AS^1, S^1X induce commuting actions on $A \square X$, such that $g \circ H_{i\lambda} \circ h = H_{g \circ i, \lambda \circ h}$ for all $i \in I$, $\lambda \in \Lambda$, $g \in G(A)$, $h \in G^*(X)$.*

PROOF. First note that $A \square X \subseteq AS^1 \cap S^1X$. Take $g = \pi(u) \in G(A)$, $m \in H_{i\lambda}$; then $m = axv$ for some $a \in A$, $x \in X$, $v \in T^*(H_{i\lambda})$. We have $g \circ ax \in H_{j\lambda}$, where $j = g \circ i$, by 4.2; since $H_{i\lambda}$, $H_{j\lambda}$ lie in the same $\mathcal{L}$-class, we have $T^*(H_{i\lambda}) = T^*(H_{j\lambda})$ and hence $g \circ m = uaxv = (g \circ ax)v \in H_{j\lambda}$. Thus $A \square X$ admits the action of $G(A)$. We have also shown that $g \circ H_{i\lambda} \subseteq H_{g \circ i, \lambda}$ and as above the equality holds. Dually, $A \square X$ also admits the action of $G^*(X)$ and $H_{i\lambda} \circ h = H_{i, \lambda \circ h}$ for all $h \in G^*(X)$. The fact that the two actions commute is proved as for 4.1 and the last part of the statement is then immediate. □

3. In comparing the four main theorems with the results of this section, we see that the former are directed towards the fine structure of the map $A \times X \longrightarrow AX$, whereas the latter give more global views of AX. For instance, when the actions of $G(A)$, $G^*(X)$ on $A \square X$ are known, one product ax determines all the others, through $(g \cdot a)(x \cdot h) = g \circ ax \circ h$. This is simpler than the corresponding part of the fourth main theorem, where we had to know all products $a_i x_\lambda$. The basic cells $Ax \cap aX$ of AX also arise as $(G(A) \circ ax) \cap (ax \circ G^*(X))$.

The approach in this section is also in some respects more natural; nevertheless it is necessarily limited in depth since all three decompositions of A which are basic for our main theorems are right coset decompositions, but not in general left coset decompositions, and hence must be expressed in terms of the action of $G^*(A)$, not $G(A)$: the choice of $G(A)$, $G^*(X)$ seems more natural at first, but it is still the wrong choice.

5. An example, and the fifth main theorem.

1. In this section we start with groups G, $C \subseteq B \subseteq A$, $Z \subseteq Y \subseteq X$ and homomorphisms γ: $B \longrightarrow G$, δ: $Y \longrightarrow G$ with the basic properties stated in the third main theorem: Ker $\gamma \subseteq C$, Ker $\delta \subseteq Z$, $\gamma C = \delta Z = \gamma B \cap \delta Y$. From this we build a semigroup in which the group model of the product of two $\mathcal{H}$-classes

consists, up to isomorphism, of the given groups and homomorphisms. This semigroup has A and X as maximal subgroups, and the products AX fall within a rectangular group built from the group G. Its construction is rather complicated and will be done in several steps.

2. The basic construction process we use, and the proof of associativity, can be stated in a somewhat more general situation as follows.

LEMMA 5.1. *Let A, X, S be pairwise disjoint semigroups and $A \times X \longrightarrow S$, $A \times S \longrightarrow S$, $S \times X \longrightarrow S$ be given "multiplications" (denoted by juxtaposition). Let P be the disjoint union $A \cup S \cup X$ together with the obvious partial multiplication of domain*

$$(A \times A) \cup (S \times S) \cup (X \times X) \cup (A \times X) \cup (A \times S) \cup (S \times X).$$

Then P is embeddable into a semigroup if and only if

(1) $a(br) = (ab)r$,
(2) $a(rs) = (ar)s$,
(3) $(rx)y = r(xy)$,
(4) $(rs)x = r(sx)$,
(5) $a(bx) = (ab)x$,
(6) $(ax)y = a(xy)$,
(7) $a(rx) = (ar)x$

hold for all $a, b \in A$, $r, s \in S$, $x, y \in X$. In this case P is a partial subsemigroup of the semigroup T obtained as follows. As a set, T is the disjoint union $A \cup S \cup X \cup (X \times S) \cup (S \times A) \cup (X \times A) \cup (X \times S \times A)$. The multiplication $*$ in T is given by the table below, in which the elements of $X \times S$, $S \times A$, $X \times A$ and $X \times S \times A$ are denoted by $(x, r]$, $[r, a)$, (x, a) and (x, r, a) respectively.

PROOF. It suffices to show that $(T, *)$ is a semigroup when (1) to (7) hold. For this we use Light's associativity test, noting that T is generated by P, so that it suffices to test the elements of $A \cup S \cup X$; since A and X play dual roles throughout, it actually suffices to test the elements of $A \cup S$.

The peculiar notation and arrangement of elements in the table was adopted to simplify this task. The three different kinds of ordered pairs in T are differentiated by the notation. We also see that the last three columns in the table arise from the second, third and fourth columns by the row-by-row adjunction of $, b)$ to each entry. The last three columns can therefore be disregarded when performing Light's test. Similarly, only the first four rows need be considered, since the last three rows arise from the previous three by the adjunction of $(x,$ to each entry.

Performing Light's test for any $c \in A$, $t \in S$ then yields the following 4×4 tables.

$*$	b	s	y	$(y, s]$
x	(x, b)	$(x, s]$	xy	$(xy, s]$
r	$[r, b)$	rs	ry $(\in S)$	$(ry)s$
a	ab	as $(\in S)$	ay $(\in S)$	$(ay)s$
$[r, a)$	$[r, ab)$	$r(as)$	$r(ay)$	$r(ay)s$
$(x, r]$	(x, r, b)	$(x, rs]$	$(x, ry]$	$(x, (ry)s]$
(x, a)	(x, ab)	$(x, as]$	$(x, ay]$	$(x, (ay)s]$
(x, r, a)	(x, r, ab)	$(x, r(as)]$	$(x, r(ay)]$	$(x, r(ay)s]$

$*$	$[s, b)$	(y, b)	(y, s, b)
x	(x, s, b)	(xy, b)	(xy, s, b)
r	$[rs, b)$	$[ry, b)$	$[(ry)s, b)$
a	$[as, b)$	$[ay, b)$	$[(ay)s, b)$
$[r, a)$	$[r(as), b)$	$[r(ay), b)$	$[r(ay)s, b)$
$(x, r]$	(x, rs, b)	(x, ry, b)	$(x, (ry)s, b)$
$(x, a]$	(x, as, b)	(x, ay, b)	$(x, (ay)s, b)$
(x, r, a)	$(x, r(as), b)$	$(x, r(ay), b)$	$(x, r(ay)s, b)$

$\textcircled{c}$				
(x, c)	(x, cb)	$(x, cs]$	$(x, cy]$	$(x, (cy)s]$
$[r, c)$	$[r, cb)$	$r(cs)$	$r(cy)$	$r(cy)s$
ac	acb	$(ac)s$	$(ac)y$	$((ac)y)s$
$[r, ac)$	$[r, acb)$	$r((ac)s)$	$r((ac)y)$	$r((ac)y)s$

$\textcircled{c}$	cb	cs	cy	$(cy)s$
	(x, cb)	$(x, cs]$	$(x, cy]$	$(x, (cy)s]$
	$[r, cb)$	$r(cs)$	$r(cy)$	$r(cy)s$
	acb	$a(cs)$	$a(cy)$	$a((cy)s)$
	$[r, acb)$	$r(a(cs))$	$r(a(cy))$	$r(a((cy)s))$

$\textcircled{t}$				
$(x, t]$	(x, t, b)	$(x, ts]$	$(x, ty]$	$(x, (ty)s]$
rt	$[rt, b)$	rts	$(rt)y$	$((rt)y)s$
at	$[at, b)$	$(at)s$	$(at)y$	$((at)y)s$
$r(at)$	$[r(at), b)$	$r(at)s$	$(r(at))y$	$((r(at))y)s$

$\textcircled{t}$	$[t, b)$	ts	ty	$(ty)s$
	(x, t, b)	$(x, ts]$	$(x, ty]$	$(x, (ty)s]$
	$[rt, b)$	rts	$r(ty)$	$r(ty)s$
	$[at, b)$	$a(ts)$	$a(ty)$	$a((ty)s)$
	$[r(at), b)$	$r(a(ts))$	$r(a(ty))$	$r(a((ty)s))$

To see that the tables agree as they should, we derive the following consequences of (1) through (7). For all $a, c \in A$, $r, s, t \in S$, $y \in X$:

(8) $((ac)y)s = (a(cy))s = a((cy)s)$ by (5), (2),

(9) $((at)y)s = (a(ty))s = a((ty)s)$ by (7), (2),

(10) $(r(at))y = r((at)y) = r(a(ty))$ by (4), (7),

(11) $((r(at))y)s = r(a(ty))s = r(a((ty)s))$ by (10), (9).

Inspection reveals that the two tables of c agree by (1), (5), (8), and those of t agree by (4), (2), (7), (9), (10), (11). □

3. By this lemma, the construction of our semigroup is reduced to the choice of S and of suitable multiplications $A \times X \longrightarrow S$, $A \times S \longrightarrow S$, $S \times X \longrightarrow X$. To construct the latter we first need to extend the homomorphisms γ: $B \longrightarrow G$ and δ: $Y \longrightarrow G$ to mappings α: $A \longrightarrow G$, β: $X \longrightarrow G$ having certain multiplicative properties.

First we select, once and for all, one element in each right coset of B in A; we write all these elements as a family $(d_i)_{i \in I}$; the family of all right cosets of B in A is then $(d_i B)_{i \in I}$. Also, B is a right coset of itself; we denote the corresponding element of I by 1 and further assume $d_1 = 1 \in A$. A left action of A on the set I is then well defined by $d_{ai}B = ad_iB$; and it is a transitive action (if $i, j \in I$, then $a = d_j d_i^{-1}$ satisfies $ai = j$). Note that $aB = ad_1B = d_{a1}B$; in particular, $B = \{a \in A; a1 = 1\}$; also $d_i 1 = i$ for all $i \in I$.

For any $a \in A$, $aB = d_{a1}B$ implies $d_{a1}^{-1}a \in B$ and we define $\alpha a \in G$ by $\alpha a = \gamma(d_{a1}^{-1}a)$.

LEMMA 5.2. *For all $i \in I$, $a, c \in A$, $b \in B$ the following hold*: (a) $\alpha b = \gamma b$; (b) $\alpha d_i = 1$; (c) $\alpha(ab) = \alpha(a)\alpha(b)$; (d) $\alpha(ac) = \alpha(ad_{c1})\alpha(c)$; (e) $\alpha(acd_i) = \alpha(ad_{ci})\alpha(cd_i)$.

PROOF. First, $b1 = 1$, so that $d_{b1}^{-1}b = b$, whereas $d_i1 = i$, so that $d_{d_i1}^{-1}d_i = 1$; therefore (a), (b) hold. Next, $d_{c1}1 = c1$, so that $ad_{c1}1 = ac1$ $(\in I)$; hence $d_{ac1}^{-1}ac = d_{ad_{c1}1}^{-1}ad_{c1}d_{c1}^{-1}c$; hence (d) follows, since γ is a homomorphism. If $c = b \in B$ then $d_{c1} = 1$, so that (c) follows from (d); finally, (e) follows from (d) since $cd_i1 = ci$. □

Dually, let Λ index the left cosets of Y in X, with 1 corresponding to Y itself; select w_λ in the left coset corresponding to λ (which can then be written as Yw_λ) with $w_1 = 1 \in X$. Then a transitive right action of X on Λ is well defined by $Yw_{\lambda x} = Yw_\lambda x$, and satisfies $Y = \{x \in X;\ 1x = 1\}$ and $1w_\lambda = \lambda$. A mapping β: $X \longrightarrow G$ is well defined by $\beta x = \delta(xw_{1x}^{-1})$; its multiplicative properties are given by

LEMMA 5.2*. *For all* $\lambda \in \Lambda$, $x, z \in X$, $y \in Y$ *the following hold*: (a) $\beta y = \delta y$; (b) $\beta w_\lambda = 1$; (c) $\beta(yx) = \beta(y)\beta(x)$; (d) $\beta(xz) = \beta(x)\beta(w_{1x}z)$; (e) $\beta(w_\lambda xz) = \beta(w_\lambda x)\beta(w_{\lambda x}z)$. □

Finally, note that $\alpha A = \alpha B = \gamma B$, $\beta X = \beta Y = \delta Y$ and hence $\alpha C = \beta Z = \alpha A \cap \beta X$.

4. We can now construct our semigroup. With A and X as before, we let R be the rectangular group $I' \times G \times \Lambda'$, where I' is any set containing I (such as I itself) and Λ' is any set that contains Λ; then S is R with a zero adjoined. Recall that the multiplication in R is given by $(i, g, \lambda)(j, h, \mu) = (i, gh, \mu)$. The multiplications $A \times X \longrightarrow S$, $A \times S \longrightarrow S$, $S \times X \longrightarrow S$ are given for all $a \in A$, $x \in X$, $i \in I'$, $g \in G$, $\lambda \in \Lambda'$ by:

$$ax = (a1, \alpha(a)\beta(x), 1x) \in R,$$

$$a(i, g, \lambda) = \begin{cases} (ai, \alpha(ad_i)g, \lambda) & \text{if } i \in I, \\ 0 & \text{if } i \in I'\backslash I, \end{cases}$$

$$(i, g, \lambda)x = \begin{cases} (i, g\beta(w_\lambda x), \lambda x) & \text{if } \lambda \in \Lambda, \\ 0 & \text{if } \lambda \in \Lambda'\backslash\Lambda, \end{cases}$$

$$a0 = 0x = 0.$$

LEMMA 5.3. *Conditions* (1) *to* (7) *of Lemma* 5.1 *hold for all* $a, b \in A$, $r, s, t \in S$, $x, y \in X$ *in the above.*

PROOF. By duality it suffices to verify (1), (2), (5), (7). Take $a, b \in A$, $r \in S$. That $a(br) = (ab)r$ is trivial if $r = 0$ and if $r = (i, g, \lambda)$ with $i \notin I$. If $r = (i, g, \lambda)$ and $i \in I$, we calculate

$$a(b(i, g, \lambda)) = a(bi, \alpha(bd_i)g, \lambda) = (abi, \alpha(ad_{bi})\alpha(bd_i)g, \lambda)$$
$$= (abi, \alpha(abd_i)g, \lambda) = (ab)(i, g, \lambda),$$

by 5.2 (e). This proves (1). The verification of (2), (5), (7) is similar and is left to the reader; 5.2 (d) is used to prove (5), but the other proofs do not depend on Lemma 5.2. □

It follows from 5.1, 5.3 that:

THEOREM 5.4. *With the notation as above, there exists a semigroup T containing A and X as maximal subgroups and R as a subsemigroup, in which A and X multiply according to* $ax = (a1, \alpha(a)\beta(x), 1x)$. □

5. We now let T be the semigroup given by 5.4 (more precisely, by 5.1 and the above), and find the group model of the product of the $\mathcal{H}$-classes A and X in T.

First we note that R is regular and hence the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ of T, restricted to R, yield precisely the corresponding Green's relations of R. In particular, $H_{i\lambda} = \{(i, g, \lambda); g \in G\}$ is an $\mathcal{H}$-class of T for each $i \in I'$, $\lambda \in \Lambda'$. [This can also be seen directly.] Clearly an $\mathcal{H}$-class of T intersects AX if and only if it is an $H_{i\lambda}$ with $i \in I$, $\lambda \in \Lambda$. Therefore I, Λ have the same meaning as in §§2, 3. Furthermore we may take $G^*(A) = A$, $G(X) = X$, with the obvious actions on A and X.

Take $i \in I$, $\lambda \in \Lambda$. When $a \in A$, $x \in X$ we see that $ax \in H_{i\lambda}$ if and only if $a1 = i$, $1x = \lambda$; since $aB = d_{a1}B$ and dually, this is equivalent to $a \in d_iB$, $x \in Yw_\lambda$. Therefore $B_i = d_iB$, $Y_\lambda = Yw_\lambda$. In particular, $G_1^* = G^*(d_iB) = B \subseteq A$ and $G_1 = Y$.

We now describe the second decomposition of A, which we recall was introduced through the equivalence relation $a \sim b \iff aY_\lambda = bY_\lambda$, where $\lambda \in \Lambda$ is arbitrary. By 2.5, $aY_\lambda = bY_\lambda$ is equivalent to $aY_\lambda \cap bY_\lambda \neq \varnothing$; hence we may pick any $x \in Y_\lambda$, for instance w_λ, and $a \sim b \iff bw_\lambda \in aY_\lambda$. When this holds, a, b lie in the same B_i and therefore $b = ac$ for some $c \in B$. Conversely, assume $a, b \in A$ and $b = ac$ for some $c \in B$. We find $bw_\lambda = (ac1, \alpha(ac)\beta(w_\lambda), 1w_\lambda) = (a1, \alpha(a)\alpha(c), \lambda)$, by 5.2 (c), 5.2* (b). Hence $bw_\lambda \in aY_\lambda$ is successively equivalent to: $bw_\lambda = ax$ for some $x \in Y_\lambda = Yw_\lambda$; $bw_\lambda = ayw_\lambda$ for some $y \in Y$; $(a1, \alpha(a)\alpha(c), \lambda) = (a1, \alpha(a)\beta(yw_\lambda), 1yw_\lambda) = (a1, \alpha(a)\beta(y), \lambda)$ for some $y \in Y$, by 5.2* (c), 5.2* (b); $\alpha(c) = \beta(y)$ for some $y \in Y$; $\gamma c \in \delta Y$; $\gamma c \in \gamma B \cap \delta Y = \gamma C$; and $c \in C$ (since $\operatorname{Ker} \gamma \subseteq C$). It follows from this that $a \cdot G_2^* = aC$ and hence $G_2^* = C$. Dually $G_2 = Z$.

It now remains to determine the homomorphisms Γ, Δ in the group model of A times X, and their codomain. The first step of this is to select a representative element in each B_i and Y_λ. We saw that $B_i = d_iB$, $Y_\lambda = Yw_\lambda$ and hence may select $d_i \in B_i$, $w_\lambda \in Y_\lambda$ as representative elements. It follows from 5.2 (b) and its dual that this yields $p_{i\lambda} = d_iw_\lambda = (i, 1, \lambda)$. We know that the $\mathcal{D}$-class which contains AX has a coherence basis which includes all $p_{i\lambda}$ above (Lemma 3.6).

We now show that G can serve as single Schützenberger group of that $\mathcal{D}$-class. First we note that G already acts, on the right and on the left, on $A \square X$, by $g \cdot (i, h, \lambda) = (i, gh, \lambda), (i, h, \lambda) \cdot g = (i, hg, \lambda)$; it is immediate that these actions satisfy all that one demands of a single Schützenberger group, including $g \cdot p_{i\lambda} = p_{i\lambda} \cdot g$, $(g \cdot r)s = g \cdot rs$ when $r \mathrel{\mathcal{R}} rs$, $t(r \cdot g) = tr \cdot g$ when $r \mathrel{\mathcal{L}} tr$, as long as $i \in I$, $\lambda \in \Lambda$ and $r, rs, tr \in A \square X$. We use this in showing that G will actually serve as well for the whole $\mathcal{D}$-class. [One can also go back to §3 and argue that the group G there is only used for his actions on $A \square X$ and hence does not need to serve for the whole $\mathcal{D}$-class. However, the main theorems do read better without this restriction on G.] Let H be the group denoted by G in §3; in particular, H acts simply and transitively on each $H_{i\lambda}$, on the right and on the left, and the three properties mentioned above for G also hold for H. Since both G and H serve as $G(H_{11})$ they must be isomorphic; more precisely, an isomorphism $\iota: H \longrightarrow G$ is well defined by $\iota h \cdot p_{11} = h \cdot p_{11}$. We see that ι also preserves the left actions of G, H on p_{11}; it also preserves their right actions on p_{11}, since $g \cdot p_{11} = p_{11} \cdot g$, $h \cdot p_{11} = p_{11} \cdot h$ for all $g \in G$, $h \in H$. Since $G \cong H$ we can use G as a single Schützenberger group of D; but its actions on D must then be redefined by $g \circ r = h \cdot r$, $r \circ g = r \cdot h$ whenever $r \in D$ and $g = \iota h$. Hence we now must show that these new actions coincide with the existing actions as long as we stay within $A \square X$. We have seen that this is already true on p_{11}. If $r \in H_{i1}$, then $r = up_{11}$ for some $u \in S^1$, since $r \mathrel{\mathcal{L}} p_{11}$, and hence

$$r \circ g = r \cdot h = up_{11} \cdot h = u(p_{11} \cdot h) = u(p_{11} \cdot g) = up_{11} \cdot g = r \cdot g$$

since $up_{11} \mathrel{\mathcal{L}} p_{11}$, whenever $g = \iota h$; hence the two right actions of G on H_{i1} agree. If now $r \in H_{i\lambda}$, then $r = p_{i1}v \mathrel{\mathcal{R}} p_{i1}$ for some $v \in S^1$ and hence

$$\begin{aligned} g \circ r = h \cdot r = h \cdot p_{i1}v = (h \cdot p_{i1})v = (p_{i1} \cdot h)v = (p_{i1} \cdot g)v \\ = (g \cdot p_{i1})v = g \cdot p_{i1}v = g \cdot r \end{aligned}$$

whenever $g = \iota h$, i.e. the two left actions of G on $H_{i\lambda}$ coincide. By duality the right actions coincide also.

Thus we can use G as Schützenberger group of D and, more important, have simple descriptions of its actions on $A \square X$. We can now calculate $\Gamma: B \longrightarrow G$ and $\Delta: Y \longrightarrow G$. We remember that $(a_i \cdot h)x_\lambda = a_i x_\lambda \cdot \Gamma h$ for all $h \in G_1^*$, where $i \in I$, $\lambda \in \Lambda$ are arbitrary (in the notation of §3); in the present notation, this reads: $(d_i b)w_\lambda = p_{i\lambda} \cdot \Gamma b = (i, \Gamma b, \lambda)$, for all $b \in B$. At the same time, $(d_i b)w_\lambda = (i, \gamma b, \lambda)$, since $d_i b1 = d_i 1 = i$, $\alpha(d_i b) = \gamma(d_{d_i b 1}^{-1} d_i b) = \gamma b$ by definition of α and $1w_\lambda = \lambda$. Therefore $\Gamma = \gamma$. Dually $\Delta = \delta$.

Thus we have proved

THEOREM 5.5 (FIFTH MAIN THEOREM). *The groups G, $C \subseteq B \subseteq A$, $Z \subseteq Y \subseteq X$ and homomorphisms $\gamma: B \longrightarrow G$, $\delta: Y \longrightarrow G$ constitute (up to isomorphism) the group model of the product of two $\mathcal{H}$-classes in some semigroup if and only if* $\operatorname{Ker} \gamma \subseteq C$, $\operatorname{Ker} \delta \subseteq Z$ *and* $\gamma C = \delta Z = \gamma B \cap \delta Y$. □

6. The fifth main theorem is the converse of the third. As a first application, it yields a partial converse to the weaker result 3.5.

COROLLARY 5.6. *The groups $C \subseteq B \subseteq A$, $Z \subseteq Y \subseteq X$ are part of the group model of the product of two $\mathcal{H}$-classes in some semigroup if and only if there exist normal subgroups $D \subseteq C$ of B and $W \subseteq Z$ of Y such that $C/D \cong Z/W$.*

PROOF. This condition is necessary by 3.5. For the converse we note that the groups B/D, Y/W contain isomorphic subgroups C/D, Z/W and hence can be embedded into their free product G amalgamating these subgroups. Then the obvious homomorphisms $\gamma: B \longrightarrow B/D \longrightarrow G$, $\delta: Y \longrightarrow Y/W \longrightarrow G$ satisfy the conditions of the fifth main theorem. □

We can also use the fifth main theorem to answer (negatively) some questions which have or may have arisen earlier. Most importantly, we can now show that the three decompositions of A may all fail to be left coset decompositions: it suffices to take $D \subseteq C \subseteq B \subseteq A$ with D normal in B but neither B, C, D normal in A (e.g. $B = C$ a subgroup of order 4 in the simple group A_5) and complete with $W = \{1\}$, $Z = Y = X = C/D$, so that the condition in 5.6 is satisfied.

We can also show that the sets $B_i Y_\lambda$ are usually not cosets. Let G be a group containing disjoint subgroups B, Y whose product BY is not a subgroup (e.g. $G = S_3$, with B, Y generated by distinct transpositions); complete with $C = Z = \{1\}$, $A = B$, $X = Y$ and the inclusion mappings, so that the conditions in 5.5 are satisfied. Using the same semigroup as before, we find $B_i Y_\lambda = \{(d_i b)(y w_\lambda); b \in B, y \in Y\} = \{(i, by, \lambda); b \in B, y \in Y\} = BY \cdot p_{i\lambda}$. If $B_i Y_\lambda$ were a coset then its left Schützenberger group (in $G = G(H_{i\lambda})$) would be BY, which is impossible.

7. In our semigroup we were careful to take the sets $I' \setminus I$ and $\Lambda' \setminus \Lambda$ arbitrary; thus our example shows that the number of $\mathcal{R}$-classes of D which do not intersect AX can be arbitrarily large (and similarly for $\mathcal{L}$-classes). However the reader will easily see that D is greater than R, so that the example does not show these two numbers can also be arbitrarily small.

This can be remedied by a much simpler example. Let A, X be any group, I be any set on which A acts transitively on the left and Λ be any set on which X acts transitively on the right. Pick $1 \in I$, $1 \in \Lambda$ and let I', Λ' be any sets such that $I \subseteq I'$, $\Lambda \subseteq \Lambda'$. Let $R = I \times \Lambda \subseteq I' \times \Lambda' = R'$ be rectangular bands (with multiplication $(i, \lambda)(j, \mu) = (i, \mu)$) and $S = A \cup X \cup R'$ together with the multiplication:

	b	y	(j, μ)
a	ab	$(a1, 1y)$	(aj, μ)
x	$(1, 1)$	xy	$(1, \mu)$
(i, λ)	$(i, 1)$	$(i, \lambda y)$	(i, μ)

where the actions of A, X on I, Λ have been extended to I', Λ' in an arbitrary fashion (e.g. having A act trivially on $I' \setminus I$ and dually). Associativity is immediate. We see that the $\mathcal{H}$-classes which intersect AX are all (i, λ) with $i \in A1 = I$, $\lambda \in 1X = \Lambda$ and therefore the number of $\mathcal{R}$-classes of D $(= R')$ that do not intersect AX is any cardinal number, and similarly for $\mathcal{L}$-classes.

The group model in the above is highly simple. The author could not find an example which would show the same as the above and would have an arbitrary group model. It is conceivable that this is due to more than technical difficulties.

6. The finite case. We now keep the notation in §§2, 3 and assume that A and X are finite (e.g. that S is finite). In what follows we let $m = |I|$ be the number of elements of I (the number of $\mathcal{R}$-classes which intersect AX) and $n = |\Lambda|$. Let also m' be the number of distinct sets aY_λ contained in B_iY_λ and n' be the number of distinct sets B_ix contained in B_iY_λ; it follows from 2.5 that there are $m'n'$ sets $aY_\lambda \cap B_ix$ in B_iY_λ; we put $r = |aY_\lambda \cap B_ix|$.

PROPOSITION 6.1. *m divides $|A|$ and n divides $|X|$. The numbers m', n', r do not depend on a, x, i, λ, and m' divides $|A|/m$, n' divides $|X|/n$, $m'r$ and $n'r$ divide $|G|$. Finally $|B_iY_\lambda| = m'n'r$ and $|AX| = mnm'n'r$.*

PROOF. By the first main theorem, $|I|$ is the index of G_1^* in $G^*(A)$ and hence divides $|G^*(A)| = |A|$; dually, $|\Lambda|$ divides $|X|$. The definition of the second decomposition of A (via 2.6) shows that m' is the index of G_2^* in G_1^*, and hence does not depend on a, λ and divides $|G_1^*| = |G^*(A)|/m$; dually, n' does not depend on x, i and divides $|X|/n$. The sets B_ix are, by 2.5, part of a left coset decomposition of $H_{i\lambda}$ and hence all have the same number of elements; similarly, the sets $aY_\lambda \cap B_ix$ form a right coset decomposition of B_ix and hence all have the same number of elements, namely $r = |B_ix|/m'$; therefore r does not depend on a, x, i, λ. This also shows $|B_ix| = m'r$ and the last two equalities in the statement are then immediate. □

Finally let $|G_3^*| = m''$, $|G_3| = n''$.

PROPOSITION 6.2. *$|A| = mm'm''r$, $|X| = nn'n''r$, so that $|AX|$ divides $|A|\,|X|$, with $|A|\,|X|/|AX| = m''n''r$.*

PROOF. Since each map $\gamma_{i,x}^*$ is surjective, it follows from 3.2 that $r = |aY_\lambda \cap B_ix| = |G^*(aY_\lambda \cap B_ix)| = |G_2^*|/|G_3^*|$; hence $|G_2^*| = m''r$ and $|A| =$

$|G_2^*|m'm = mm'm''r$. Dually, $|X| = nn'n''r$. Thus, $|A|\,|X| = mm'm''nn'n''r^2$ is a multiple of $|AX| = mm'nn'r$, with the quotient as indicated. □ [In fact it is easy to derive from 3.5 that all classes modulo the equivalence relation induced by $A \times X \longrightarrow AX$ have the same number of elements $m''n''r$.]

PROPOSITION 6.3. *The numbers $r, m, m', m'', n, n', n''$ can be any seven positive integers.*

PROOF. For any given $r, m, m', m'', n, n', n''$, let

$$C = \mathbf{Z}(r) \oplus \mathbf{Z}(m''), \quad B = C \oplus \mathbf{Z}(m'), \quad A = B \oplus \mathbf{Z}(m),$$

$$Z = \mathbf{Z}(r) \oplus \mathbf{Z}(n''), \quad Y = Z \oplus \mathbf{Z}(n'), \quad X = Y \oplus \mathbf{Z}(n).$$

It is clear from 5.6 that $C \subseteq B \subseteq A$ and $Z \subseteq Y \subseteq X$ are part of the group model of some semigroup. □ [It is of interest that this semigroup can be chosen of finite orders. In the proof of 5.6, we may use a pushout construction in the category of abelian groups, rather than in that of all groups, in order to produce G; then G is finite. The semigroup provided by 5.4 is also finite if we choose I', Λ' finite.]

It follows from 6.3 that the divisibility relationships expressed, directly or indirectly, by 6.1, 6.2, are the most general that occur in general.

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