CONTINUOUS COHOMOLOGY FOR
COMPACTLY SUPPORTED VECTORFIELDS ON $\mathbb{R}^n$

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ABSTRACT. In this paper we study the Gelfand-Fuks cohomology of the Lie algebra of compactly supported vectorfields on $\mathbb{R}^n$ and establish the degeneracy of a certain spectral sequence at the $E_1$ level. We apply this result to the study of another spectral sequence introduced by Ressetnikov for the cohomology of the algebra of vectorfields on $S^m$.

Let $L$ be the Lie algebra of compactly supported smooth vectorfields on a manifold $M$. For $U$ a precompact open subset of $M$ let $L_U$ be the set of vectorfields supported in $U$ with the $C^\infty$ topology, then $L = \bigcup_{U \subset M} L_U$ and we give $L$ the topology of a strict inductive limit. Let $C^q(L)$ be the vectorspace of all continuous skewsymmetric $\mathbb{R}$-multilinear functions from $L \times \cdots \times L$ ($q$ times) into $\mathbb{R}$. Define

$$d^q: C^q(L) \rightarrow C^{q+1}(L),$$

$$(d^q\lambda)(\xi_1, \ldots, \xi_{q+1}) = \sum (-1)^{i+j+k} \lambda([\xi_i, \xi_j], \ldots, \hat{\xi_i}, \ldots, \hat{\xi_j}, \ldots, \xi_{q+1})$$

where $[,]$ denotes the Lie bracket of vectorfields and $\ ^\wedge$ indicates omission.

Then $d^{q+1} \circ d^q = 0$ and $C^*(L) = \bigoplus_{q=0,\ldots,\infty} C^q(L)$ is a differential complex with differential $d = \bigoplus d^q$. The cohomology of $(C^*(L), d)$ is known as the Gelfand-Fuks cohomology of $L$ with coefficients in $\mathbb{R}$.

Let $\text{pr}_i: M^q \rightarrow M$ be the projection on the $i$th factor of the $q$-fold cartesian product of $M$ and let $\text{pr}^*_i T$ be the pull-back of the tangent bundle to $M$ along $\text{pr}_i$. Define $T^q = \text{pr}^*_1 T \otimes \cdots \otimes \text{pr}^*_q T$ as a bundle over $M^q$. A vectorfield $\xi$ on $M$ defines a section $\text{pr}^*_i T$ in a natural way and a $q$-tuple $(\xi_1, \ldots, \xi_q)$ of vectorfields defines a section $\text{pr}^*_1 \xi_1 \otimes \cdots \otimes \text{pr}^*_q \xi_q$ of $T^q$ over $M^q$. Linear combinations of sections of this type are dense in the space of compactly supported sections of $T^q$, denoted $[T^q]_C$, with the inductive limit topology defined similarly to that on $L = [T]_C$. Thus an element $\lambda \in C^q(L)$ defines a continuous function
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$\tilde{\lambda}: \left[T^q\right]_C \to \mathbb{R}$. If $\text{Hom}_\mathbb{R}(\left[T^q\right]_C, \mathbb{R})$ denotes the set of continuous $\mathbb{R}$ multilinear functions, then we have a map $C^q(L) \to \text{Hom}_\mathbb{R}(\left[T^q\right]_C, \mathbb{R})$. If we let $B^q(L)$ denote the set of not necessarily skewsymmetric continuous $\mathbb{R}$-multilinear functions $L \times \cdots \times L \to \mathbb{R}$, then we have an isomorphism:

(1) $B^q(L) \cong \text{Hom}_\mathbb{R}(\left[T^q\right]_C, \mathbb{R})$.

Let $\Sigma_q$ be the permutation group on $q$-letters and corresponding to $\sigma \in \Sigma_q$ and $\lambda \in B^q(L)$ let $\sigma \circ \lambda \in B^q(L)$ be defined by

$$(\sigma \circ \lambda)(\xi_1, \ldots, \xi_q) = e_\sigma \lambda(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(q)})$$

where $e_\sigma$ is the sign of $\sigma$ as a permutation. With these definitions $C^q(L)$ is the subspace of $\Sigma_q$ invariants in $B^q(L)$.

(2) $B^q(L)^{\Sigma_q} = C^q(L)$.

Let $\mathcal{D}(M^q)$ be the space of distributions on $M^q$,

$$\mathcal{D}(M^q) = \text{Hom}_\mathbb{R}(C^\infty(M^q), \mathbb{R}) = \text{Hom}_\mathbb{R}(\left[1\right]_C, \mathbb{R}).$$

Consider $C^\infty(M^q)$ as a left $C^\infty(M^q)$ module making $\mathcal{D}(M^q)$ a right $C^\infty(M^q)$ module. Then

$$\text{Hom}_\mathbb{R}(\left[T^q\right]_C, \mathbb{R}) = \text{Hom}_\mathbb{R}(\left[T^q\right] \otimes_{C^\infty(M^q)} \left[1\right]_C, \mathbb{R})$$

(3) $= \text{Hom}_{C^\infty(M^q)}(\left[T^q\right], \text{Hom}(\left[1\right]_C, \mathbb{R}))$

$= \text{Hom}_{C^\infty(M^q)}(\left[T^q\right], \mathcal{D}(M^q)) \cong \mathcal{D}(M^q) \otimes_{C^\infty(M^q)} \left[T^q*\right]_C$. 

Let $\Sigma_q$ act on $M^q$ by permuting factors $\sigma(x_1, \ldots, x_q) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(q)})$.

This induces an action on $C^\infty(M^q)$ and by duality on $\mathcal{D}(M^q)$. Let $\Sigma_q$ act on $T^q*$ by permuting factors and multiplying by $e_\sigma$, then for $\omega_1 \otimes \cdots \otimes \omega_q \in [T^q*]$, $\xi_1 \otimes \cdots \otimes \xi_q \in \left[T^q\right]_C$ and $u \in \mathcal{D}(M^q)$,

$$\sigma(u \otimes \omega_1 \otimes \cdots \otimes \omega_q)[\xi_1 \otimes \cdots \otimes \xi_q] = e_\sigma(u \otimes \omega_{\sigma^{-1}(1)} \otimes \cdots \otimes \omega_{\sigma^{-1}(q)})[\xi_1 \otimes \cdots \otimes \xi_q]$$

(4) $= e_\sigma(u)[\langle \omega_{\sigma^{-1}(1)}, \xi_1 \rangle_{x_1} \otimes \cdots \otimes \langle \omega_{\sigma^{-1}(q)}, \xi_q \rangle_{x_q}]$

$= e_\sigma u[\langle \omega_{\sigma^{-1}(1)}, \xi_1 \rangle_{x_1} \otimes \cdots \otimes \langle \omega_{\sigma^{-1}(q)}, \xi_q \rangle_{x_q}]$

Therefore

(4) $= e_\sigma u \otimes \omega_1 \otimes \cdots \otimes \omega_q[\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(q)}]$. 

Therefore

$$\left(\mathcal{D}(M^q) \otimes_{C^\infty(M^q)} \left[T^q*\right]_C\right)^{\Sigma_q} \cong C^q(L).$$
To compute the cohomology of $C^q(L)$ we use the spectral sequence defined as follows. Let $\mathcal{D}(M^q)|_{M^q}$ be the distributions with support on the subset $M^q_k = \{(x_1, \ldots, x_q)\}$ at most $k$ of the points $x_i \in M$. Set

$$C_k^q(L) = (\mathcal{D}(M^q)|_{M^q_k} \otimes C^\infty(M^q)[T^q \ast])^\ast,$$

then $C_k^q(L) \subset C_{k+1}^q(L)$ and $d^2 C_k^q(L) \subset C_k^{q+1}(L)$. If we define $F^{-k}C^q = C_k^q$ we have a decreasing filtration preserved by the differential and thus a cohomology spectral sequence.

Note that $M^q_k$ is a union of submanifolds. In fact if $S$ is a partition of $q$ elements into $k$ sets, let $M^q_S$ be the set of points in $M^q$ consisting of $(x_1, \ldots, x_q)$ such that if $i, j$ are in the same subset of the partition then $x_i = x_j$. There is an obvious diffeomorphism of $M^k$ and $M^q_S$, and $M^q_S = \bigcup_S$ a partition of $kM^q_S$. Any element of $\mathcal{D}(M^q)|_{M^q_S}$ can be written as a sum of normal derivatives of distributions on $M^q_S$, see Schwartz [4]. P. Trauber in his Princeton thesis [6] has used the isomorphism (4) and this fact to give a nice description of the $E_0$ term of the spectral sequence and then applied the methods of relative homological algebra to compute $E_1$. We summarize his results below, making the obvious extension to the case of compactly supported vectorfields. Let $D(M)$ be the differential operators on $M$, not necessarily of finite order, topologized as follows. For $U$ a precompact open subset of $M$, let $D^k(U)$ be the differential operators of at most order $k$ on smooth functions with support in $U$. As sections of a vector bundle $D^k(U)$ has a nuclear locally convex topology and so the inductive limit $D(U) = \lim_k D^k(U)$ does also. For $U \subset V$ there is a restriction map $D(V) \to D(U)$ and the precompact open subsets of $M$ together with these restriction maps form a directed system. Let $D(M) = \lim_{U \subset M} D(U)$, as a projective limit of nuclear spaces it is a nuclear space. If we use the cofinal family $U^q = U \times \cdots \times U$ ($q$ times) of precompact open sets on $M^q$ to define the topology on $D(M^q)$, then because

$$D^k(U^q) \cong D^k(U) \otimes \cdots \otimes D^k(U)$$

and $\otimes$ is an exact functor we have $D(U^q) \cong D(U) \otimes \cdots \otimes D(U)$ and $D(M^q) \cong D(M) \otimes \cdots \otimes D(M)$. Similarly $[T^q \ast] \cong [T^\ast] \otimes \cdots \otimes [T^\ast]$. Let $D(M^q)|_{M^q_S}$ be the differential operators $C^\infty_0(M^q) \to C^\infty_0(M^q_S)$. Composition on the left defines a left $D(M^q_S)$ module structure on $D(M^q)|_{M^q_S}$ and $C^\infty(M^q_S) \subset D(M^q_S)$. Relative to these structures we have the following

**Proposition (Trauber [6]).**

(a) $\mathcal{D}(M^q)|_{M^q_S} \cong \mathcal{D}(M^q_S) \otimes_{D(M^q_S)} D(M^q)|_{M^q_S}$,

(b) $D(M^q)|_{M^q_S} \cong C^\infty(M^q_S) \otimes_{C^\infty(M^q)} D(M^q)$,

where the $C^\infty(M^q)$ module structure on $C^\infty(M^q_S)$ is restriction followed by multiplication. Using these isomorphisms we have
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\[ \mathcal{V}(M^q)_{M^q_3} \otimes_{C^\infty(M^q)} [T^q^*] \]
\[ \cong \mathcal{V}(M^q_3) \otimes_{D(M^q_3)} C^\infty(M^q_3) \otimes_{C^\infty(M^q)} D(M^q) \otimes_{C^\infty(M^q)} [T^q^*] \]
\[ \cong \mathcal{V}(M^q_3) \otimes_{D(M^q_3)} C^\infty(M^q_3) \otimes_{C^\infty(M^q)} (D(M) \otimes \cdots \otimes D(M)) \]
\[ \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} ([T^*] \otimes \cdots \otimes [T^*]) \]
\[ \cong \mathcal{V}(M^q_3) \otimes_{D(M^q_3)} C^\infty(M^q_3) \otimes_{C^\infty(M^q)} D(M) \otimes_{C^\infty(M)} [T^*] \]
\[ \otimes \cdots \otimes D(M) \otimes_{C^\infty(M)} [T^*]. \]

Let $D \otimes T^* = D(M) \otimes_{C^\infty(M)} [T^*]$ and let $X$ be the elements of positive degree in the exterior algebra over $C^\infty(M)$ of $D \otimes T^*$ let $X^k = X \otimes \cdots \otimes X$ ($k$ times) and let $X^k(q)$ be the subspace of $X^k$ consisting of elements with $q$ factors of $T^*$. Trauber proves the following

**THEOREM (Trauber [6]).**

(a) $C^k_q(L) \cong (\mathcal{V}(M^q))_{M^q_3} \otimes_{C^\infty(M^q)} [T^q^*]) \cong (\mathcal{V}(M^k_3) \otimes_{D(M^k)} X^k(q))^{\Sigma^k}$,

(b) $\frac{F^{-k}C^*(L)}{F^{-k+1}C^*(L)} \cong \left( \mathcal{V}(M^k_3) \otimes_{D(M^k)} X^k_1 \right)^{\Sigma^k}.$

He also points out the following interpretation of the isomorphism (a).

Let $J^k(T)$ be the bundle of $k$-jets on $M$, for $U$ a precompact open set let $[J^k(T)]_U$ be the sections with support in $U$, this is a Fréchet nuclear space. Define $[J^\infty(T)]_C = \lim_{\rightarrow U} \lim_{\leftarrow U} [J^k(T)]_U$. This is a nuclear l.c.s. such that

\[ D \otimes T^* = \text{Hom}_{C^\infty(M)}([J^\infty(T)]_C, C^\infty(M)). \]

There is a continuous function $j^\infty: [T]_C \rightarrow [J^\infty(T)]_C$ which associates to any compactly supported vectorfield its infinite jet at each point. The bundle $J^\infty(T)$ has a canonical connection $\nabla: [J^\infty(T)]_C \rightarrow [T^* \otimes J^\infty(T)]_C$ introduced by Spencer, see [2]. If $\widetilde{\xi} \in [J^\infty(T)]_C$ then $\widetilde{\xi} = j^\infty(\xi)$ for some $\xi \in [T]_C$ if and only if $\nabla \widetilde{\xi} = 0$ in $[T^* \otimes J^\infty(T)]_C$. The connection $\nabla$ has 0 curvature and thus gives a representation of $D(M)$ on $[J^\infty(T)]_C$. The image of $j^\infty$ is the subspace of $D(M)$ invariants in $[J^\infty(T)]_C$. Using the isomorphism $D(M^q) \cong D(M) \otimes \cdots \otimes D(M)$ we get a representation of $D(M^q)$ on $[J^\infty(T)]_C \otimes \cdots \otimes [J^\infty(T)]_C$.

(2) For any vector bundle $E$ with connection $\nabla: E \rightarrow T^* \otimes E$ we write $\nabla_X$ for the germ of a differential operator $(\nabla_X)(\rho) = (\nabla \rho)(X)$ is a vector field $X \in \mathfrak{X}_p$ and $S \in E_p$. If $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0$ we say the connection has curvature zero and we get a Lie algebra representation of $[T]_C \rightarrow [\text{Diff } E] = \text{differential operators on } E$. This extends to a representation $D(M) \rightarrow [\text{Diff } E]$. 


which we will also denote by \( \nabla \) also. For \( \xi_1 \otimes \cdots \otimes \xi_q \in [\mathcal{U}^\infty(T)]_C \otimes \cdots \otimes [\mathcal{U}^\infty(T)]_C \) and \( \eta_1 \otimes \cdots \otimes \eta_q \in D(M) \otimes \cdots \otimes D(M), \)
\[
\nabla_{\eta_1 \otimes \cdots \otimes \eta_q} \xi_1 \otimes \cdots \otimes \xi_q = \nabla_{\eta_1} \xi_1 \otimes \cdots \otimes \nabla_{\eta_q} \xi_q.
\]

Now \( L_C : [\mathcal{U}^\infty(T)]_C \to [\mathcal{U}^\infty(T)]_C \) is a Lie algebra map; therefore there is a cochain map \( C^q([\mathcal{U}^\infty(T)]_C) \to C^q(L) \) which is the same as

\[
\mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [\mathcal{U}^\infty(T)]_C \otimes \cdots \otimes [\mathcal{U}^\infty(T)]_C \xrightarrow{(j^\infty)^*} \mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [T^q]^* \]

or equivalently

\[
\mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} D \otimes T^* \otimes \cdots \otimes D \otimes T^*
\]

\[
(6) \quad \xrightarrow{(j^\infty)^*} \mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [T^q]^*.
\]

Since the image of \( j^\infty \) is the subspace of \( D(M) \) invariants it is not hard to see that \( (j^\infty)^* \) factors through the tensor product over \( D(M^q) \) to give an isomorphism

\[
\mathcal{U}'(M^q) \otimes_{D(M^q)} D \otimes T^* \otimes \cdots \otimes D \otimes T^* \to \mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [T^q]^*.
\]

This allows us to identify the differential on the complex \( X \) appearing in the previous theorem: \( X \) is the exterior algebra on \( [\mathcal{U}^\infty(T)]_C \) and the differential \( d_X \) on \( X \) is the usual coboundary operator in the cochain complex on the dual of a Lie algebra. We can restate the previous theorem

\[
(\mathcal{U}'(M^k) \otimes_{D(M^k)} \Lambda^+ [\mathcal{U}^\infty(T)]_C \otimes \cdots \otimes \Lambda^+ [\mathcal{U}^\infty(T)]_C)^{\Sigma k}
\]

\[
(7) \quad \cong F^{-k}C^\infty(L)/F^{-k+1}C^\infty(L)
\]

as cochain complexes with the isomorphism induced by \( (j^\infty)^* \).

To compute \( H^*(F^{-k})/F^{-k+1}C^\infty(L) \) we note that \( X^k \) is flat as a \( D(M^k) \) module since \( X = \Lambda^+ D \otimes T^* \) is flat as a \( D \) module in each degree of the exterior power. Therefore the higher derived functors of \( \otimes_{D(M^k)} X^k \) in the category of differential complexes vanish.

\[
\text{Tor}_p^{D(M^k)}(A, X^k) = 0, \quad p > 0,
\]

\[
(8) \quad \text{Tor}_0^{D(M^k)}(A, X^k) = H^*(A \otimes_{D(M^k)} X^k, d_{X^k}).
\]

However we can also compute the differential derived functor by resolving \( X^k \).

Let \( Y_p = D(M^k) \otimes \Lambda^p [T(M^k)] \) define \( \partial_p : Y_p \to Y_{p-1} \) by

\[
\partial_p(u \otimes \xi_1 \wedge \cdots \wedge \xi_p) = \sum_i (-1)^{i-1} u \xi_i \otimes \xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_p
\]

\[
\cdot \sum_{ij} (-1)^{i+j} u \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_j
\]

\[
\wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_p.
\]
Then $Y = \bigoplus Y_p$ gives a resolution of $C^\infty(M^k)$ as a left $D(M^k)$ module and tensoring on the right over $C^\infty(M^k)$ with $X^k$ we get a resolution:

$$
D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k
$$

(9)

$$
\epsilon_0
\downarrow
X^k
$$

Let $A$ be a right $D(M^k)$ module then tensoring on the left over $D(M^k)$ with $A$

$$
A \otimes_{C^\infty(M^k)} \Lambda^*[T(M^k)] \otimes_{C^\infty(M^k)} X^k
$$

(10)

$$
id \otimes \epsilon_0
\downarrow
A \otimes_{D(M^k)} X^k
$$

as an augmented complex with homology (making $X^k$ a chain complex using negative indexing) equal to

$$
\text{Tor}^D(M^k)(A, X^k) = H_*(A \otimes_{D(M^k)} X^k).
$$

Computing the $\partial$ spectral sequence of the double complex we have

$$
E^1_{p,-q} \cong A \otimes \Lambda^p[T(M^k)] \otimes_{C^\infty(M^k)} H^{-q}(X^k).
$$

Here we need an additional fact. Let $L$ be the algebra of formal power series vectorfields, i.e., the fiber of $J^\infty(T)$ over a point of $M$, $L = \lim_k J^k(T)$. Let $L^* = \lim J^k(T)^*$, then $H(X) \cong C^\infty(M) \otimes_R H(\Lambda^+L^*)$ and the $D(M)$ module structure on $H(X)$ is trivial, see [5] or [1a, pp. 205–206]. Therefore, we have

$$
H(X^k) \cong C^\infty(M^k) \otimes H(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)
$$

with $D(M^k)$ acting trivially. Hence

$$
E^2_{p,-q} \cong H(A \otimes_{C^\infty(M^k)} \Lambda[T(M^k)]) \otimes_R H(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*),
$$

(11)

$$
E^\infty_{p} \cong \text{Gr}_H(A \otimes_{D(M^k)} X^k).
$$

Let $\mathcal{U}(M^k - M^k_{k-1})$ be the distributions on $M^k - M^k_{k-1}$ which extend to distributions on $M^k$. The inclusion $i: C^\infty_0(M^k - M^k_{k-1}) \to C^\infty_0(M^k)$ induces an isomorphism

$$
\mathcal{U}(M^k)/\mathcal{U}(M^k)|_{M^k_{k-1}} \cong \mathcal{U}(M^k - M^k_{k-1}).
$$

Since $\mathcal{U}(M^k - M^k_{k-1})$ is dense in $\mathcal{U}(M^k - M^k_{k-1})$ and $\mathcal{U}(M^k - M^k_{k+1}) \otimes_{C^\infty(M^k)}$
\( \Lambda[T(M^k)] \) is dual to \( \Omega_c(M^k - M_{k-1}^k) \) the de Rham complex of compactly supported differential forms we have a nondegenerate pairing

\[
\mathcal{D}(M^k - M_{k-1}^k) \otimes C^\infty(M^k) \Lambda^p[T(M^k)] \times \Omega_c^c(M^k - M_{k-1}^k) \rightarrow \mathbb{R}.
\]

Moreover, the differential \( \partial_p \) on the left factor is dual to the de Rham differential. Thus if \( A = \mathcal{D}(M^k)/\mathcal{D}'(M_{k-1}^k) \)

\[
H_p(A \otimes C^\infty(M^k) \Lambda[T(M^k)]) \cong H^p_c(M^k - M_{k-1}^k)^*.
\]

Putting all this together we conclude

**Theorem 1.** Let \( F^{-k}C^*(L)/F^{-k+1}C^*(L) \) be considered as a chain complex using negative indexing; then there is a homology spectral sequence with

\[
E_2^{p-q} = (H^p_c(M^k - M_{k-1}^k))^* \otimes H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*))^{\Sigma k}
\]

and

\[
E^{\infty}_{p-q} = G\text{r}_p(H^{q-p}(F^{-k}/F^{-k+1})).
\]

In the special case when \( M = \mathbb{R}^n \) we have \( X \cong C^\infty(M) \otimes_R \Lambda^+L^* \) and \( X^k \cong C^\infty(M^k) \otimes_R \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^* \). This gives the following isomorphism

\[
\mathcal{D}(M^k - M_{k-1}^k) \otimes C^\infty(M^k) \Lambda[T(M^k)] \otimes C^\infty(M^k) X^k
\]

\[
\cong \mathcal{D}(M^k - M_{k-1}^k) \otimes C^\infty(M^k) \Lambda[T(M^k)] \otimes C^\infty(M^k) C^\infty(M^k)
\]

\[
\otimes_R \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*
\]

\[
\cong (\mathcal{D}'(M^k - M_{k-1}^k) \otimes C^\infty(M^k) \Lambda[T(M^k)]) \otimes_R (\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*).
\]

One can apply the Kunneth theorem to the latter complex, therefore its homology is

\[
H(\mathcal{D}'(M^k - M_{k-1}^k) \otimes \Lambda[T(M^k)]) \otimes_R H^* (\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)
\]

and we conclude that \( E^2 = E^{\infty} \).

**Theorem 2.** If \( L \) is the Lie algebra of compactly supported vectorfields on \( \mathbb{R}^n \), then with respect to the filtration defined earlier there is a spectral sequence with

\[
E^{-k,l+k}_1 = H^l \left( \frac{F^{-k}C^*(L)}{F^{-k+1}C^*(L)} \right)
\]

\[
\cong \bigoplus_{q-p=l} [H^p_c((\mathbb{R}^n)^k - (\mathbb{R}^n)^{k-1})^* \otimes_R H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)]^{\Sigma k}.
\]
We will give an explicit expression for this isomorphism and show that the spectral sequence collapses at $E_1$.

When $M = \mathbb{R}^n$ we can find a global basis $[T(M^k)]$ as a $C^\infty(M^k)$ module which consists of commuting vectorfields; then

$$[T(M^k)] \cong C^\infty(\mathbb{R}^{nk}) \otimes \mathbb{R}^{nk}, \quad \Lambda[T(M^k)] \cong C^\infty(\mathbb{R}^{nk}) \otimes_R \Lambda\mathbb{R}^{nk}.$$  

Let $\tilde{X}^k = C^\infty(\mathbb{R}^{nk}) \otimes \Lambda(L \oplus \cdots \oplus L)^*$, i.e., the full exterior algebra. It is clear that $X^k$ is a direct summand of $\tilde{X}^k$ as a $D(M^k)$ module. Let $j$ be the inclusion and $\pi$ the projection $X^k \xrightarrow{j} \tilde{X}^k \xrightarrow{\pi} X^k$. Both $i$ and $\pi$ are cochain maps. Since $L \cong \mathbb{R}^n \oplus L^0$ we have $L \oplus \cdots \oplus L \cong \mathbb{R}^{nk} \oplus L^0 \oplus \cdots \oplus L^0$ and there is an obvious interior product $\Lambda\mathbb{R}^{nk} \otimes_R \tilde{X}^k \to \tilde{X}^k$. Using the isomorphisms given above we get a map

$$\tilde{i} : \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} \tilde{X}^k \to \tilde{X}^k.$$  

Composing on the right with $\text{id} \otimes j$ and on the left with $\pi$ we get

$$i : \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \to X^k$$  

which we will denote

$$i : \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha \longmapsto \xi_1 \wedge \cdots \wedge \xi_p \perp \alpha.$$  

Tensoring on the left over $C^\infty(M^k)$ with $D(M^k)$

$$\text{id} \otimes i : D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \to D(M^k) \otimes_{C^\infty(M^k)} X^k.$$  

Composition with the left module structure on $X^k$ with $D(M^k) \otimes X^k \to X^k$ gives

$$\psi : D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \to X^k,$$

$$u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha \longmapsto u(\xi_1 \wedge \cdots \wedge \xi_p \perp \alpha).$$  

We will show that $\psi$ is a cochain map. Passing to $\Sigma_k$ invariants we get an explicit isomorphism for the $E_1$ term of the spectral sequence given in the previous theorem.

The map $\psi$ is defined with respect to a fixed parallelisation of $T(M^k)$, with respect to which we have

$$D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \cong D(\mathbb{R}^{nk}) \otimes_R \Lambda\mathbb{R}^{nk} \otimes_R \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*.$$  

The differential is given by

$$d(u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) = \sum (-1)^{j-1} u\xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \otimes \alpha$$

$$+ (-1)^p u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes d_L \alpha$$
where \( d_L \) is the differential in \( \Lambda L^* \otimes \cdots \otimes \Lambda L^* \),

\[
d\psi(u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) \\
= d(u(\xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha)) = ud(\xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) \\
= u \left( \sum (-1)^{j-1} (\xi_1 \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \xi_p \otimes \text{ad} \, \xi \alpha) \\
+ (-1)^p (\xi_1 \wedge \cdots \wedge \xi_p \otimes d_L \alpha) \right).
\]

By definition \( \text{ad} \) is the adjoint representation of \( L \otimes \cdots \otimes L \) on \( \Lambda(L \otimes \cdots \otimes L)^* \) dual to the adjoint representation of \( L \otimes \cdots \otimes L \) on \( \Lambda(L \otimes \cdots \otimes L) \). For \( \alpha \in \Lambda(L \otimes \cdots \otimes L)^* \) and \( \xi_1, \ldots, \xi_p \in \mathbb{R}^{nk} \) we have \( \text{ad} \, \xi \alpha = \xi \cdot \alpha \) where \( \cdot \) indicates the module structure and \( \xi \) are considered as constant coefficient differential operators. Furthermore \( (\xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_p \otimes \text{ad} \, \xi \alpha) = \text{ad} \, \xi_i(\xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_p \otimes \alpha) \), thus \( \psi \) is a cochain map.

We can represent the induced map on cohomology

\[
[H^p_c(\mathbb{R}^{nk} - (\mathbb{R}^n)_{k-1}^c) \otimes \mathbb{R} \Lambda^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*))]^{\Sigma k} \rightarrow H^{q-p}(F^{-k}/F^{-k+1})
\]

more conveniently as follows. For \( \eta \in \mathcal{L}, j^\infty(\eta) \in C_0^\infty(M) \otimes \mathcal{L} \) so if \( \alpha \in \Lambda L^* \) we can form \( j^\infty(\eta) \otimes \alpha \in C_0^\infty(M) \otimes \Lambda L^* \). For \( \alpha = \sum a_1^j \otimes \cdots \otimes a_k^j \in \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^* \) and for \( S \) a partition \((a_1, \ldots, a_q)(a_1, \ldots, b_{s_2}) \cdots (c_1, \ldots, c_{s_k})\) of \( q \) into \( k \) sets it makes sense to partition a set of \( q \) vectorfield \( \eta_1, \ldots, \eta_q \) into \( \eta_{a_1}, \ldots, \eta_{a_{s_1}}, \ldots, \eta_{b_1}, \ldots, \eta_{b_{s_2}}, \ldots, \eta_{c_1}, \ldots, \eta_{c_{s_k}} \) and form

\[
\sum_i (f^\infty(\eta_{a_1}^i) \wedge \cdots \wedge f^\infty(\eta_{a_{s_1}}^i) \otimes \alpha_i^1) \wedge (f^\infty(\eta_{b_1}^i) \wedge \cdots \wedge f^\infty(\eta_{b_{s_2}}^i) \otimes \alpha_i^2) \\
\wedge \cdots \wedge (f^\infty(\eta_{c_1}^i) \wedge \cdots \wedge f^\infty(\eta_{c_{s_k}}^i) \otimes \alpha_i^k).
\]

We will write \( j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_q) \otimes \alpha \) to mean the interior product just defined. Let \( i: \mathbb{R}^{nk} \rightarrow L \oplus \cdots \oplus L \) be the injection defined earlier and \( \Lambda(L \oplus \cdots \oplus L)^* \overset{i^*}{\rightarrow} \Lambda \mathbb{R}^{nk}^* \) the extension of the dual map to exterior algebras. Let \( \phi \) be the isomorphism

\[
C_0^\infty(\mathbb{R}^{nk}) \otimes \Lambda \mathbb{R}^{nk}^* \overset{\phi}{\rightarrow} \Omega_c(\mathbb{R}^{nk})
\]

given by the choice of a parallelism. Finally for \( S \), the partition above, let \( \varepsilon_S \) be the sign of the permutation

\[
(1 \cdots S_1 \cdots k - S_k + 1 \cdots k)
\]

\[
a_1 \cdots a_{S_1} \cdots c_1 \cdots c_{S_k}.
\]

Then for \( \lambda \in \mathcal{F}(\mathbb{R}^{nk} - (\mathbb{R}^n)_{k-1}^c) \otimes \Lambda^p [T(M^k)] \alpha \in (\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)^q \) we have
$\psi(\lambda \otimes \alpha)(\eta_1, \ldots, \eta_{q-p})$

(14) $= \sum \varepsilon_{S^k} [\phi^* (f^\omega(\eta_1) \wedge \cdots \wedge f^\omega(\eta_{q-p})) \cup S^k \alpha]$

and

(15) $\psi(d\lambda \otimes \alpha) + (-1)^{q-p} \psi(\lambda \otimes d_L \alpha) = d_0(\psi(\lambda \otimes \alpha))$

where $d$ is the differential in $\mathcal{F}(M^k - M^{k-1}) \otimes \Lambda[T(M^k)]$, $d_L$ is the differential in $\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*$ and $d_0$ is the differential in $F^k C^*(L)/F^{-k+1} C^*(L)$.

Let $\eta \in \mathbb{R}^n$ and $(\eta_1, \ldots, \eta_k) \in \mathbb{R}^{nk}$ and let $R^k_{(i,j)} = \{(\eta_1, \ldots, \eta_k) | \eta_i = v_j\}$ then $(R^n)^k_{k-1} = \bigcup_{i<j} R^k_{(i,j)}$. Let $R^k \cup \{\infty\} = S^k$ and $R^k_{(i,j)} \cup \{\infty\} = S^k_{(i,j)}$, then

$$H_c^p(R^k - (R^n)^k_{k-1}) = H_c^p \left( \bigcup_{i<j<k} R^k_{(i,j)} \right)$$

$$= H_c^p \left( \bigcup_{i<j<k} S^k_{(i,j)} \right)$$

$$\cong H^p \left( S^k, \bigcup_{i<j<k} S^k_{(i,j)} \right).$$

Hence

$$H_c^p(R^k - (R^n)^k_{k-1})^* \cong H^p \left( S^k, \bigcup_{i<j<k} S^k_{(i,j)} \right)$$

and composing these isomorphisms with $\psi$ we have

$$\Phi: \left( H^p \left( S^k, \bigcup_{i<j<k} S^k_{(i,j)} \right) \otimes H^q(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*)^p \right) \rightarrow E^{-k,q-p+k}$$

(16)

For $\Sigma_{i=1}^m [\sigma_i] \otimes [\alpha_i]$ an element of the left-hand side if we choose representative cycles $\sigma_i$ and representative cocycles $\alpha_i$ we get a representative element of $\Phi(\Sigma_{i=1}^m [\sigma_i] \otimes [\alpha_i])$.

$$(\eta_1, \ldots, \eta_{q-p})$$

(17) \rightarrow \sum_{j=1}^m \sum_{\text{partitions}} \varepsilon_{S^k} \phi^* (f^\omega(\eta_1) \wedge \cdots \wedge f^\omega(\eta_{q-p}) \cup S^k \alpha_j).$

If we pull back $d_1: E^{-k,h+k} \rightarrow E^{-k+1,h+k}$ by the isomorphism $\Phi$ we get a mapping for $q - p = h$.
\[
\left[ H_p \left( S^{n_k}, \bigcup_{i<j<k-1} S^{n_{k-n}}_{i,j} \right) \otimes H^q(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^* ) \right]^{\Sigma_k}_{k-1}
\]

It is computed as follows. For \( \eta_1, \eta_2, \ldots, \eta_{n+1} \in L, \)
\[
\Phi \left( \sum_{i=1}^m [\alpha_i] \otimes [\alpha_i] \right)(\eta_1, \ldots, \eta_{n+1})
\]
\[
= \sum_{i<j<h+1} (-1)^{i+j} \Phi \left( \sum_{i=1}^m [\alpha_i] \otimes [\alpha_i] \right)
\]
\[
= \sum_{i<j<h+1} \sum_s \int_{\sigma_1} \epsilon_S \phi^{*}(\phi^n([\eta_i, \eta_j]) \wedge f^n(\eta_1) \wedge \cdots \wedge f^n(\eta_l) \wedge \cdots \wedge f^n(\eta_{n+1}) \wedge S \alpha_i)
\]
\[
= \sum_{i<j<h+1} \sum_{i=1}^m \sum_s \int_{\sigma_1} \epsilon_S \phi^{*}([f^n(\eta_i), f^n(\eta_j)]) \wedge f^n(\eta_1) \wedge \cdots \wedge f^n(\eta_l) \wedge \cdots \wedge f^n(\eta_{n+1}) \wedge S \alpha_i
\]

Now \( \alpha_i \) is a tensor product of \( k \) cycles \( \alpha_{i,j} \in Z(\Lambda^+ L^*). \) To compute the last term we see what is happening to each \( \alpha_{i,j}. \) For \( \alpha \in Z^t(\Lambda L^*) \) and \( \eta_1, \ldots, \eta_s \in L \)
\[
\sum_{i<j<s} \phi^n([f^n(\eta_i), f^n(\eta_j)]) \wedge f^n(\eta_1) \wedge \cdots \wedge f^n(\eta_l) \wedge \cdots \wedge f^n(\eta_{s+1}) \wedge S \alpha
\]
\[
= \sum_{i<j<s} \sum_{i_1<j_2<\cdots<i_{t-s}<n} (-1)^{i+j} \alpha([f^n(\eta_i), f^n(\eta_j)], f^n(\eta_1) \cdots f^n(\eta_l) \cdots f^n(\eta_s), e_{t_1} \cdots e_{t-s})
\]
\[
dx_{i_1} \wedge \cdots \wedge dx_{i_{t-s}}
\]
This shows what happens to each factor of $\alpha_i$; hence the end product is

$$\Phi(\bar{\alpha}_1 \sum [\alpha_1] \otimes [\alpha_1])(\eta_1, \ldots, \eta_{n+1})$$

$$= \sum_i \sum_{S'} \int_{\partial \sigma_i} e_S^* d\phi^*(\pi^*_i \rangle \cdots \rangle \pi^*_i) \cup_S^* \alpha$$

$$= \sum_i \sum_{S'} \int_{\partial \sigma_i} e_S^* \phi^*(\pi^*_i \rangle \cdots \rangle \pi^*_i) \cup_S^* \alpha$$

where $S'$ ranges over partitions of $h + 1$ elements into $k$ sets. We can decompose $\partial \sigma_i$ into a sum of $\partial_{(i,j)} \sigma_i$ where $|\partial_{(i,j)} \sigma_i| \subset S_{(i,j)}^{k-n}$. When $\phi^*(\pi^*_i \rangle \cdots \rangle \pi^*_i) \cup_S^* \alpha$ is integrated over $S_{(i,j)}^{k-n}$, the $i$th and $j$th factors are identified by restricting to the diagonal in the product of the $i$th and $j$th factors. This gives a mapping

$$H(L^+* \otimes \cdots \otimes L^+*) \cong H(L^+* \otimes \cdots \otimes H(L^+*))$$

by multiplying the $i$th and $j$th factors, just as restriction to the diagonal induces the cup product in singular cohomology. Therefore the $\bar{\alpha}_1$ operator involves multiplication in the cohomology algebra of the formal Lie algebra. It is known that this multiplication is trivial [5], [7], so $\bar{\alpha}_1 = 0$. In a similar way one can see that all the higher differentials involve multiplication in the formal algebra so we have

**Theorem 3.** There is a spectral sequence for the continuous cohomology of the algebra of compactly supported vectorfields on $\mathbb{R}^n$ which collapses at the $E_1$ level.

$$E^{-k,l+k} \cong \bigoplus_{q-p-l} H_p\left(S^{nk}, \bigcup_{i<j} S_{i,j}^{nk-n} \right) \otimes \bigotimes_{p=1}^k \hat{H}^*(L) \otimes \Sigma^k.$$
Let $L$ be the algebra of vectorfields on the $n$ sphere $S^n$, let $p \in S^n$ and let $\widetilde{L}$ be the ideal of vectorfields flat at $p$ in some, hence any, coordinate system. Let $C^*(L)$ be the Gelfand-Fuks complex for the continuous cohomology of $L$, and define a filtration

$$F^kC^q(L) = \{ \lambda \in C^q(L) \mid \lambda(\xi_1, \ldots, \xi_q) = 0 \text{ if } q - k + 1 \text{ of } \xi_i \text{ are in } \widetilde{L} \},$$

then $F^k \supset F^{k+1}$ and $dF^k \subset F^k$. This is the filtration defining the Hochschild-Serre spectral sequence for $H(L)$ with respect to the ideal $\widetilde{L}$.

$$E_2^{p,q} = H^p(L/\widetilde{L}, H^q(\widetilde{L})), \quad E_\infty^{p,q} = Gr_p(H^{p+q}(L)).$$

There is an exact sequence of Lie algebras

$$0 \rightarrow \widetilde{L} \rightarrow L \rightarrow L \rightarrow 0.$$ 

Thus $E_2^{p,q} \cong H^p(L, H^q(\widetilde{L}))$. The action of $L$ on $H^q(\widetilde{L})$ is defined as follows: for $\eta \in L$ let $\tilde{\eta} \in \widetilde{L}$ be a vectorfield such that $i^\infty(\tilde{\eta})_p = \eta$ then Lie derivation with respect to $\tilde{\eta}$ defines a map $D_{\tilde{\eta}}: \widetilde{L} \rightarrow \widetilde{L}$ which in turn defines a cochain map $D_{\tilde{\eta}}: C^*(\widetilde{L}) \rightarrow C^*(\widetilde{L})$ and therefore a map $D_{\tilde{\eta}}: H^*(\widetilde{L}) \rightarrow H^*(\widetilde{L})$. If $i^\infty(\tilde{\eta}_1)_p = i^\infty(\tilde{\eta}_2)_p$ then $\tilde{\eta}_1 - \tilde{\eta}_2 \in \widetilde{L}$ and as is well known $D_{\tilde{\eta}_1 - \tilde{\eta}_2}$ induces the trivial map in cohomology, so $D_{\tilde{\eta}_1} = D_{\tilde{\eta}_2}$. Reshetnikov [3] has stated the following theorem for arbitrary $M$ but it is not clear to us that his proof is correct.

**Theorem.** Since $L$ acts trivially on $H^*(\widetilde{L})$, the $E_2$ term of the previous spectral sequence is $E_2^{p,q} \cong H^p(L) \otimes H^q(\widetilde{L})$. Furthermore if $L_C$ is the algebra of compactly supported vectorfields on $\mathbb{R}^n$, then $H^q(\widetilde{L}) \cong H^q(L_C)$.

**Proof.** Let $\{U_i\}$ be a decreasing sequence of open sets which form a neighborhood basis at $p$. Let $K_i = S^n - U_i$; then $K$ is compact, and if we define $\phi: S^n - \{p\} \rightarrow \mathbb{R}^n$ by stereographic projection with $p$ as north pole then the $\phi(K_i)$ form a compact exhaustion of $\mathbb{R}^n$. Let $L_i$ be the algebra of vectorfields on $S^n$ with support in $K_i$, there are inclusions $\iota^i_j: L_j \rightarrow L_i$; therefore, we can define $L_\infty = \lim L_i$. Clearly $L_\infty \cong L_C$, compactly supported vectorfields. Let $\psi^i: L_i \rightarrow \widetilde{L}$ be the inclusion; then $\psi^i \cdot \iota^i_j = \psi^j$ so we can define $\psi: L_\infty \rightarrow L$. This induces $\psi^*: H(\widetilde{L}) \rightarrow H(L_\infty)$. For $\eta \in L$ let $\tilde{\eta}_i \in \widetilde{L}$ be vectorfield such that $i^\infty(\tilde{\eta}_i)_p = \eta$ and $\text{supp } \tilde{\eta}_i \subset U_i$; then for $\lambda \in H^*(\widetilde{L})$ we have $\eta \cdot [\lambda] = [D_{\tilde{\eta}_i}\lambda]$ for any $i$. Clearly $\psi^* [D_{\tilde{\eta}_i}\lambda] = 0$ and from the fact that $\psi^* \eta \cdot [\lambda] = 0$ if and only if $(\psi^i)^* \eta \cdot [\lambda] = 0$ for all $i$ we conclude $\psi^* \eta \cdot [\lambda] = 0$. To conclude the proof it is sufficient to show that $\psi^*$ is injective. In fact, $\psi^*$ is an isomorphism. To see this, look at the spectral sequences defined at the beginning of the paper. Since $\widetilde{L}$ can be thought of as rapidly decreasing vectorfields on $\mathbb{R}^n$, the space that arises in defining $C^*(\widetilde{L})$ is $S'(\mathbb{R}^n)$. From this observation we see that the spectral sequence converging to $H^*(F^{-k}C^*(\widetilde{L})/F^{-k+1}C^*(\widetilde{L}))$, which is $E_1$ of
another spectral sequence, has \( E^2_{p, -q} \):

\[
[H_p(S'(R^{n^k})/S'(R^{n^k}))_{(R^n)^{k-1}} \otimes \Lambda[T(R^{n^k})]) \otimes H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)]^{\Sigma k}.
\]

We can identify the factor on the left from the following exact sequences (see Schwartz [4]). If \( p \) is the north pole of \( S^{n^k} \),

\[
\begin{align*}
0 &\rightarrow E'(S^{n^k})_p \rightarrow E'(S^{n^k}) \rightarrow S'(R^{n^k}) \rightarrow 0 \\
0 &\rightarrow E'(S^{n^k})_p \rightarrow E'(S^{n^k}) | \bigcup_{i \leq j \leq k} S^{n_{k-n}}_{(i,j)} \rightarrow S'(R^{n^k}) | (R^n)_{k-1} \rightarrow 0.
\end{align*}
\]

Thus

\[
H_p(S'(R^{n^k})/S'(R^{n^k}))_{(R^n)^{k-1}} \otimes \Lambda[T(R^{n^k})]) \cong H_p(S^{n^k}, \bigcup S^{n_{k-n}}_{(i,j)})
\]

and \( E^2_{p, -q} (L) \cong E^2_{p, -q} (L_\infty) \).

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