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ON THE JUMP OF AN α-RECURSIVELY ENUMERABLE SET(1)

BY

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ABSTRACT. We discuss the proper definition of the jump operator in α-recursion theory and prove a sample theorem: *There is an incomplete α-r.e. set with jump $0''$ unless there is precisely one nonhyperregular α-r.e. degree.* Thus we have a theorem in the first order language of Turing degrees with the jump which fails to generalize to all admissible α.

The jump operator has been somewhat problematical in α-recursion theory. It was not even clear for some time what the correct definition should be, while good results for any of the reasonable definitions were just not available. The main obstacles to finding both a satisfactory definition and nontrivial results were of course nonregularity and nonhyperregularity. Indeed all the reasonable definitions of α-jump agree on sets which are regular and hyperregular and some nice results are known about such sets. The best of these are in [11] where Simpson proves the following:

THEOREM (Simpson). *Let* $\mathbf{b}$ *be an α-degree* $\geqslant \mathbf{0}'$. *Then* $\mathbf{b}$ *is regular if and only if there is a regular, hyperregular α-degree* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \vee \mathbf{0}' = \mathbf{b}$.

He then uses this result to find admissible ordinals α for which Friedberg's theorem that every degree above $\mathbf{0}'$ is a jump holds by showing that for many α every degree above $\mathbf{0}'$ is regular. We should also note that, although Simpson makes a definite choice of definition for the α-jump in [11], his results do not depend on this choice as all sets produced are regular and hyperregular.

In this paper we want to make a brief case for a definition of "α-recursively enumerable in" and so of "α-jump" which is equivalent to that of [11] and investigate the α-jump of the α-r.e. degrees. In particular we will consider generalizing a theorem of Sacks in ordinary recursion theory: there is an incomplete r.e. degree whose jump is $0''$ (see [6, §16]). Thus we will still be dealing only with the α-jump of regular α-degrees (every α-r.e. degree is regular [4]). On the other hand, we will have to consider nonhyperregular α-degrees whose jump need not

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be regular. (Indeed for some admissible α, $\mathbf{0}''$ is not regular.)

In addition to forcing us to confront these problems with the α-jump this theorem is also of methodological interest in terms of priority argument constructions. Its proof in ordinary recursion theory is one of the simplest applications of the infinite injury priority method and so would seem to be a good testing ground for this method in α-recursion theory. One's hope, of course, is to apply the methods developed in [8] to prove that the α-r.e. degrees are dense to this problem as well. However difficulties arise when one attempts to carry out the proof.

As one might guess the source of the difficulties is nonhyperregularity. All goes well if α is Σ_2-admissible and so every α-r.e. degree is hyperregular. If α is not Σ_2-admissible, however, the proof breaks down. As we shall see, the difficulties are insurmountable and the theorem actually fails for some admissible α. Indeed we shall show that if $\mathbf{0}'$ is the only nonhyperregular α-r.e. degree then $A' \equiv_\alpha \mathbf{0}'$ for every α-r.e. $A <_\alpha \mathbf{0}'$. On the other hand if A is α-r.e. and nonhyperregular we shall see that $A' \equiv_\alpha \mathbf{0}''$. Thus the theorem fails just in case there is exactly one nonhyperregular α-r.e. degree. (See [9] for a characterization of these ordinals.)

We feel that the implications of these results for the infinite injury priority method are that one will have to stay entirely within the realm of the α-r.e. degrees to achieve general success in generalizing theorems by this method. On the other hand this result is the first example of a theorem about the r.e. degrees with the jump operator which is known to fail for some admissible α. The proofs also supply us with an example for all the degrees by showing that for some α's, e.g. $\aleph^L_\omega$, every incomplete α-degree $\mathbf{a}$ has jump $\mathbf{0}'$. The important question here is whether one can find theorems not involving the jump operator which fail to generalize to all admissible α. We expect so but examples seem hard to come by.

The plan of the paper is as follows: We sketch the basic definitions in §1 paying special attention to relative α-recursive enumerability and the α-jump. In §2 we consider those α which are not Σ_2-admissible and prove the results mentioned above. We also note some implications of the proofs for alternate notions of α jump and for some results about the jump of non-α-r.e. degrees below $\mathbf{0}'$. Finally in §3 we use a much simplified version of the priority method of [7] and [8] to construct an α-r.e. set $A <_\alpha \mathbf{0}'$ such that $A' \equiv_\alpha \mathbf{0}''$ for all Σ_2-admissible α. The simplifications are due to the assumption of Σ_2-admissibility, the fact that we are not working relative to an arbitrary incomplete α-r.e. set as in [8] and the adoption of the method introduced by Lachlan of considering nondeficiency stages to replace the bookkeeping devices adopted from Shoenfield [6] that were used in [8]. We suggest [5], [8] or [11] for general background in α-recursion theory and priority arguments. The reader should also be adept at standard recur-

sion theoretic manipulations with quantifiers and the *s-m-n* theorem.

1. Definitions and other preliminaries. We first summarize the standard definitions of α-recursion theory in terms of the levels L_α of Gödel's constructible universe and the usual (strict) Σ_n hierarchy of formulas. α is *admissible* if L_α satisfies the replacement axiom schema of *ZF* for Σ_1 formulas. Thus we are thinking of L_α as a model of a weak set theory. All the usual set-theoretic terms (cardinal, cofinality, etc.) will therefore have their usual definitions but interpreted inside L_α.

A set $A \subseteq \alpha$ is *α-recursively enumerable* (*α-r.e.*) if it has a Σ_1 definition over L_α while a partial function f is *partial α-recursive* if its graph is α-r.e. It is *α-recursive* if its domain is α. (Note that since there is a one-one α-recursive map of α onto L_α it suffices for recursion theoretic purposes to restrict our attention to subsets of α and functions on α.) Of course an $A \subseteq \alpha$ is α-recursive if its characteristic function is while it is *α-finite* if it is a member of L_α. Finally we say that $A \subseteq \alpha$ is *regular* if $A \cap \beta$ is α-finite for every $\beta < \alpha$.

The basic recursion theoretic fact about admissible ordinals is that one can perform Δ_1 (= α-recursive) recursions in L_α to produce α-recursive functions. Thus for example we can α-recursively Gödel number the α-finite sets K_γ ($\gamma < \alpha$) and the Σ_0/L_α formulas with two free variables $\varphi_\epsilon(x, y)$. This immediately gives a Gödel numbering for the α-r.e. sets, $R_\epsilon = \{x \mid L_\alpha \models \exists y \varphi_\epsilon(x, y)\}$, and *a standard simultaneous α-recursive enumeration* of these sets, $R_\epsilon^\sigma = \{x \mid (\exists y \in L_\sigma)\varphi_\epsilon(x, y)\}$.

We now use this enumeration to define relative recursiveness beginning of course with an approximation: $[\epsilon]_\sigma^C(\gamma) = \delta$ iff

$$(\exists \rho)(\exists \eta)[\langle \gamma, \delta, \rho, \eta \rangle \in R_\epsilon^\sigma \;\&\; K_\rho \subseteq C \cap \sigma \;\&\; K_\eta \subseteq (\alpha - C) \cap \sigma].$$

(We employ some α-recursive coding $\langle , \ldots , \rangle$ of n-tuples.) We then say that $[\epsilon]^C(\gamma) = \delta$ if $[\epsilon]_\sigma^C(\gamma) = \delta$ for some σ. (Note that this makes $[\epsilon]^C$ a possibly multivalued function.) This enables us to define the notion of *weakly α-recursive in* ($\leqslant_{w\alpha}$) for a partial function f and a set C: $f \leqslant_{w\alpha} C$ iff $f = [\epsilon]^C$ for some ϵ (and so in particular $[\epsilon]^C$ is single valued). Of course for a set B, we say that $B \leqslant_{w\alpha} C$ iff the characteristic function of B is weakly α-recursive in C. We now use weak α-recursiveness to define two key notions. The *recursive cofinality* of a set A (*rcf* A) is the least $\gamma \leqslant \alpha$ such that there is an $f \leqslant_{w\alpha} A$ with domain γ and range unbounded in α. A is *hyperregular* iff rcf $A = \alpha$ otherwise it is *nonhyperregular.*

Although $\leqslant_{w\alpha}$ is a useful tool, we are really interested in recovering α-finite amounts of information rather than just single values. We therefore define *α-recursive in* ($\leqslant_\alpha$) by saying that $B \leqslant_\alpha C$ iff there is an ϵ such that for all α-finite sets K_γ

$$K_\gamma \subseteq B \leftrightarrow (\exists \rho)(\exists \eta)(\exists \sigma)(\langle \rho, \eta, \gamma, 0\rangle \in R_\epsilon^\sigma \ \&\ K_\rho \subseteq C \ \&\ K_\eta \subseteq \alpha - C)$$

and

$$K_\gamma \subseteq \alpha - B \leftrightarrow (\exists \rho)(\exists \eta)(\exists \sigma)(\langle \rho, \eta, \gamma, 1\rangle \in R_\epsilon^\sigma \ \&\ K_\rho \subseteq C \ \&\ K_\eta \subseteq \alpha - C).$$

As $\leqslant_\alpha$ is transitive and reflexive it gives us a notion of *α-degree*: $\deg(A) = \{B \mid B \leqslant_\alpha A \leqslant_\alpha B\}$. As usual the α-degrees form an upper semilattice ordered by $\leqslant_\alpha$ when the *join* of two degrees, $\deg(A) \vee \deg(B)$, is $\deg(C)$ where

$$C = \{2 \cdot \gamma \mid \gamma \in A\} \cup \{2 \cdot \gamma + 1 \mid \gamma \in B\}.$$

We call an α-degree *α-r.e., regular, irregular, hyperregular* or *nonhyperregular* if it contains an α-r.e., regular, nonregular, hyperregular or nonhyperregular set respectively. (Note that if an α-degree is (non)hyperregular then every set in it is (non) hyperregular. An α-degree can however be both regular and irregular. It is called *nonregular* if no member is regular.)

As in ordinary recursion theory there are also finer reducibilities which are of some interest. Thus for example we say that *A is α-many-one reducible to B*, $A \leqslant_{m\alpha} B$, if there is an α-recursive function f such that for every x, $x \in A$ iff $f(x) \in B$. ($A \leqslant_{m\alpha} B$ clearly implies that $A \leqslant_\alpha B$.) We then have the corresponding notions of completeness: Given a class $\mathcal{C}$ of sets $C \in \mathcal{C}$ is said to be an *mα*-complete (α-complete) $\mathcal{C}$ set if every $B \in \mathcal{C}$ is *mα*-(α)-reducible to C. Thus for example the set $\{\langle x, y\rangle \mid x \in R_y\}$ is both an *mα*- and α-complete α-r.e. set. Of course any two *mα*- or α-complete sets are of the same α-degree.

The notion of completeness is as usual essential for the definition of the jump operator. A', the jump of A, is intended to be an *mα*-complete (and so α-complete) set recursively enumerable in A. Thus the problem of defining A' is really one of defining relative α-recursive enumerability. We suggest that "B is α-r.e. in A" means that there is an enumeration, i.e. an α-r.e. set R which enumerates B when we use A to determine which elements are correctly enumerated by R. To be more precise, for any α-r.e. set of triples R_ϵ we say that R_ϵ enumerates x relative to A if $(\exists \zeta, \eta)(\langle x, \zeta, \eta\rangle \in R \ \&\ K_\zeta \subseteq A \ \&\ K_\eta \subseteq \bar{A})$. We write $R_\epsilon^A = \{x \mid \exists \zeta, \eta(\langle x, \zeta, \eta\rangle \in R_\epsilon \ \&\ K_\zeta \subseteq A \ \&\ K_\eta \subseteq \bar{A})\}$. Note that as in the definition of $\leqslant_\alpha$ we allow ourselves to use α-finitely much information about A to determine which elements are enumerated. Thus B is α-r.e. in A if $B = R_\epsilon^A$ for some ϵ.

Given this definition of relative α-recursive enumerability it is easy to define the *α-jump operator* by the completeness requirement: $A' = \{\langle x, \epsilon\rangle \mid x \in R_\epsilon^A\}$. Clearly if B is α-r.e. in A then $B \leqslant_{m\alpha} A'$. Indeed it is not hard to see that the converse holds as well: If $B \leqslant_{m\alpha} A'$ then for some α-recursive f, $x \in B$ iff $f(x) \in A'$. But $f(x) \in A'$ iff $(\exists y, \epsilon)(f(x) = \langle y, \epsilon\rangle \ \&\ \exists \zeta, \eta(\langle y, \zeta, \eta\rangle \in R_\epsilon \ \&\ K_\zeta \subseteq A \ \&\ K_\eta \subseteq \bar{A}))$. Let ϵ' be such that $R_{\epsilon'} = \{\langle x, \zeta, \eta\rangle \mid (\exists y, \epsilon)(f(x) = \langle y, \epsilon\rangle \ \&\ \exists \zeta, \eta(\langle y, \zeta, \eta\rangle \in R_\epsilon))$. It is then clear that $B = R_{\epsilon'}^A$. Thus A' has the primary

property of the jump. The other basic properties are also easy to establish.

We first note that our definition of α-jump is equivalent to that of [11] in that the sets produced are $m\alpha$-equivalent. This can be seen by unravelling the definitions and applying standard transformations. The following facts are then also easily verified:

1.1. B is α-r.e. in A iff B is the domain or range of a partial function $w\alpha$-recursive in A.

1.2. $A' \not\leqslant_{w\alpha} A$ (and so $A' \not\leqslant_{\alpha} A, A' \not\leqslant_{m\alpha} A$).

1.3. $A \leqslant_{\alpha} B$ iff $A' \leqslant_{m\alpha} B'$.

(And so the α-jump is well defined and increasing on α-degrees.) Note that we use $\mathbf{0}$ for the degree of the recursive sets and so $\mathbf{0}'$ is the complete α-r.e. degree.

Before leaving the definition of the α-jump we would like to mention two other proposals for relative α-recursive enumerability. The first, suggested by Jhu [2], argues that we should require our enumeration to list all α-finite subsets rather than just single elements. Unfortunately we shall see in §2 that for many α the only sets r.e. in $0'$ would then be those recursive in it. Thus this proposal should be completely rejected.

The second suggestion involves enlarging our notion of enumeration. Roughly speaking it allows us to build new enumerations from the given set A in a way not restricted by existing procedures. To be slightly more precise one introduces an equation calculus and calls a function f α-calculable from A ($f \leqslant_{c\alpha} A$) if its graph can be deduced from a finite set of equations plus the diagram of A. This gives a stronger reducibility than α-recursiveness and is more closely connected with model-theoretic ideas than $\leqslant_{\alpha}$. (See [3] for the details.) Accordingly we would say that B is $c\alpha$-enumerable in A if it is the range of a function $f \leqslant_{c\alpha} A$. A corresponding notion of jump $A^{c\alpha}$ would then be defined by taking a complete set $c\alpha$-enumerable in A.

Although we reject this notion of relative enumerability for much the same reasons that we prefer $\leqslant_{\alpha}$ to $\leqslant_{c\alpha}$ (all computations are α-finite (and of length $<\alpha$) for $\leqslant_{\alpha}$ but not for $\leqslant_{c\alpha}$), we want to point out another reason why we feel $A^{c\alpha}$ is too strong. (These remarks are intended only for those familiar with $\leqslant_{c\alpha}$.) Of course if A is regular and hyperregular $A' = A^{c\alpha}$ and all notions coincide. If A is not hyperregular, however, the true power of the $c\alpha$-jump appears:

PROPOSITION 1.4. *If A is not hyperregular and $B \leqslant_{c\alpha} A$ then $B' \leqslant_{c\alpha} A$.*

PROOF. Let $\gamma = \operatorname{rcf} A$ and let $h \leqslant_{w\alpha} A$ map γ unboundedly into α. Define $\varphi(x, \epsilon, \delta)$ for $\delta < \gamma$ by

$$\varphi(x, \epsilon, \delta) = \begin{cases} 0 & \text{if } (\exists \langle x, \zeta, \eta\rangle \in R_{\epsilon}^{h(\delta)})(K_{\zeta} \subseteq B \ \&\ K_{\eta} \subseteq \bar{B}), \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $\varphi \leqslant_{c\alpha} A$ as h and B are α-calculable from A. Now $\langle x, \epsilon\rangle \in B'$ iff $(\exists \delta < \gamma)\varphi(x, \epsilon, \delta) = 0$ and $\langle x, \epsilon\rangle \notin B'$ iff $(\forall \delta < \gamma)\varphi(x, \epsilon, \delta) = 1$. As α-calculability degrees are obviously closed under bounded quantification we see that $B' \leqslant_{c\alpha} A$ as required. □

Of course $A^{c\alpha} \not\leqslant_{c\alpha} A$ and so the $c\alpha$-jump is much stronger than the α-jump. It absorbs all iterations (α-recursive) of the α-jumps. Indeed it also absorbs iterations of quantification i.e. the complete set $\Sigma_1(A)$ is α calculable from A if A is nonhyperregular by an argument like that above. Moreover there is only one $c\alpha$-enumerable degree above any nonhyperregular set. Thus the $c\alpha$-jump seems to act more like the hyperjump in ordinary recursion theory than like the Turing jump.

To conclude our list of basic definitions we consider the notions of projection and cofinality. We define the Σ_n-*projectum* of α, written $\sigma np(\alpha)$ as the least $\beta \leqslant \alpha$ for which there is a one-one Σ_n map of α into β. The key fact here is that this is also the least β such that there is a Σ_n subset of β which is not in L_α[1]. Note that the usual notation for the $\sigma 1p(\alpha)$ is α^* and the key fact says that any α-r.e. set bounded below α^* is α-finite. Similarly the Σ_n *cofinality* of $\delta \leqslant \alpha$, written $\sigma ncf(\delta)$ is the least β such that there is a map of β onto an unbounded subset of δ which is Σ_n.

As an example in definition chasing note that α is Σ_2-*admissible* (i.e. L_α satisfies the replacement axiom schema of ZF for Σ_2 formulas) iff $\sigma 2cf(\alpha) = \alpha$ iff $\mathbf{0}'$ is hyperregular.

2. **α is not Σ_2 admissible.** In this section we take care of those α which are not Σ_2-admissible, i.e. $\mathbf{0}'$ is nonhyperregular. We have two cases to consider: (1) $\mathbf{0}'$ is the only nonhyperregular α-r.e. degree; (2) there is a nonhyperregular α-r.e. set $A <_\alpha 0'$. We begin our attack on the first case with a lemma.

LEMMA 2.1. *If* $\mathbf{a}$ *is regular and hyperregular and* $\mathbf{c}$ *is α-r.e. in* $\mathbf{a}$ *then there is a regular* $C \in \mathbf{c}$ *which is α-r.e. in* $\mathbf{a}$.

PROOF. This is just the relativization of Sacks' result [4] that every α-r.e. degree contains a regular α-r.e. set. One just relativizes the proof to a regular hyperregular set A of degree $\mathbf{a}$. These properties of A guarantee that the proof goes through. □

Now for case 1.

THEOREM 2.2. *If* $\mathbf{0}'$ *is the only nonhyperregular α-r.e. degree and* $\mathbf{a} \leqslant_\alpha \mathbf{0}'$ *is α-r.e. then* $\mathbf{a}' \equiv_\alpha \mathbf{0}'$.

PROOF. Let A be a regular α-r.e. set of degree $\mathbf{a}$ and let $C = R_\epsilon^A$ (for some ϵ) be a regular set of degree $\mathbf{a}'$. (These exist by Lemma 2.1.) Let $\gamma = \sigma 2cf(\alpha)$ and let h be the associated Σ_2 map. (As h is total on γ it is in fact Δ_2.) We can

now define a relation P by $\langle\beta, \delta\rangle \in P \equiv \beta < \gamma \;\&\; \delta < \gamma \;\&\; (\forall x < h(\beta))(x \in C \longrightarrow x$ is enumerated in C by an element $< h(\delta))$. To be more precise

$$\langle\beta, \delta\rangle \in P \equiv \beta < \gamma \;\&\; \delta < \gamma \;\&\; (\forall x)(\forall y)(\forall z)(y = h(\delta) \;\&\; z = h(\delta) \;\&\; x < y \longrightarrow$$
$$[(\exists \zeta, \eta)(\langle x, \zeta, \eta\rangle \in R_\epsilon \;\&\; K_\zeta \subseteq A \;\&\; K_\eta \subseteq \bar{A}) \longrightarrow$$
$$(\exists \zeta, \eta < z)(\langle x, \zeta, \eta\rangle \in R_\epsilon^z \;\&\; K_\zeta \subseteq A \;\&\; K_\eta \subseteq \bar{A})].$$

As h is Δ_2 and A is Σ_1 and regular (and so $\Sigma_0(A) \subseteq \Delta_2$) we see that P is a Π_2 subset of $\gamma \times \gamma$. Now by Theorem 2.1 of [9] our hypothesis guarantees that $\sigma 2cf(\alpha) < \sigma 2p(\alpha)$ and so P is α-finite by the key fact about projecta in §1. We can therefore uniformize P by an α-finite function g. ($g(\beta) =$ the least δ such that $\langle\beta, \delta\rangle \in P$.)

We claim that g is total on γ. For any $\beta < \gamma$, $C \cap h(\beta)$ is α-finite by the regularity of C. The map taking an $x \in C \cap h(\beta)$ to the least z such that $(\exists \zeta, \eta < z)(\langle x, \zeta, \eta\rangle \in R_\epsilon^z \;\&\; K_\zeta \subseteq A \;\&\; K_\eta \subseteq \bar{A})$ is clearly $w\alpha$-recursive in A. Moreover it totals on $C \cap h(\beta)$ by the definition of $C = R_\epsilon^A$. Thus by the hyperregularity of A its range is bounded on $C \cap h(\beta)$ say by w. Now if δ is any ordinal less than γ such that $w < h(\delta)$, we see that $\langle\beta, \delta\rangle \in P$ and so $(\forall \beta < \gamma)(\exists \delta < \gamma)(\langle\beta, \delta\rangle \in P)$. Thus g which uniformizes P must be total.

We can now conclude the proof by using g to compute C from $0'$: To find $C \cap w$ we first find a $\beta < \gamma$ such that $w < h(\beta)$. (The point here is that $h \leqslant_{w\alpha} 0'$.) We then compute $hg(\beta)$ again α-recursively in $0'$. By the definition of g we see that

$$x \in C \cap w \text{ iff } (\exists \zeta, \eta < hg(\beta))(\langle x, \zeta, \eta\rangle \in R_\epsilon^{hg(\beta)} \;\&\; K_\zeta \subseteq A \;\&\; K_\eta \subseteq \bar{A}).$$

To answer such questions however we only need $A \cap hg(\beta)$ which we can recover α-recursively from $0'$ since $A <_\alpha 0'$. Thus using only α-finitely much information about $0'$ we have computed $C \cap w$ for an arbitrary w and so $C \leqslant_\alpha 0'$. As $C \equiv_\alpha A'$ we have our conclusion. □

Before proceeding with case (2) we would like to point out some corollaries of this proof. First, note that by using the reduction of A to $0'$ to write $\Sigma_0(A)$ as $\Delta_1(0') \subseteq \Delta_2$, the proof really shows that if $\mathbf{a} <_\alpha \mathbf{0}'$ is regular and hyperregular then $\mathbf{a}' \equiv_\alpha \mathbf{0}'$. Thus if every $\mathbf{a} < \mathbf{0}'$ is regular and hyperregular, e.g. $\alpha = \aleph_\omega^L$, then there is no α-degree less than $\mathbf{0}'$ whose jump is $\mathbf{0}''$. Next we note that a similar argument disposes of Jhu's suggestion for relative recursive enumerability [2]. This proposal would imply that B is α-r.e. in A iff

$$(\forall K)(K \subseteq B \longleftrightarrow \exists \zeta, \eta)(\langle K, \zeta, \eta\rangle \in R_\epsilon \;\&\; K_\zeta \subseteq A \;\&\; K_\eta \subseteq \bar{A})$$

for some ϵ (where K ranges over α-finite sets). Taking $\alpha = \aleph_\omega^L$ as an example we show that if B is α-r.e. in $0'$ (via this definition) then $B \leqslant_\alpha 0'$: suppose B is α-r.e. in $0'$ via ϵ. Define $f\colon \omega \longrightarrow \omega$ by

$$f(n) = \mu m(\exists \zeta, \eta < \aleph_m^L)(\langle B \cap \aleph_n^L, \zeta, \eta\rangle \in R_\epsilon^{\aleph_m^L} \ \& \ K_\zeta \subseteq 0' \ \& \ K_\eta \subseteq \overline{0'}).$$

As every constructible set bounded below $\alpha = \aleph_\omega^L$ is α-finite $B \cap \aleph_n^L$ is α-finite for each n. Thus f is total by the proposed definition for α-r.e. Of course it too is α-finite and so we can compute B from $0'$ as follows: To find $B \cap w$ first find an n such that $w < \aleph_n^L$ (the point is that $\{\aleph_n^L | n < \omega\} \leqslant_\alpha 0'$). Then calculate $f(n) = m$ and find α-recursively in $0'$ the largest subset K of $\aleph_n$ such that

$$(\exists \zeta, \eta(\langle K, \zeta, \eta\rangle \in R_\epsilon^{\aleph_m^L} \ \& \ K_\zeta \subseteq 0' \ \& \ K_\eta \subseteq \overline{0'}).$$

We must have $K = B \cap \aleph_n^L$. This type of argument was first used in [10] to show that $\aleph_\omega^L$ has precisely one nonhyperregular α-r.e. degree.

We now handle the case in which there are nonhyperregular α-r.e. degrees other than $0'$ by proving the following:

THEOREM. *If A is α-r.e. and nonhyperregular then $A' \equiv_\alpha 0''$.*

PROOF. It is easy to see that $0''$ is a Σ_2 set (just write out the definition and note that the admissibility of α allows us to find a Σ_1 equivalent for a formula of the form $\forall x < y \varphi$ where φ is Σ_1). We can therefore choose a $\varphi(x, y, s)$ in Δ_0 such that $s \in 0'' \equiv \exists x \forall y \varphi(x, y, s)$. By Lemma 2.1 and the remarks in 1.3 and the definition of nonhyperregular we may assume that A is regular. Of course if $A \equiv_\alpha 0'$ we are done and so we assume that $A <_\alpha 0'$. By 1.4 of [9] $\rho = \sigma_1 p_A(\alpha) \leqslant \operatorname{rcf} A = \beta$. Let $\exists w \theta(x, y, w)$ define a $\Sigma_1(A)$ map of a subset of ρ onto α and let g be a $w\alpha$-recursive in A (and so $\Delta_1(A)$) map from β onto an unbounded subset of α. We can now define a map h: $\beta \longrightarrow \alpha$ which is onto α and $\Delta_1(A)$ by

$$h(\langle x, \delta\rangle) = \begin{cases} y & \text{if } (\exists w < h(\delta))\theta(x, y, w), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that $s \in 0'' \equiv \exists x \forall y \varphi(x, y, s) \Longleftrightarrow (\exists x < \beta)(\forall z)(h(x) = z \longrightarrow \forall y \varphi(z, y, s)) \Longleftrightarrow (\exists x < \beta)(\forall w)\psi(x, w, s)$ for an appropriate ψ which is $\Delta_0(A)$. As in ordinary recursion theory it is clear that there is an α-recursive function $l(x, s)$ such that $\forall w \psi(x, w, s) \Longleftrightarrow l(x, s) \notin A'$. Thus $s \in 0'' \Longleftrightarrow (l''\beta \times \{s\}) \cap A' = \emptyset$. Indeed for any α-finite set K_y we have $K_y \subseteq 0'' \Longleftrightarrow (l''\beta \times K_y) \cap A' = \emptyset$. (Note that $l''\beta \times K_y$ is α-finite as l is α-recursive.) We also have that for any α-finite set K_x, $K_x \cap 0'' = \emptyset \Longleftrightarrow (\forall y \in K_x)(y \notin 0'') \Longleftrightarrow k(x) \notin 0''$ for an appropriate α-recursive function k. (Just choose k so that $k(x) = (0, w)$ where $R_w = \{\langle 0, \zeta, \eta\rangle | (\exists \langle z, \epsilon\rangle \in K_x)(\langle z, \zeta, \eta\rangle \in R_\epsilon)\}$.) Thus $K_x \cap 0'' = \emptyset \Longleftrightarrow (\exists y)(y \in l''\beta \times \{k(x)\} \ \& \ y \in A')$. It is now routine to convert these conditions on $K_y \subseteq 0''$ and $K_x \cap 0'' = \emptyset$ to those required by the definition of $0'' \leqslant_\alpha A'$. $\square$

3. **α is Σ_2-admissible.** Our goal in this section is to prove the analog of Sacks' theorem for Σ_2-admissible ordinals α (L_α satisfies the replacement axiom

schema for Σ_2 formulas). We use a very simple infinite injury priority argument much like the one in ordinary recursion theory. The Σ_2-admissibility of α lets us avoid most of the difficulties that would otherwise be encountered. We want to construct a regular α-r.e. set $A <_\alpha 0'$ such that $A' \equiv_\alpha 0''$. The plan is to choose a regular set C of degree $\mathbf{0}''$ and to guarantee that for every z, $(\exists x)(\forall s < z)(s \notin C \longrightarrow (\forall y > x)(\langle s, y\rangle \in A))$ and $(\exists x)(\forall s < z)(s \in C \longrightarrow (\forall y > x)(\langle s, y\rangle \notin A))$. We then prove that this implies that $C \leqslant_\alpha A'$. As $A <_\alpha 0'$ tells us that $A' \leqslant_\alpha 0''$ we will have our desired result.

The procedure will entail trying to enumerate elements $\langle s, y\rangle$ in A if it appears that s is not in C (via some approximation) so that if s is not in C we will try to put every $\langle s, y\rangle$ into A while if $s \in C$ we will try to put in only an α-finite number of elements $\langle s, y\rangle$ into A. These attempts correspond to our positive requirements. We will also have negative requirements for each ϵ to make sure that $[\epsilon]^A \neq c_B$ (c_B is the characteristic function of a regular α-r.e. set B of degree $\mathbf{0}'$). These requirements actually attempt to preserve computations showing that $[\epsilon]^A = c_B$. The point however is that the positive requirements of higher priority will form an α-recursive set. Thus we can tell α-recursively if a negative requirement is permanent. If there were then unboundedly many of them (as would be the case if $[\epsilon]^A = c_B$) we could compute B α-recursively for a contradiction. We will use the technique introduced by Lachlan of looking at nondeficiency stages of the construction to show that the positive requirements succeed unless they are thwarted by permanent negative requirements of higher priority. As there will be only α-finitely many of these we will in fact get the condition on A stated above. We give the priority argument in a lemma.

LEMMA 3.1. *Let C be a regular Σ_2 set and B a regular α-r.e. set of degree $\mathbf{0}'$. There is a regular α-r.e. set A such that $B \not\leqslant_{w\alpha} A$ and*

(1) $(\forall z)(\exists x)(\forall s < z)(s \notin C \longrightarrow (\forall y > x)(\langle s, y\rangle \in A))$ *and*

(2) $(\forall z)(\exists x)(\forall s < z)(s \in C \longrightarrow (\forall y > x)(\langle s, y\rangle \notin A))$.

PROOF. Choose a Δ_0 formula φ such that $s \in C \equiv \exists x \forall y \varphi(x, y, s)$ and an α-recursive function b enumerating B (i.e. $B = \text{range } b$). We let $B^\sigma = \{b(\delta) \mid \delta < \sigma\}$ and will use A^σ to mean the set of elements enumerated in A by stage σ. We begin by describing the creation of the requirements that guide our construction at stage σ.

The positive requirements. We see (for each $\langle s, \gamma\rangle < \sigma$) if $(\forall x < \gamma)(\exists y < \sigma) \neg \varphi(x, y, s)$. If so we *create a positive requirement* (of priority s) for $\langle s, \gamma\rangle$.

The negative requirements. For each active $\epsilon < \sigma$ we find the least x for which we have no current negative requirement associated with x. We then ask if for $i = c_{B^\sigma}(x)$ there are ordinals ζ and $\eta < \sigma$ such that $\langle x, i, \zeta, \eta\rangle \in R_\epsilon^\sigma$ & $K_\eta \subseteq \bar{A}^\sigma$. If so we take the least such $\langle \zeta, \eta\rangle$ and *create a negative requirement*

N of priority ϵ associated with x where $N = (\bigcup K_\eta) \cup (\bigcup M)$ ($M = \{N' | N$ is a current negative requirement of priority $\epsilon\}$). Note that N is an initial segment of α. It tries to protect all the information used about $\bar{A}$ in all computations of $c_B(y)$ from A via ϵ for $y \leqslant x$. It will be destroyed at any stage $\tau > \sigma$ at which we put one of its elements into A. N is *current* as long as it has not been destroyed. If it is never destroyed it is *permanent*; otherwise it is *temporary*. Finally a reduction procedure ϵ is *active* at stage σ unless we have a current negative requirement of priority ϵ associated with some x which was created at a stage $\tau < \sigma$ and $c_{B^\sigma}(x) \neq c_{B^\tau}(x)$. The point is that if ϵ is not active we need not worry about it since we are preserving a computation which shows that $c_B \neq [\epsilon]^A$. If ϵ is not active we call it *inactive*.

The construction. We begin each stage σ by going through (by induction on s) the positive requirements and putting any element $\langle s, \gamma\rangle$ not yet in A with a positive requirement into A^σ unless it belongs to a negative requirement of priority $e < s$. We of course destroy other negative requirements as necessary. The second part of each stage σ consists of creating positive and negative requirements as dictated by the instructions above.

The priority argument. We prove some lemmas about the construction to establish the required properties of the set constructed.

SUBLEMMA 3.2. *Any element $\langle s, \gamma\rangle$ with a positive requirement is eventually put into A unless it is in a permanent negative requirement of priority $e < s$.*

PROOF. Let σ be the first stage after $\langle s, \gamma\rangle$ gets a positive requirement at which an element x is enumerated in A^σ and at no stage $\tau > \sigma$ is any element $y < x$ enumerated in A^τ. (Such stages are called *nondeficiency* stages.) Any negative requirement not destroyed by the end of the first part of stage σ must be permanent. (It cannot contain any element $\geqslant x$ or it would be destroyed at stage σ while no element $< x$ is enumerated at any later stage.) On the other hand $\langle s, \gamma\rangle$ would have been enumerated in A^σ if it were not in a negative requirement of priority $e < s$. As this requirement survives stage σ it is permanent.

SUBLEMMA 3.3. *For each x there are α-finitely many permanent negative requirements of priority $< x$. Moreover, for each $s < x$, $\{\gamma | \langle s, \gamma\rangle \in A\}$ is a final segment of α if $s \notin C$ while the set $\{\langle s, \gamma\rangle \in A | s < x \ \& \ s \in C\}$ is α-finite.*

PROOF. We proceed by induction. First note that, for any s, positive requirements are created for $\langle s, \gamma\rangle$ for every $\gamma < \alpha$ if $s \notin C$ and only for a proper initial segment of ordinals $\gamma < \alpha$ if $s \in C$. By Sublemma 3.2 each $\langle s, \gamma\rangle$ with a positive requirement is enumerated in A unless it belongs to a permanent negative requirement of priority $e < s$. By induction however there are only α-finitely many such requirements for $e < s < x$. Thus for $s < x$ and $s \notin C$ all but α-fi-

nitely many $\langle s, \gamma\rangle$ are in A. Of course for $s \in C$ only α-finitely many can be put in at all.

By the regularity of C the set $\{s < x | s \in C\}$ is α-finite. The map taking $s \in C$ to the least γ such that there is no positive requirement for $\langle s, \gamma\rangle$ is Σ_2 and so bounded say by δ for $s < x$ (α is Σ_2-admissible). The set $\{\langle s, \gamma\rangle | s \in C \;\&\; \gamma < \delta\} \cap A$ is then a bounded Σ_1 set which is therefore α-finite. (Σ_2-admissibility easily implies that $\alpha^* = \alpha$.) For *each* $s < x$ with $s \notin C$ we of course know that every $\langle s, \gamma\rangle$ gets a positive requirement and all but α-finitely many of them succeed. (We do not yet know that the set of all the ones which fail is α-finite if e.g. x is a limit ordinal.)

We claim now that we can uniformly decide α-recursively if a given positive requirement $\langle s, \gamma\rangle$ with $s < x$ succeeds and if a given negative requirement of priority $\epsilon < x$ is permanent. We begin after all elements $\langle s, \gamma\rangle$ in A with $s \in C$ have been enumerated in A. Upon being presented with a positive requirement $\langle s, \gamma\rangle$ with $s < x$ (and $s \notin C$ of course) we carry out the construction until either $\langle s, \gamma\rangle$ is enumerated in A or a negative requirement of priority $\epsilon < s$ containing it is found to be permanent. We check a given negative requirement N of priority ϵ for permanence by checking the elements $\langle s, \gamma\rangle$ in N with $s \leqslant \epsilon$ ($s \notin C$) by induction on s to see if any of them get into A (and so destroy N). We must eventually either discover that every such $\langle s, \gamma\rangle$ is in a permanent negative requirement of priority $\epsilon' < \epsilon$ or destroy N. As we know that for both types of requirements one of the alternatives that we are looking for must occur, the well-foundedness of the priority listing and the uniformity of the search guarantee that each such check eventually ends.

Finally using this procedure we can show that there are only α-finitely many permanent negative requirements of priority $< x$. We must first consider those $\epsilon < x$ which become inactive because of a permanent requirement. As checking for permanence is α-recursive this set is clearly a Σ_1 subset of x and so α-finite. We can therefore wait until a stage τ by which all such ϵ have become inactive via permanent requirements. From now on any permanent negative requirement of priority $\epsilon < x$, associated with some y, corresponds to a computation giving the correct value of $c_B(y)$. (Otherwise ϵ would become inactive when $c_{B^\sigma}(y)$ changed.) If the set of permanent negative requirements of priority $< x$ created after stage τ were α-infinite the associated y's would form a final segment of α. (If all requirements of priority ϵ associated with y are temporary so are all those associated with any $z > y$ by our including the information used for y with that used for z.) Were this true, however, we could calculate B on this final segment α-recursively as follows: To find out if $z \in B$ just look for a permanent negative requirement of priority $< x$ associated with z created at a stage $\sigma > \tau$. Then $z \in B$ just in case $z \in B^\sigma$ by our choice of τ. Of course the search is α-recursive

by the above argument while it terminates by the assumption that the set of such requirements is α-infinite. As this contradicts the non-α-recursiveness of B there are only α-finitely many permanent negative requirements of priority $< x$. This completes the inductive proof of Sublemma 3.3. $\square$

As these sublemmas clearly guarantee conditions (1) and (2) of Lemma 3.1 all that remains to do to prove this lemma is to show that $B \not\leqslant_{w\alpha} A$. Assume for the sake of a contradiction that $c_B = [\epsilon]^A$. Let y be the least x for which there is no permanent negative requirement of priority ϵ associated with x. As $c_B = [\epsilon]^A$ there is a least pair $\langle \zeta, \eta \rangle$ such that $\langle y, i, \zeta, \eta \rangle \in R_\epsilon$ & $K_\zeta \subseteq A$ & $K_\eta \subseteq \bar{A}$ where $i = c_B(y)$. We can choose σ large enough so that there are permanent requirements for all $z < y$, $\langle y, i, \zeta, \eta \rangle \in R_\epsilon^\sigma$, $A^\sigma \cap \cup K_\eta = A \cap \cup K_\eta$ and $c_{B^\sigma}(y) = c_B(y)$ (of course $K_\eta \subseteq \overline{A^\sigma}$ for every σ). Moreover we can make σ large enough so that all smaller pairs $\langle \zeta', \eta' \rangle$ such that $\langle y, i, \zeta', \eta' \rangle \in R_\epsilon$ with $K_{\zeta'} \subseteq A$ have been found not to satisfy $K_{\eta'} \subseteq \bar{A}$. Now none of the permanent requirements can make ϵ inactive since they correspond to correct computations of $[\epsilon]^A$ and $[\epsilon]^A = c_B$ by assumption. Thus at the first stage $\tau > \sigma$ at which there is no negative requirement of priority ϵ associated with y (as any such is temporary there is such a stage τ) we will create a requirement corresponding to $\langle \zeta, \eta \rangle$. As all the requirements associated with $z < y$ are permanent and $A^\sigma \cap \cup K_\eta = A \cap \cup K_\eta$ this requirement is permanent. Since this contradicts our choice of y we conclude that $c_B \neq [\epsilon]^A$. $\square$

We can now settle the case that α is Σ_2-admissible.

THEOREM 3.4. *If α is Σ_2-admissible there is an α-r.e. set $A <_\alpha 0'$ such that $A' \equiv_\alpha 0''$.*

PROOF. By Lemma 2.1 we can choose regular sets B and C of degree $\mathbf{0}'$ and $\mathbf{0}''$ respectively. (Σ_2-admissibility is equivalent to $\mathbf{0}'$ being regular and hyperregular.) Let A be the α-r.e. set constructed in Lemma 3.1. (As C is α-r.e. in $\mathbf{0}'$ it is clearly Σ_2.) As the lemma guarantees that $B \not\leqslant_{w\alpha} A$ we immediately have that $A <_\alpha 0'$. Thus $A' \leqslant 0''$ and it suffices to show that $C \leqslant_\alpha A'$:

Consider any α-finite set K_δ bounded say by z. By condition (1) of lemma 3.1 we see that

$$K_\delta \subseteq \bar{C} \text{ iff } (\exists x)(\forall s \in K_\delta)(\forall y > x)(\langle s, y \rangle \in A).$$

standard manipulations (using the completeness of A' and the *s-m-n* theorem) now show that there is an α-recursive function k such that $K_\delta \subseteq \bar{C}$ iff $(\exists x)(k(x) \notin A')$. Similarly condition (2) guarantees that

$$K_\delta \subseteq C \text{ iff } (\exists x)(\forall s \in K_\delta)(\forall y > x)(\langle s, y \rangle \notin A) \text{ iff } (\exists x)(f(x) \notin A')$$

for some α-recursive function f. It is now easy to translate these conditions into the ones required in the definition of $C \leqslant_\alpha A'$. $\square$

This then finishes the last part of the complete result:

THEOREM 3.5. *There is an α-r.e. set $A <_\alpha 0'$ such that $A' \equiv_\alpha 0''$ unless there is precisely one nonhyperregular α-r.e. degree.* □

As we noted before a characterization of these ordinals can be found in [9]. To fulfill our promise to those interested in $\leqslant_{c\alpha}$ we recall that every nonhyperregular α-r.e. set falls in the same α-calculability degree [10]. (This is also an immediate corollary of Proposition 1.4.) Combining this with the fact that $A' \equiv_\alpha A^{c\alpha}$ for regular hyperregular A we can translate our theorem to one about the $c\alpha$-jump.

COROLLARY 3.6. *There is an α-r.e. set $A <_{c\alpha} 0^{c\alpha}$ such that $A^{c\alpha} \equiv_{c\alpha} (0^{c\alpha})^{c\alpha}$ if and only if α is Σ_2-admissible.*

PROOF. If α is Σ_2-admissible then every α-r.e. degree is regular and hyperregular and so $A' = A^{c\alpha}$ for A α-r.e. thus Theorem 3.5 immediately supplies the desired set. If α is not Σ_2-admissible then $0' = 0^{c\alpha}$ is not hyperregular while $A <_{c\alpha} 0^{c\alpha}$ implies that A is hyperregular. Thus $A <_{c\alpha} 0^{c\alpha}$ means that $A' = A^{c\alpha}$ but by Proposition 1.4 $A' \leqslant_{c\alpha} 0^{c\alpha} <_{c\alpha} (0^{c\alpha})^{c\alpha}$. □

Note that this gives us an example of a difference between the theory (with jump operator) of α-degree and that of α-calculability degrees for some α (including $\alpha = \omega_1^{CK}$).

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