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MEAN CONVERGENCE OF GENERALIZED WALSH-FOURIER SERIES

BY

WO-SANG YOUNG

ABSTRACT. Paley proved that Walsh-Fourier series converges in L^p $(1 < p < \infty)$. We generalize Paley's result to Fourier series with respect to characters of countable direct products of finite cyclic groups of arbitrary orders.

1. **Introduction.** It is known that the Walsh functions are characters of the countable direct product of groups of order 2. In this note we consider characters of $\Pi_{i=0}^{\infty} Z_{p_i}$, where Z_{p_i} is a cyclic group of order p_i, $p_i \geq 2$. Various Fourier properties of this generalized Walsh system have been studied in [8], [7], [9], [5], [3], [4], [2], and others. Many of these results are obtained only for the case where $\sup_i p_i < \infty$. In fact, Price [7] showed that some basic properties no longer hold when $\sup_i p_i = \infty$. We will show that results concerning mean convergence, however, are still valid even if the orders p_i are unbounded. The bounded case was first obtained by Watari [9]. See also Gosselin [2].

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Let $\{p_i\}_{i \geq 0}$ be a sequence of integers, $p_i \geq 2$. Let $G = \Pi_{i=0}^{\infty} Z_{p_i}$ be the direct product of cyclic groups of order p_i, and μ the Haar measure on G normalized by $\mu(G) = 1$. Each element of G can be considered as a sequence $\{x_i\}$, with $0 \leq x_i < p_i$. Set $m_0 = 1$, $m_k = \Pi_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We can identify G with the unit interval $(0, 1)$. This identification consists in associating with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\Sigma_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto and measure preserving.

We define an orthonormal system of functions $\{\phi_k\}$ on G. For each $x = \{x_i\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, \ldots$. We enumerate the set of all finite products of $\{\phi_k\}$ using a scheme of Paley. We express each nonnegative integer n as a finite sum $n = \Sigma_{k=0}^{\infty} \alpha_k m_k$, with $0 \leq \alpha_k < p_k$, and define $\chi_n = \Pi_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G, and they form a complete orthonormal system on G. For the case $p_i = 2$, $i = 0, 1, \ldots$, G is the dyadic group, $\{\phi_k\}$ are the Rademacher functions, and $\{\chi_n\}$ the Walsh functions.

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We consider Fourier series with respect to $\{\chi_n\}$. Let $D_n = \Sigma_{j=0}^{n-1}\chi_j$, $n = 1, 2, \ldots$, be the nth Dirichlet kernel. For $f \in L^1(G)$,

$$S_n f(x) = \int_G f(t) D_n(x - t)\, d\mu(t), \qquad n = 1, 2, \ldots,$$

denotes the nth partial sum of the Fourier series of f. We have the following uniform estimates on $\{S_n f\}$.

THEOREM 1. *There are absolute constants C and C_p such that, for $n = 1, 2, \ldots$,*

$$\|S_n f\|_p \leq C_p \|f\|_p, \qquad f \in L^p(G), 1 < p < \infty, \tag{1}$$

$$\mu\{|S_n f| > y\} \leq Cy^{-1}\|f\|_1, \qquad f \in L^1(G), y > 0. \tag{2}$$

These results and the density of the generalized Walsh polynomials imply the mean convergence of $S_n f$ to f in $L^p(G)$, $1 < p < \infty$.

The constants C and C_p in the above theorem are independent of the orders p_i of the cyclic groups.

If $p_i = 2$, $i = 0, 1, \ldots$, Theorem 1 is Paley's result for the Walsh-Fourier series [6]. On the other hand, if $p_0 \longrightarrow \infty$, $S_n f$ resembles the nth trigonometric partial sum. Thus, when restricted to one cyclic group, Theorem 1 can be viewed as a discrete analogue of M. Riesz's theorem for the trigonometric Fourier series [10, I, p. 266].

In what follows C will denote an absolute constant, which may vary from line to line.

2. **Modified partial sums and conjugate functions.** We will use the following notation. Let $\{G_k\}$ be a sequence of subgroups of G defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \qquad k = 1, 2, \ldots.$$

Then $\mu(G_k) = m_k^{-1}$. Let F_k be the σ-algebra generated by the cosets of G_k. On the interval $(0, 1)$, atoms of F_k are intervals of the form $(jm_k^{-1}, (j+1)m_k^{-1})$, $j = 0, 1, \ldots, m_k - 1$. We note that ϕ_k is measurable with respect to F_{k+1}.

It is proved in [8] that

$$D_{m_k}(x) = \begin{cases} m_k & \text{if } x \in G_k, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

From (3) it follows that

$$S_{m_k} f(x) = \frac{1}{\mu(I)} \int_I f\, d\mu,$$

where $I = x + G_k$.

It is also proved in [8] that if $n = \Sigma_{k=0}^{\infty}\alpha_k m_k$, $0 \leq \alpha_k < p_k$,

$$D_n = \chi_n \sum_{k=0}^{\infty} D_{m_k}\phi_k^{-\alpha_k}\left(\sum_{j=0}^{\alpha_k-1} \phi_k^j\right), \tag{4}$$

with the interpretation that $\Sigma_{j=0}^{\alpha_k-1}\phi_k^j = 0$ if $\alpha_k = 0$. It is convenient to consider the modified Dirichlet kernel D_n^* defined by $D_n^* = \overline{\chi}_n D_n$. From (4) we have

$$D^*_{\alpha_k m_k} = D_{m_k}\phi_k^{-\alpha_k}\left(\sum_{j=0}^{\alpha_k-1} \phi_k^j\right) = D_{m_{k+1}} - D_{(p_k-\alpha_k)m_k}, \tag{5}$$

and

$$D_n^* = \sum_{k=0}^{\infty} D^*_{\alpha_k m_k}. \tag{6}$$

Let $S_n^*f(x) = \int_G f(t)D_n^*(x-t)\,d\mu(t)$ be the nth modified partial sum. Since $S_n^*f = \overline{\chi}_n S_n(f\chi_n)$, Theorem 1 is equivalent to

THEOREM 1*. *There are absolute constants C and C_p such that, for $n = 1, 2, \ldots$,*

$$\|S_n^*f\|_p \leq C_p\|f\|_p, \qquad f \in L^p(G),\ 1 < p < \infty, \tag{7}$$

$$\mu\{|S_n^*f| > y\} \leq Cy^{-1}\|f\|_1, \qquad f \in L^1(G),\ y > 0. \tag{8}$$

We will prove Theorem 1*. The following facts concerning the modified partial sums will be needed. First of all we have, by (5) and (6),

$$S_n^*f = \sum_{k=0}^{\infty} S^*_{\alpha_k m_k}f, \tag{9}$$

with $S^*_{\alpha_k m_k}f = S_{m_{k+1}}f - S_{(p_k-\alpha_k)m_k}f$. Moreover, it follows from (5) and (3) that

$$S^*_{\alpha_k m_k}f(x) = \frac{1}{\mu(I)}\int_I f(t)\phi_k^{-\alpha_k}(x-t)\left(\sum_{j=0}^{\alpha_k-1} \phi_k^j(x-t)\right)d\mu(t), \tag{10}$$

where $I = x + G_k$. Now, for $f \in L^1(G)$,

$$\frac{1}{\mu(I)}\int_I f(t)\left(\sum_{j=0}^{\alpha_k-1} \phi_k^j(x-t)\right)d\mu(t)$$

resembles the α_kth partial sum of the trigonometric Fourier series of f on the coset I. The relation between the trigonometric partial sum and conjugate function leads to our definition of the conjugate function H_kf of $f \in L^1(G)$. Let $x = \{x_k\} \in G$. We define

$$H_kf(x) = \frac{1}{2}\frac{1}{\mu(I)}\int_{I\cap\{x_k \neq t_k\}} f(t)\cot(\pi(x_k - t_k)/p_k)\,d\mu(t),$$

where $I = x + G_k$. Since

$$\phi_k^{-\alpha_k}(t) \sum_{j=0}^{\alpha_k - 1} \phi_k^j(t) = \begin{cases} \alpha_k & \text{if } t_k = 0, \\ \tfrac{1}{2}\phi_k^{-\alpha_k}(t) - \tfrac{1}{2} + \tfrac{1}{2} i \phi_k^{-\alpha_k}(t) \cot(\pi t_k/p_k) \\ \qquad - \tfrac{1}{2} i \cot(\pi t_k/p_k) & \text{if } t_k \neq 0, \end{cases}$$

(10) implies

$$\begin{aligned} S^*_{\alpha_k m_k} f(x) &= \frac{\alpha_k}{\mu(I)} \int_{I \cap \{x_k = t_k\}} f(t)\, d\mu(t) \\ &\quad + \frac{1}{2} \phi_k^{-\alpha_k}(x) \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq t_k\}} f(t) \phi_k^{\alpha_k}(t)\, d\mu(t) \\ &\quad - \frac{1}{2} \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq t_k\}} f(t)\, d\mu(t) \\ &\quad + i \phi_k^{-\alpha_k}(x) H_k(f \phi_k^{\alpha_k})(x) - i H_k f(x). \end{aligned} \tag{11}$$

(9) and (11) will be used later in the proof of Theorem 1*.

3. A decomposition lemma. For the proof of Theorem 1* we need a modified form of the Calderón-Zygmund decomposition lemma [1, p. 91]. The following may best be described on the interval (0, 1).

LEMMA 2. *Let f belong to $L^1(G)$ and $y > 0$ with $\|f\|_1 \leq y$. Let $\{\alpha_k\}_{k \geq 0}$ be a sequence of integers with $0 \leq \alpha_k < p_k$. Then there are L^1 functions g and b, and a collection $\mathcal{C} = \{\omega_j\}$ of disjoint intervals such that*

(12) $f = g + b$.

(13) $|g| \leq Cy$ *a.e.*

(14) $\|g\|_1 \leq C\|f\|_1$.

(15) $\mathcal{C} = \bigcup_{k=0}^{\infty} \mathcal{C}_k$ *where each $\omega_j \in \mathcal{C}_k$ is measurable with respect to $\mathcal{F}_{k+1}$ and is a proper subset of a coset of G_k.*

(16) $b(x) = 0$ *if $x \notin \bigcup_j \omega_j$.*

(17) $\int_{\omega_j} b\, d\mu = 0$ *for every $\omega_j \in \mathcal{C}$ and $\int_{\omega_j} b \phi_k^{\alpha_k} d\mu = 0$ for every $\omega_j \in \mathcal{C}_k$, $k = 0, 1, \ldots$.*

(18) $\int_{\omega_j} |b|\, d\mu \leq C \int_{\omega_j} |f|\, d\mu$ *for every $\omega_j \in \mathcal{C}$.*

(19) $\Sigma_j \mu(\omega_j) \leq y^{-1} \|f\|_1$.

PROOF. We first construct the collection $\mathcal{C}$ of disjoint intervals. We divide (0, 1) into two subintervals I_1 and I_1', with $I_1, I_1' \in \mathcal{F}_1$ and $\mu(I_1) - m_1^{-1} \leq \mu(I_1') \leq \mu(I_1)$. If $(1/\mu(I_1)) \int_{I_1} |f|\, d\mu > y$, then I_1 is in $\mathcal{C}$. Otherwise we repeat the above process with (0, 1) replaced by I_1. We do the same with I_1'. Finally we reach a stage where the subinterval I is an atom of $\mathcal{F}_1$ and $(1/\mu(I)) \int_I |f|\, d\mu \leq y$.

We then divide I into subintervals I_2 and I_2', with $I_2, I_2' \in \mathcal{F}_2$ and $\mu(I_2) - m_2^{-1} \leqslant \mu(I_2') \leqslant \mu(I_2)$, and proceed as before. In this way we obtain a collection $\mathcal{C} = \{\omega_j\}$ of disjoint intervals which has the properties that

$$y < \frac{1}{\mu(\omega_j)} \int_{\omega_j} |f| \, d\mu \leqslant 3y, \qquad \omega_j \in \mathcal{C}, \tag{20}$$

and

$$|f(x)| \leqslant y \quad \text{for a.e. } x \notin \bigcup_j \omega_j. \tag{21}$$

The first inequality of (20) implies (19). Set

$$\mathcal{C}_0 = \{\omega_j \in \mathcal{C} \colon \omega_j \in \mathcal{F}_1\},$$

and

$$\mathcal{C}_k = \left\{\omega_j \in \mathcal{C} \setminus \bigcup_{l=0}^{k-1} \mathcal{C}_l \colon \omega_j \in \mathcal{F}_{k+1}\right\},$$

$k = 1, 2, \ldots$. Then $\{\mathcal{C}_k\}$ satisfies (15).

Next we decompose f as $f = g + b$, with

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_j \omega_j, \\ a_{kj} + b_{kj}\phi_k^{-\alpha_k}(x) & \text{if } x \in \omega_j \in \mathcal{C}_k, \end{cases} \tag{22}$$

where a_{kj}, b_{kj} are constants chosen in such a way that

$$\int_{\omega_j} f \, d\mu = \int_{\omega_j} (a_{kj} + b_{kj}\phi_k^{-\alpha_k}) \, d\mu, \tag{23}$$

and

$$\int_{\omega_j} f\phi_k^{\alpha_k} \, d\mu = \int_{\omega_j} (a_{kj} + b_{kj}\phi_k^{-\alpha_k})\phi_k^{\alpha_k} \, d\mu. \tag{24}$$

Then $b = g - f$ automatically satisfies (16) and (17). The proof will be completed if we show

$$|g(x)| \leqslant \frac{C}{\mu(\omega_j)} \int_{\omega_j} |f| d\mu, \qquad x \in \omega_j, \ \omega_j \in \mathcal{C}, \tag{25}$$

for then (25) together with (20) and (21) will imply (13), (14) and (18).

To prove (25) we write $\beta_k = \alpha_k$ if $0 \leqslant \alpha_k \leqslant p_k/2$ and $\beta_k = \alpha_k - p_k$ if $p_k/2 < \alpha_k < p_k$. Then $-p_k/2 < \beta_k \leqslant p_k/2$ and $\phi_k^{\alpha_k} = \phi_k^{\beta_k}$. Let $\omega_j \in \mathcal{C}_k$. If ω_j is a coset of G_{k+1}, or if $\beta_k = 0$, then ϕ_k is constant in ω_j. In this case we set $a_{kj} = (\mu(\omega_j))^{-1} \int_{\omega_j} f \, d\mu$ and $b_{kj} = 0$. (25) follows immediately.

Now suppose $\beta_k \neq 0$ and ω_j is not a coset of G_{k+1}, that is $\mu(\omega_j)m_{k+1} \geqslant 2$. Then $|(\mu(\omega_j))^{-1} \int_{\omega_j} \phi_k^{\beta_k} \, d\mu| \neq 1$. Solving (23), (24) for a_{kj}, b_{kj} and substituting into (22) we obtain, for $x \in \omega_j$,

$$g(x) = \left[\frac{1}{\mu(\omega_j)}\int_{\omega_j} f\,d\mu - \frac{1}{\mu(\omega_j)}\int_{\omega_j} \phi_k^{-\beta_k}\,d\mu \frac{1}{\mu(\omega_j)}\int_{\omega_j} f\phi_k^{\beta_k}\,d\mu \right.$$
$$+ \frac{1}{\mu(\omega_j)}\int_{\omega_j} f\phi_k^{\beta_k}\,d\mu\phi_k^{-\beta_k}(x)$$
$$\left. - \frac{1}{\mu(\omega_j)}\int_{\omega_j} \phi_k^{\beta_k}\,d\mu \frac{1}{\mu(\omega_j)}\int_{\omega_j} f d\mu\, \phi_k^{-\beta_k}(x)\right]$$
$$\times \left[1 - \left|\frac{1}{\mu(\omega_j)}\int_{\omega_j} \phi_k^{\beta_k}\,d\mu\right|^2\right]^{-1}$$
$$= \left[\frac{1}{\mu(\omega_j)}\int_{\omega_j} f(y) \frac{1}{\mu(\omega_j)}\int_{\omega_j} (\phi_k^{\beta_k}(y) - \phi_k^{\beta_k}(t)) \right.$$
$$\left. \times (\phi_k^{-\beta_k}(x) - \phi_k^{-\beta_k}(t))\,d\mu(t)\,d\mu(y)\right]$$
$$\times \left[1 - \left|\frac{1}{\mu(\omega_j)}\int_{\omega_j} \phi_k^{\beta_k}\,d\mu\right|^2\right]^{-1}.$$

Observe that for $s, t \in \omega_j$,

$$|\phi_k^{\beta_k}(s) - \phi_k^{\beta_k}(t)| \leqslant |2\pi\beta_k/p_k|\; |s_k - t_k|$$
$$\leqslant (2\pi|\beta_k|/p_k)\mu(\omega_j)m_{k+1} = 2\pi|\beta_k|\mu(\omega_j)m_k,$$

and

$$|\phi_k^{\beta_k}(s) - \phi_k^{\beta_k}(t)| \leqslant 2.$$

Also,

$$\left|\frac{1}{\mu(\omega_j)}\int_{\omega_j} \phi_k^{\beta_k}\,d\mu\right| = \left|\frac{1 - \exp(2\pi i\beta_k\mu(\omega_j)m_k)}{\mu(\omega_j)m_{k+1}(1 - \exp(2\pi i\beta_k/p_k))}\right|.$$

Therefore, for $x \in \omega_j$,

$$|g(x)| \leqslant \frac{1}{\mu(\omega_j)}\int_{\omega_j} |f|\,d\mu \min(4, (2\pi\beta_k\mu(\omega_j)m_k)^2)$$
$$\times \left[1 - \left|\frac{1 - \exp(2\pi i\beta_k\mu(\omega_j)m_k)}{\mu(\omega_j)m_{k+1}(1 - \exp(2\pi i\beta_k/p_k))}\right|^2\right]^{-1}. \tag{26}$$

A direct calculation shows that for any integer $n \geqslant 2$ and any number θ with $-\pi < \theta \leqslant \pi$, we have

$$(n\theta)^2[1 - |(1 - e^{in\theta})/n(1 - e^{i\theta})|^2]^{-1} \leqslant C \tag{27}$$

for $n|\theta| \leqslant \pi/10$, and

(28) $$[1 - |(1 - e^{in\theta})/n(1 - e^{i\theta})|^2]^{-1} \leqslant C$$

for $n|\theta| \geqslant \pi/10$. (25) now follows immediately from (26), (27) and (28). This concludes the proof of the lemma.

4. **Proof of Theorem 1*.** The case $p = 2$ of (7) is a consequence of Plancherel's formula. It therefore suffices to prove (8), for then (7) will follow by the Marcinkiewicz interpolation theorem [10, II, p. 112] and a duality argument.

For the proof of (8) we note that there is nothing to prove if $\|f\|_1 > y$, so we can assume $\|f\|_1 \leqslant y$. Decompose f as in Lemma 2. Since

$$\mu\{|S_n^* f| > y\} \leqslant \mu\{|S_n^* g| > y/2\} + \mu\{|S_n^* b| > y/2\},$$

(8) will follow if we can show that each term on the right is bounded by $Cy^{-1}\|f\|_1$.

Using the fact that $\{S_n^*\}$ is uniformly bounded in L^2, we obtain

$$\mu\{|S_n^* g| > y/2\} \leqslant Cy^{-2}\|S_n^* g\|_2^2 \leqslant Cy^{-2}\|g\|_2^2 \leqslant Cy^{-1}\|f\|_1,$$

by (13) and (14).

To estimate $|S_n^* b|$ we use the following notation. Let $\omega_j \in F_{k+1}$, with ω_j contained in the coset I of G_k. We consider I as a circle, and let ω_j^* denote the interval inside I which contains ω_j at its center and $\mu(\omega_j^*) = 3\mu(\omega_j)$. Let $\Omega^* = \bigcup_j \omega_j^*$. We have, by (19),

$$\mu(\Omega^*) \leqslant 3 \sum_j \mu(\omega_j) \leqslant 3y^{-1}\|f\|_1.$$

Therefore it suffices to prove

(29) $$\mu\{x \notin \Omega^*: |S_n^* b| > y/2\} \leqslant Cy^{-1}\|f\|_1.$$

To do this we expand $S_n^* b$ as in (9) and (11). Moreover, we observe that for $x \notin \Omega^*$ the first three terms in (11) vanish. This can be seen as follows. Let $I = x + G_k$ and $I' = x + G_{k+1}$. Then neither I nor I' is contained in $\bigcup_j \omega_j$. For the first term in (11), we have

$$\int_{I \cap \{x_k = t_k\}} b(t)\, d\mu(t) = \sum_{\omega_j \subset I'} \int_{\omega_j} b\, d\mu = 0,$$

by (16) and (17). For the second term,

$$\begin{aligned}\int_{I \cap \{x_k = t_k\}} b(t)\phi_k^{\alpha k}(t)\, d\mu(t) &= \sum_{\omega_j \subset I; \omega_j \not\subset I'} \int_{\omega_j} b(t)\phi_k^{\alpha k}(t)\, d\mu(t) \\ &= \sum_{\omega_j \subset I; \omega_j \in C_k} \int_{\omega_j} b(t)\phi_k^{\alpha k}(t)\, d\mu(t) \\ &\quad + \sum_{\omega_j \subset I; \omega_j \not\subset I'; \omega_j \notin C_k} \int_{\omega_j} b(t)\phi_k^{\alpha k}(t) d\mu(t).\end{aligned}$$

If $\omega_j \in C_k$, then $\int_{\omega_j} b\phi_k^{\alpha_k}\,d\mu = 0$, by (17). If $\omega_j \subset I$ and $\omega_j \notin C_k$, then $\phi_k^{\alpha_k}$ is constant on ω_j, so $\int_{\omega_j} b\phi_k^{\alpha_k}\,d\mu = 0$ by (17). Hence $\int_{I\cap\{x_k\neq t_k\}} b(t)\phi_k^{\alpha_k}(t)\,d\mu(t) = 0$. Similarly $\int_{I\cap\{x_k\neq t_k\}} b(t)\,d\mu(t) = 0$. Therefore we have

$$S^*_{\alpha_k m_k} b(x) = i\phi_k^{-\alpha_k}(x) H_k(b\phi_k^{\alpha_k})(x) - iH_k b(x), \qquad x \notin \Omega^*. \tag{30}$$

Thus, if $x \notin \Omega^*$,

$$|S_n^* b(x)| \leqslant \sum_{k=0}^{\infty} |S^*_{\alpha_k m_k} b(x)| \leqslant \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| + \sum_{k=0}^{\infty} |H_k b(x)|.$$

(29) will be proved if we can show

$$\mu\left\{x \notin \Omega^*\colon \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| > \frac{y}{4}\right\} \leqslant Cy^{-1}\|f\|_1 \tag{31}$$

and

$$\mu\left\{x \notin \Omega^*\colon \sum_{k=0}^{\infty} |H_k b(x)| > \frac{y}{4}\right\} \leqslant Cy^{-1}\|f\|_1. \tag{32}$$

We will demonstrate (31). (32) can be proved similarly.

Suppose $x \notin \Omega^*$. Let $I = x + G_k$ and $I' = x + G_{k+1}$. Then, as before, we have

$$\begin{aligned} H_k(b\phi_k^{\alpha_k})(x) &= \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j \subset I;\, \omega_j \not\subset I'} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) d\mu(t) \\ &= \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j \subset I;\, \omega_j \in C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) d\mu(t) \\ &\quad + \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j \subset I;\, \omega_j \not\subset I';\, \omega_j \notin C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) d\mu(t). \end{aligned}$$

Again, if $\omega_j \subset I$ and $\omega_j \notin C_k$, $\phi_k^{\alpha_k}(t)\cot(\pi(x_k - t_k)/p_k)$ is constant on ω_j. Therefore the last term on the right vanishes by (17). Moreover, if $\omega_j \in C_k$, $\int_{\omega_j} b\phi_k^{\alpha_k}\,d\mu = 0$, also by (17). Consequently,

$$\begin{aligned} H_k(b\phi_k^{\alpha_k})(x) = \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j \subset I;\, \omega_j \in C_k} \int_{\omega_j} & b(t)\phi_k^{\alpha_k}(t) \\ & \times \left[\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right] d\mu(t), \end{aligned}$$

where $t^j = \{t_k^j\}_{k\geqslant 0}$ is any fixed point in ω_j. Thus for any coset I of G_k,

$$\int_{I\cap{}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})(x)|\, d\mu(x)$$

$$\leqslant \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j\subset I;\omega_j\in C_k} \int_{\omega_j} |b(t)| \int_{I\cap{}^c\Omega_j^*} \left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(x)\, d\mu(t).$$

A simple calculation shows that, for $t \in \omega_j$,

$$\frac{1}{\mu(I)} \int_{I\cap{}^c\omega_j^*} \left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(x) \leqslant C,$$

so we obtain

$$\int_{I\cap{}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})|\, d\mu \leqslant C \sum_{\omega_j\subset I;\omega_j\in C_k} \int_{\omega_j} |b|\, d\mu \leqslant C \sum_{\omega_j\subset I;\omega_j\in C_k} \int_{\omega_j} |f|\, d\mu,$$

by (18). Therefore

$$\mu\left\{x \notin \Omega^*: \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| > y/4\right\}$$

$$\leqslant Cy^{-1} \sum_{k=0}^{\infty} \int_{{}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})|\, d\mu \leqslant Cy^{-1} \sum_{k=0}^{\infty} \sum_{\omega_j\in C_k} \int_{\omega_j} |f|\, d\mu$$

$$= Cy^{-1} \sum_j \int_{\omega_j} |f|\, d\mu \leqslant Cy^{-1} \|f\|_1.$$

This establishes (31), and hence completes the proof of Theorem 1*.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201

Current address: Department of Mathematics, University of Chicago, Chicago, Illinois 60637