BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS IN CONVEX SUBSETS OF A BANACH SPACE

BY

KLAUS SCHMITT(1) AND PETER VOLKMANN

ABSTRACT. Let $E$ be a real Banach space, $C$ a closed, convex subset of $E$ and $f: [0, 1] \times E \times E \to E$ be continuous. Let $u_0, u_1 \in C$ and consider the boundary value problem

\[ u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1. \]

We establish sufficient conditions in order that (*) have a solution $u: [0, 1] \to C$.

Introduction. Let $C$ be a closed, convex subset of the real Banach space $E$ and let $f: [0, 1] \times C \times E \to E$ be a function with the property

\[ \phi \in E^* \quad (2), \quad x \in C, \quad \phi(x) = \max_{q \in C} \phi(q) \bigg\} \to \phi(f(t, x, y)) \geq 0. \]

In this paper we show that under some additional (sometimes rather restrictive) assumptions the boundary value problem (BVP)

\[ u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1, \]

($u_0, u_1 \in C$) has a solution $u: [0, 1] \to C$. We note that (1) describes the behavior of $f$ on the boundary $\partial C$ of $C$, for if $\phi \neq 0$, then condition (1) implies $x \in \partial C$. In case $E = \mathbb{R}^n$, $n$-dimensional Euclidean space, and $C$ is bounded with int $C(3) \neq \emptyset$, various results of this type exist in the literature (see e.g. [5] for a survey of such results). In this finite dimensional situation the general case may easily be obtained by projection methods. On the other hand, if $E$ is infinite dimensional, certain additional assumptions, either on $E$ or on $f$ seem to be needed to pass from the case int $C \neq \emptyset$ to the general case.

The paper is divided into two parts. In the first part we assume $f(t, x, y)$

1. Received by the editors August 23, 1974.

(1) Research was performed while the first named author was a Visiting Professor at Universität Karlsruhe. His research was supported in part by U. S. Army research grant OAH-C-04-74-G-0208.

(2) $E^*$ denotes the space of all bounded linear functionals on $E$.

(3) "int" denotes the interior of a set.

Copyright © 1976, American Mathematical Society 397
to be completely continuous and satisfy a Nagumo type growth condition with respect to \( y \). Then it is known [6] that if \( C \) is bounded and \( \text{int} C \neq \emptyset \), the BVP (2) has a solution \( u: [0, 1] \rightarrow C \). In Theorem 1 we show that the same conclusion holds in case \( C \) is a closed, bounded, convex subset of a uniformly convex space \( E \), or in case \( C \) is a compact convex subset. (The existence of a solution \( u: [0, 1] \rightarrow C \) of (2) for certain compact convex \( C \) in \( l^p \), \( 1 < p < \infty \), has already been treated by Thompson [7]; his methods, however, are quite different from ours.) In the second part we assume \( f \) in (2) to be independent of \( u' \), continuous on \( [0, 1] \times E \) and satisfy a Lipschitz condition

\[
\|f(t, x) - f(t, y)\| \leq L \|x - y\|, \quad x, y \in E,
\]

where \( L < \pi^2 \). Under these assumptions the existence of a unique solution \( u: [0, 1] \rightarrow E \) of (2) follows easily by means of the contraction mapping principle, see e.g. [1] where the one dimensional case is treated, so one only needs to show that \( u: [0, 1] \rightarrow C \). This is done (Theorem 2) by using results and techniques formerly used by Redheffer and Walter [4] and in [8], [9], [10] in the study of invariance properties of sets relative to initial value problems for first order equations. A final result (Theorem 3) shows that it suffices to assume \( f \) to be defined on \( [0, 1] \times C \), provided the continuity of \( f \) relative to \( t \) is uniform with respect to \( x \in C \).

1. Completely continuous right-hand sides. Throughout this section we assume that \( f: [0, 1] \times C \times E \rightarrow E \) is completely continuous.

**Theorem 1.** Let \( C \) be a closed, bounded, convex subset of \( E \) and assume there exists a continuous projection \( P: E \rightarrow C \) assigning to each \( x \in E \) a nearest point \( Px \in C \) (i.e., \( \|x - Px\| = \text{dist}(C, x) = \inf_{q \in C} \|q - x\| \); such \( P \) always exists if the Banach space \( E \) is uniformly convex in the sense of Clarkson [2]), or assume \( C \) is compact. Let \( u_0, u_1 \in C \) and let \( f \) satisfy (1) and the growth condition

\[
\|f(t, x, y)\| \leq \omega(t\|y\|) \quad (0 \leq t \leq 1, x \in C, y \in E),
\]

where \( \omega: [0, \infty) \rightarrow (0, \infty) \) is a continuous nondecreasing function with

\[
\lim_{s \to \infty} s^2 / \omega(s) = \infty.
\]

Then the BVP (2) has a solution \( u: [0, 1] \rightarrow C \).

**Proof.** 1. If \( C \) is closed, bounded, convex and \( \text{int} C \neq \emptyset \), the above result holds without further assumptions on \( C \), [6, Theorem 4.1].

2. A further result [6, Lemma 2.1] which is needed in what is to follow and which makes use of the properties of \( \omega \) is the following: For each \( R > 0 \) there exists \( M \) (depending only on \( R \) and \( \omega \)) such that: if \( u: [0, 1] \rightarrow E \) is twice continuously differentiable and
3. Let $C$ be such that there exists a continuous projection $P: E \to C$ as in the statement of Theorem 1. Define $\tilde{f}: [0, 1] \times E \times E \to E$ by

$$
\tilde{f}(t, x, y) = f(t, Px, y).
$$

For each $\varepsilon > 0$ the set $C_\varepsilon$ defined by

$$
C_\varepsilon = \{ x \in E : \text{dist}(C, x) \leq \varepsilon \}
$$

is a closed, bounded, convex subset of $E$ with $\text{int} C_\varepsilon \neq \emptyset$. We shall show next that the result of [6] stated in 1. above may be applied to $\tilde{f}$ and $C_\varepsilon$.

Obviously $\tilde{f}$ is completely continuous and verifies the estimate

$$
||\tilde{f}(t, x, y)|| \leq C(||y||) \quad (0 \leq t \leq 1, x, y \in E).
$$

Let us show (1) with $C$ and $f$ replaced by $C_\varepsilon$ and $\tilde{f}$, respectively, i.e.

$$
\varphi \in E^*, x \in C_\varepsilon, \varphi(x) = \max_{q \in C_\varepsilon} \varphi(q)
$$

(6)

$$
y \in E, \varphi(y) = 0, 0 \leq t \leq 1
$$

Let $x \in C_\varepsilon$, then $||x - Px|| \leq \varepsilon$. Thus, if $q \in C$, we have that $q + x - Px \in C_\varepsilon$. The hypotheses of (6) consequently imply

$$
\varphi(x) \geq \varphi(q + x - Px) = \varphi(q) + \varphi(x) - \varphi(Px),
$$

and since $q \in C$ was arbitrary, it follows that

$$
\varphi(Px) = \max_{q \in C} \varphi(q).
$$

Using (1), we therefore obtain

$$
\varphi(\tilde{f}(t, x, y)) = \varphi(f(t, Px, y)) \geq 0,
$$

proving (6).

Using Theorem 4.1 of [6] we conclude the existence of a solution $u_\varepsilon: [0, 1] \to C_\varepsilon$ of the BVP

(7)

$$
u_\varepsilon^n = \tilde{f}(t, u_\varepsilon^n, u_\varepsilon^n') \quad u_\varepsilon(0) = u_0, \quad u_\varepsilon(1) = u_1.
$$

4. We now employ a limiting process (letting $\varepsilon \to 0$) to obtain the desired conclusion.

Let $\{ \varepsilon_n \}$ be a monotone decreasing sequence of real numbers with

$$
\lim_{n \to \infty} \varepsilon_n = 0.
$$

Denote by $u_n = u_{\varepsilon_n}$, where $u_{\varepsilon_n}: [0, 1] \to C_{\varepsilon_n}$ is a solution of (7), with $\varepsilon$ replaced by $\varepsilon_n$. Choose $R > 0$ such that $||u_n(t)|| \leq R, 0 \leq t \leq 1, n = 1, 2, \ldots$. Using (5) and 2. we obtain the existence of a constant $M > 0$ such that $||u_n(t)|| \leq M, 0 \leq t \leq 1, n = 1, 2, \ldots$.

Let $G$ denote the Green's function
\[ G(t, s) = \begin{cases} -s(1 - t), & 0 \leq s \leq t \leq 1, \\ -t(1 - s), & 0 \leq t \leq s \leq 1; \end{cases} \]

then

\[ u_n(t) = \int_0^1 G(t, s) \tilde{f}(s, u_n(s), u_n'(s)) \, ds + (1 - t)u_0 + tu_1 \tag{8} \]

and

\[ u_n'(t) = \int_0^1 \frac{\partial}{\partial s} G(t, s) \tilde{f}(s, u_n(s), u_n'(s)) \, ds + u_1 - u_0. \tag{9} \]

Using the complete continuity of \( \tilde{f} \), the uniform boundedness of \( \{u_n'\} \) and (8), (9) we conclude that \( \{u_n\} \) and \( \{u_n'\} \) are equicontinuous sequences and that there exists a compact set \( K \subseteq E \) such that \( u_n(t), u_n'(t) \in K, 0 \leq t \leq 1, \ n = 1, 2, \ldots. \)

We may thus employ the theorem of Ascoli-Arzelà to obtain a subsequence of \( \{u_n\} \) which converges to a solution \( u \) of

\[ u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1. \]

Since, further, \( \text{dist}(C, u_n(t)) \leq \epsilon_n \) and \( \lim_{n \to \infty} \epsilon_n = 0 \), we obtain \( \text{dist}(C, u(t)) = 0 \), from which follows that \( u : [0, 1] \to C \) and \( \tilde{f}(t, u, u') = f(t, u, u') \), proving that \( u \) is a solution of (2).

5. We next consider the case where \( C \) is a compact convex subset of \( E \) (here no additional assumptions on \( E \) are needed). Choose \( R > 0 \) such that: \( x \in C \Rightarrow \|x\| \leq R \). Determine \( M = M(R, \omega) \) according to 2. above. Define \( Q : E \to E \) by

\[ Qy = \begin{cases} y, & \|y\| \leq M, \\ My/\|y\|, & \|y\| > M, \end{cases} \]

and put

\[ \tilde{f}(t, x, y) = f(t, x, Qy) \quad (0 \leq t \leq 1, x \in C, y \in E). \]

The complete continuity of \( f \) implies that of \( \tilde{f} \). Hence the range of \( \tilde{f} \) is contained in some compact set \( K \subseteq E \), and (1) and (4) are satisfied by \( \tilde{f} \).

Let \( E_1 \) denote the closed linear span of \( C, K \) and restrict \( \tilde{f} \) to \( \tilde{f} : [0, 1] \times C \times E_1 \to E_1 \). Since \( C \) and \( K \) are compact, \( E_1 \) is a separable Banach space. Using a result of Clarkson [2] we may equip \( E_1 \) with a new norm \( \| \cdot \|_1 \), equivalent to \( \| \cdot \| \), such that \( E_1 \) becomes strictly convex. Hence to each \( x \in E_1 \) there corresponds a unique nearest point (with respect to \( \| \cdot \|_1 \)) \( Px \) in \( C \). Since (1) holds with \( E, f \) replaced by \( E_1, \tilde{f} \) (where \( \varphi(x) = \max_{q \in C} \varphi(q) \) is extendable to a \( \Phi \in E^* \) with the same property) and since \( \tilde{f} \) is bounded and the projection \( P \), just defined, is continuous, we may apply the arguments of 3. and 4. to obtain a
solution \( u: [0, 1] \rightarrow C \) of

\[
(10) \quad u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1.
\]

Returning to the original norm we have that \( \|u(t)\| \leq R, 0 \leq t \leq 1 \), and by the monotonicity of \( \omega \) we find \( \|u''(t)\| \leq \omega(\|u'(t)\|) \), implying \( \|u'(t)\| \leq M, 0 \leq t \leq 1 \). Hence the definition of \( \tilde{f} \) shows that \( u \) is a solution of (2).

2. Right-hand sides satisfying a Lipschitz condition. Throughout this section we shall assume that \( f \) is independent of \( u' \) and satisfies a Lipschitz condition

\[
(11) \quad \|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad (0 \leq t \leq 1; x, y \in E).
\]

**Theorem 2.** Let \( C \) be a closed, convex subset of \( E \) and let \( u_0, u_1 \in C \). Assume that \( f: [0, 1] \times E \rightarrow E \) is continuous and satisfies the Lipschitz condition (11) with \( L \leq \pi^2 \). Further assume

\[
(12) \quad (0 \leq t \leq 1, x \in C, \varphi(x) = \max_{q \in C} \varphi(q) \Rightarrow \varphi(f(t, x)) \geq 0.
\]

Then the BVP

\[
(13) \quad u'' = f(t, u), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1,
\]

has a unique solution \( u: [0, 1] \rightarrow C \).

**Proof.** 1. For our proof we need a formula first established for closed, convex cones by Redheffer and Walter [4] equivalent to (12):

\[
(14) \quad \lim_{h \to 0^+} \frac{1}{h} \text{dist}(C, x - hf(t, x)) = 0 \quad (0 \leq t \leq 1, x \in C)
\]

(see [8]). Letting (for \( \xi \geq 0 \))

\[
C_\xi = \{ x \in E: \text{dist}(C, x) \leq \xi \}
\]

\( (C_0 = C) \), using (11) and (12) and a result from [10] we obtain

\[
(15) \quad \lim_{h \to 0^+} \frac{1}{h} \text{dist}(C_\xi, x - hf(t, x)) \leq L\xi \quad (0 \leq t \leq 1, x \in C_\xi).
\]

(In [10] this formula is written with lim sup in place of lim, however, since \( C \) is the limit exists.)

2. Let \( \tilde{E} = E \oplus \mathbb{R} \) normed by \( \|(x, \xi)\| = \max(\|x\|, |\xi|) \). With \( p = (0, 1) \) \( (\theta = \text{zero element of } E) \) we may write

\[
\tilde{E} = E \oplus \mathbb{R} = \{ x + \xi p: x \in E, \xi \in \mathbb{R} \}.
\]

Via the natural embedding, we consider \( E \) as a subspace of \( \tilde{E} \). Let

\[
\tilde{C} = \{ x + \xi p: \text{dist}(C, x) \leq \xi \} ,
\]

then \( \tilde{C} \) is a closed, convex subset of \( \tilde{E} \) with nonempty interior. Define \( \tilde{f}: [0, 1] \)
\[ \mathcal{C} \rightarrow \tilde{E} \] by
\[ \tilde{f}(t, x + \xi p) = f(t, x) - L\xi p \quad (0 \leq t \leq 1, x + \xi p \in \mathcal{C}). \]

Then \( \tilde{f} \) is continuous and satisfies a Lipschitz condition with Lipschitz constant \( L \) with respect to its second argument:
\[ \| \tilde{f}(t, \tilde{x}) - \tilde{f}(t, \tilde{y}) \| \leq L\| \tilde{x} - \tilde{y} \| \quad (0 \leq t \leq 1, \tilde{x}, \tilde{y} \in \tilde{C}). \]

Our method of proof requires a condition analogous to (12) for \( \tilde{f} \) and \( \tilde{C} \), namely:
\[ (0 < f < 1, \xi \in E^*, x \in \tilde{C}, \varphi(\tilde{x}) = \max_{\tilde{q} \in \tilde{C}} \tilde{\varphi}(\tilde{q})) \Rightarrow \tilde{\varphi}(\tilde{f}(t, \tilde{x})) > 0. \]

That (18) follows from (12) has already been sketched in [9] for the case where \( \mathcal{C} \) is a closed, convex cone; our proof to follow is patterned after the one in [9].

(For general closed, convex \( \mathcal{C} \) (18) has been established in [8] for \( f \) defined by \( \tilde{f}(t, x + \xi p) = f(t, x) - 4L\xi p \). That result, however, is not sufficient for our purposes.)

3. To prove (18) we use the equivalence of (12) and (14) (applied to \( \tilde{C} \) and \( \tilde{f} \)) and verify
\[ \lim_{h \to 0+} \frac{1}{h} \text{dist}(\tilde{C}, \tilde{x} - \tilde{h}\tilde{f}(t, \tilde{x})) = 0 \quad (0 \leq t \leq 1, \tilde{x} \in \tilde{C}). \]

Let \( t \in [0, 1] \) and \( \tilde{x} = x + \xi p \in \tilde{C} \), i.e., \( x \in C_\xi \). Then (15) implies that for \( e > 0 \) there exists \( h_0(e) \) such that
\[ h^{-1} \text{dist}(C_\xi, x - hf(t, x)) < L\xi + e \quad (0 < h < h_0(e)). \]

Thus there exists \( y_h \in C_\xi \) (i.e. \( y_h + \xi p \in \tilde{C} \)) such that
\[ \| x - hf(t, x) - y_h \| < hL\xi + he, \]
implying
\[ x - hf(t, x) - y_h + h(L\xi + e)p \in \tilde{C} \equiv \{ y + \eta p: y \in E, \| y \| \leq \eta \}. \]

Now \( \tilde{C} + \tilde{K} \subseteq \tilde{C} \) and \( y_h + \xi p \in \tilde{C} \), yielding
\[ x + \xi p - h[f(t, x) - L\xi p] + hep \in \tilde{C}, \]
from which, in turn, it follows that
\[ h^{-1} \text{dist}(\tilde{C}, \tilde{x} - h\tilde{f}(t, \tilde{x})) < e \quad (0 < h < h_0(e)), \]
implying (19).

4. Define \( P: \tilde{E} \rightarrow \tilde{C} \) by
\[ P(x + \xi p) = \begin{cases} x + \xi p, & \text{dist}(C, x) \leq \xi, \\ x + \text{dist}(C, x)p, & \text{dist}(C, x) > \xi. \end{cases} \]

Then it is easily seen that
Extending $\tilde{f}$ to $[0, 1] \times \tilde{E}$ by setting
\begin{equation}
\tilde{f}(t, \tilde{x}) = \tilde{f}(t, \tilde{P}\tilde{x}) \quad (0 \leq t \leq 1, \tilde{x} \in \tilde{E}),
\end{equation}
we see by (21) that (17) remains valid for the extended function (with the same Lipschitz constant).

Letting
\begin{equation}
C_\eta = C - \eta p = \{x - \eta p: x \in C\} \quad (\eta > 0; C_0 = \overline{C})
\end{equation}
we see that (18) holds with $C$ replaced by $C_\eta$, i.e.,
\begin{equation}
(0 < t < 1, \tilde{x} \in \tilde{E}^*, \tilde{x} \in \tilde{C}_\eta, \tilde{\varphi}(\tilde{x}) = \max_{\tilde{q} \in \tilde{C}_\eta} \tilde{\varphi}(\tilde{q})) \Rightarrow \tilde{\varphi}(\tilde{f}(t, \tilde{x})) > 0,
\end{equation}
for if $\tilde{x} = x + \xi p$ and $\varphi \neq 0$ satisfy the hypotheses of (23), then $\tilde{x} \in \partial \tilde{C}_\eta$ and therefore $x + (\xi + \eta)p = \tilde{x} + \eta p \in \partial \tilde{C}$. Thus dist$(C, x) = \xi + \eta$, which combined with (20) yields $\tilde{P}\tilde{x} = x + (\xi + \eta)p = \tilde{x} + \eta p$. Therefore $\tilde{\varphi}(\tilde{P}\tilde{x}) = \max_{\tilde{q} \in \tilde{C}} \tilde{\varphi}(\tilde{q})$. Using (18) we obtain $\tilde{\varphi}(\tilde{f}(t, \tilde{P}\tilde{x})) > 0$, which by (22) implies (23).

5. The function $\sigma: \tilde{E} \rightarrow \mathbb{R}$, defined by
\begin{equation}
\sigma(x + \xi p) = \begin{cases} 0, & \text{dist}(C, x) \leq \xi, \\ \text{dist}(C, x) - \xi, & \text{dist}(C, x) > \xi, \end{cases}
\end{equation}
satisfies a Lipschitz condition with Lipschitz constant $2$. Choose $e > 0$ such that $L_1 = L + 2e < \pi^2$. Then
\begin{equation}
\hat{f}(t, \tilde{x}) = \tilde{f}(t, \tilde{x}) - e\sigma(\tilde{x})p
\end{equation}
satisfies
\begin{equation}
\|\hat{f}(t, \tilde{x}) - \hat{f}(t, \tilde{y})\| \leq L_1 \|\tilde{x} - \tilde{y}\| \quad (0 \leq t \leq 1, \tilde{x}, \tilde{y} \in \tilde{E});
\end{equation}
further it follows from (23) and (24) that
\begin{equation}
(0 < t < 1, \eta > 0, \tilde{\varphi} \in \tilde{E}^*, \tilde{\varphi} \neq 0, \tilde{x} \in \tilde{C}_\eta, \tilde{\varphi}(\tilde{x}) = \max_{\tilde{q} \in \tilde{C}_\eta} \tilde{\varphi}(\tilde{q}))
\end{equation}
\begin{equation}
= \tilde{\varphi}(\hat{f}(t, \tilde{x})) > 0.
\end{equation}
Because $L_1 < \pi^2$, the BVP
\begin{equation}
\tilde{u}'' = \hat{f}(t, \tilde{u}), \quad \tilde{u}(0) = u_0, \quad \tilde{u}(1) = u_1,
\end{equation}
has a unique solution $\tilde{u}: [0, 1] \rightarrow \tilde{E}$ (this fact has already been mentioned in the introduction). It is the purpose of the next paragraphs to show that $\tilde{u}$ is a solution of (13) with values in $C$. 

6. There exists a smallest $\eta > 0$ such that $\tilde{u} : [0, 1] \to \overline{C}_\eta$ ($\tilde{u}$ is the solution of (26)). Suppose $\eta > 0$. Then there exists $t_0 \in (0, 1)$ such that $\tilde{u}(t_0) \in \partial \overline{C}_\eta$ ($\tilde{u}(0), \tilde{u}(1) \in \text{int } C_\eta$). We may thus choose $\varphi \in \tilde{E}^*, \varphi \neq 0$, such that $\varphi(\tilde{u}(t_0)) = \max_{\eta \in C_\eta} \varphi(\eta)$. By (25)

\begin{equation}
\varphi(\tilde{f}(t_0, \tilde{u}(t_0))) > 0.
\end{equation}

On the other hand, the scalar function $\rho(t) = \varphi(\tilde{u}(t)), 0 \leq t \leq 1$, attains its maximum at $t_0$, hence $\rho''(t_0) \leq 0$. But

$$\rho''(t_0) = \varphi''(\tilde{u}(t_0)) = \varphi(\tilde{f}(t_0, \tilde{u}(t_0))),$$

contradicting (27). Thus $\tilde{u} : [0, 1] \to \overline{C}_0 = \overline{C}$.

7. It now follows from the definition of $\tilde{f}$ that $\tilde{f}(t, \tilde{u}(t)) = \tilde{f}(t, \tilde{u}(t))$. Thus $\tilde{u}$ is the solution of the BVP

\begin{equation}
\tilde{u}'' = \tilde{f}(t, \tilde{u}), \quad \tilde{u}(0) = u_0, \quad \tilde{u}(1) = u_1.
\end{equation}

Using the notation

$$\tilde{u}(t) = u(t) + \xi(t)\rho \quad (u(t) \in E, \xi(t) \in \mathbb{R}, 0 \leq t \leq 1),$$

we may decompose (28) into

\begin{align}
\tilde{u}'' & = \tilde{f}(t, u), \quad u(0) = u_0, \quad u(1) = u_1, \\
\xi'' & = -L\xi, \quad \xi(0) = 0, \quad \xi(1) = 0,
\end{align}

with the further constraint

\begin{equation}
\text{dist}(C, u(t)) \leq \xi(t).
\end{equation}

Since, however, $L < \pi^2$, it follows that $\xi(t) \equiv 0$, and thus $\text{dist}(C, u(t)) = 0$, i.e., $u : [0, 1] \to C$. This completes the proof of Theorem 2.

**Theorem 3.** Theorem 2 remains valid if $f(t, x)$ is only defined on $[0, 1] \times C$, but is uniformly continuous in $t$ with respect to $x$, i.e.,

\begin{equation}
\sup_{x \in C} ||f(t_n, x) - f(t, x)|| \to 0 \quad \text{as } t_n \to t.
\end{equation}

**Proof.** We embed $E$ via an isometric isomorphism in some Banach space $B(S)$ of bounded functions on some set $S$ (e.g. $S = \{ \varphi \in E^*: ||\varphi|| \leq 1 \}$). Then (12) remains valid with $B(S)^*$ in place of $E^*$. Thus we may consider the problem in $B(S)$ instead of $E$; in particular we may consider $f : [0, 1] \times C \to B(S)$, where $C \subseteq B(S)$. By adopting the coordinate conventions and writing the elements $z \in B(S)$ as $z = (z_\sigma)_{\sigma \in S}$ ($z_\sigma \in \mathbb{R}, ||z|| = \sup_{\sigma \in S} |z_\sigma|$), we define $f_\sigma : [0, 1] \times C \to \mathbb{R}$ ($\sigma \in S$) by

$$f_\sigma(t, x) = f(t, x)_\sigma \quad (0 \leq t \leq 1, x \in C, \sigma \in S).$$
The Lipschitz continuity of $f$ implies that of $f_\sigma$, i.e.,

$$|f_\sigma(t, x) - f_\sigma(t, y)| \leq L\|x - y\| \quad (0 \leq t \leq 1, x, y \in C, \sigma \in S).$$

A result of McShane [3] implies that the function

$$\tilde{f}_\sigma(t, x) = \sup_{q \in C} (f_\sigma(t, q) - L\|q - x\|) \quad (x \in B(S))$$

is an extension of $f_\sigma$ to $[0, 1] \times B(S)$, such that

$$|\tilde{f}_\sigma(t, x) - \tilde{f}_\sigma(t, y)| \leq L\|x - y\| \quad (0 \leq t \leq 1, x, y \in B(S), \sigma \in S).$$

Define $\tilde{f} : [0, 1] \times B(S) \rightarrow B(S)$ by

$$\tilde{f}(t, x) = \tilde{f}_\sigma(t, x) \quad (0 \leq t \leq 1, x \in B(S), \sigma \in S).$$

Then $\tilde{f}$ is an extension of $f$ to $[0, 1] \times B(S)$ and satisfies (11). By (32) $\tilde{f}(t, x)$ is also continuous with respect to $t$. We may therefore use Theorem 2 to conclude that the BVP

$$u'' = \tilde{f}(t, u), \quad u(0) = u_0, \quad u(1) = u_1 \quad (u_0, u_1 \in C)$$

has a solution $u : [0, 1] \rightarrow C$. Since $\tilde{f}$ is an extension of $f$, $u$ is a solution of the original problem.

REFERENCES


