CERTAIN CONTINUA IN $S^n$ OF THE SAME SHAPE
HAVE HOMEOMORPHIC COMPLEMENTS($^1$)

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ABSTRACT. As a consequence of Theorem 1 of this paper, we see that
if $X$ and $Y$ are globally 1-alg continua in $S^n$ ($n \geq 5$) having the shape of the
real projective space $P^k$ ($k \neq 2, 2k + 2 < n$), then $S^n - X \approx S^n - Y$. (For $P^1$
$= S^1$, this establishes the last case of such a result for spheres.) We also show
that if $X$ and $Y$ are globally 1-alg continua in $S^n$, $n \geq 6$, which have the shape
of a codimension $> 3$, closed, $0 < (2m - n + 1)$-connected, PL-manifold $M^m$,
then $S^n - X \approx S^n - Y$.

1. Introduction. The problem of classifying the shape of compacta in $S^n$
(or $E^n$) in terms of their complements in $S^n$ (or $E^n$) has been studied by a num-
ber of people.

In [3], Chapman proved that two $Z$-sets in the Hilbert cube have the same
shape if and only if their complements are homeomorphic. Working with $Z_k$-
sets, Geoghegan and Summerhill [10] improved the finite dimensional theorem
of Chapman [4] by reducing the condition $n \geq 3k + 3$ to $n \geq 2k + 2$.

Rushing, in [16], proved that for a continuum $X$ in $S^n$ ($n \geq 5$), $\text{Sh}(X) =
\text{Sh}(S^k)$ ($S^k$ is the standard $k$-sphere in $S^n$) is equivalent to $S^n - X \approx S^n - S^k$,
if $X$ is globally 1-alg in $S^n$ and $k \neq 1$ ($S^n - X$ must have homotopy type of $S^1$
if $k = n - 2$). He also gave an example to show that $S^n - X \approx S^n - S^1$ is not
sufficient to imply $\text{Sh}(X) = \text{Sh}(S^1)$.

In [5], Coram, Daverman and Duvall proved that if $\dim X \leq n - 3, X \subset
E^n$ satisfies small loop condition (SLC) and $X$ has the shape of a finite complex
$K$ in the trivial range, then $X$ has a neighborhood $N$ in $E^n$ such that $N - X \approx
\partial N \times [0, 1)$, where $N$ is also a regular neighborhood of a copy of $K$ in $E^n$ ($n \geq 5$).

Recently, Coram and Duvall have proved [6] the equivalence of $S^n - X \approx
S^n - Y$ and $\text{Sh}(X) = \text{Sh}(Y)$, where $X, Y$ are sphere-like continua in $S^n$ ($n \geq 5$)
(see definition in [15]) satisfying SLC and $\max \{\dim X, \dim Y\} \leq n - 4$.

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Received by the editors February 17, 1975.

AMS (MOS) subject classifications (1970). Primary 57A15, 57A35; Secondary 57C20,
57C30, 57C40.

Key words and phrases. Stable end, $H$-cobordism, regular neighborhood, shape, glo-
ally 1-alg.

($^1$) This research will constitute a part of the author's doctoral dissertation under the
direction of Professor T. B. Rushing at the University of Utah.

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However, notice that the class of sphere-like continua is much smaller than the class of continua having the shape of sphere-like continua.

In this note, we consider globally 1-alg continua in $S^n$ having either the shape of finite complexes in the trivial range or the shape of closed, simply connected PL-manifolds of codimension $\geq 3$. As a result, we solve the $S^1$-case of Rushing [16]. Thus, we generalize the main result of Daverman [8].

I am very grateful to Professor T. B. Rushing for many helpful questions and discussions concerning this paper. I also thank Dr. R. Stern for his help.

2. Notation and definitions. Throughout this note, we use the following notations:

$\approx$ Homeomorphic or isomorphic
$\cong$ Homotopy equivalence or homotopic
$\sim$ Homologous
$\partial V$, Int $V$ Boundary, interior of a manifold $V$
i or $A \subset B$ Inclusion map
$f_\ast, f_\#$ Induced maps on homotopy, homology groups
$H_\ast$ Singular homology, $\mathbb{Z}$ coefficients
$\tilde{H}_\ast$ Čech cohomology, $\mathbb{Z}$ coefficients

For basic shape theory results, we refer to [1] and [15]. For convenience, in this paper we use both shape theories [1] and [15] as is justified in [22].

A continuum $X$ in $S^n$ is said to be globally 1-alg in $S^n$ if for every neighborhood $U$ of $X$ in $S^n$, there is a neighborhood $V$ of $X(V \subset U)$ such that if $f : S^1 \to V - X$, $f \sim 0$ in $V - X$, then $f \simeq 0$ in $U - X$.

An inverse sequence of groups

$$G_1 \overset{i_1}{\leftarrow} G_2 \overset{i_2}{\leftarrow} \cdots$$

is said to be constant if we have

$$\text{Im } i_1 \overset{i_1}{\sim} \text{Im } i_2 \overset{i_2}{\sim} \text{Im } i_3 \overset{i_3}{\sim} \cdots.$$

An inverse sequence of groups

$$G_1 \overset{i_1}{\leftarrow} G_2 \overset{i_2}{\leftarrow} \cdots$$

is said to be stable if it has a constant subsequence.

Let $X$ be a continuum in $S^n$. If $S^n - X$ is connected, then $S^n - X$ has a unique end $e$. According to Siebenmann [17, Chapter III], the fundamental group $\pi_1(e)$ is stable if there is a nested sequence $\{V_j\}$ of connected neighborhoods of $X$ such that the inverse sequence

$$\pi_1(V_1 - X) \overset{i_1}{\leftarrow} \pi_1(V_2 - X) \overset{i_2}{\leftarrow} \cdots$$

is constant, where $i_q : (V_{q+1} - X) \subset (V_q - X)$.

In this case, $\pi_1(e)$ is said to be isomorphic to Im $i_{1\ast}$. 

By a closed manifold, we mean a compact manifold without boundary.

For definitions of regular neighborhood, PL-embedding, PL-homeomorphism, etc., we refer to Hudson [12].

A complex $\mathcal{K}$ in $S^n$ (or $E^n$) is said to be in trivial range if $2 \dim \mathcal{K} + 2 \leq n$.

A continuum is a compact, connected space.

Let $\{\gamma_1, \ldots, \gamma_q\}$ be a family of pairwise disjoint simple closed curves in the interior of a 2-simplex $\Delta^2$. Let $F$ be the closure of the component of $\Delta^2 - \bigcup_{i=1}^q \gamma_i$ which contains $\partial \Delta^2$. The components of $\partial F$ other than $\partial \Delta^2$ are the outermost elements of $\{\gamma_1, \ldots, \gamma_q\}$.


**Theorem 1.** Let $X$ be a globally 1-alg continuum in $S^n$, $n \geq 5$, having the shape of a finite complex $\mathcal{K}$ ($2k + 2 \leq n$) such that $\pi_1(\mathcal{K})$ is abelian. If $\pi_1(\mathcal{K}) = 0$ or $\pi_2(\mathcal{K}) = 0$, then $X$ has a neighborhood $N$, which is a regular neighborhood of a copy $\mathcal{K}_1$ of $\mathcal{K}$ in $S^n$, such that $N - X \approx \partial N \times [0, 1)$ ($\approx N - \mathcal{K}_1$).

As an immediate consequence of Theorem 1, Lemma 4.3 of Geoghegan and Summerhill [10], and the unknottedness of trivial-range complexes [11], we obtain the following result.

**Theorem 2.** Let $X, Y$ be globally 1-alg continua in $S^n$, $n \geq 5$, having the shape of finite complexes $\mathcal{K}, \mathcal{L}$ (respectively) in trivial range such that $\pi_1(\mathcal{K}), \pi_1(\mathcal{L})$ are abelian. If either $\pi_1(\mathcal{K}) = \pi_1(\mathcal{L}) = 0$ or $\pi_2(\mathcal{K}) = \pi_2(\mathcal{L}) = 0$, then $\text{Sh}(X) = \text{Sh}(Y)$ if and only if $S^n - X \approx S^n - Y$.

**Corollary 1.** Let $X, Y$ be globally 1-alg continua in $S^n$, $n \geq 5$, having the shape of the projective space $P^k$, $2k + 2 \leq n, k \neq 2$. Then, $S^n - X \approx S^n - Y$.

**Corollary 2.** Let $X$ be a globally 1-alg continuum in $S^n$, $n \geq 5$, then $\text{Sh}(X) = \text{Sh}(T^k)$ if and only if $S^n - X \approx S^n - T^k$ and $X$ has the shape of a trivial-range finite complex $\mathcal{K}$, with $\pi_2(\mathcal{K}) = 0$ and $\pi_1(\mathcal{K})$ is abelian, where $2k + 2 \leq n$.

**Remark 1.** This corollary generalizes the weakly flat 1-spheres theorem of Daverman [8, Theorem 1].

**Remark 2.** Using J. Stallings' theorem in M. A. Kervaire [14, Theorem V] and imitating an example in [16], we can construct a globally 1-alg connected polyhedron $X$ in $S^n$ such that $\text{Sh}(X) \neq \text{Sh}(S^1)$ but $S^n - S^1 \approx S^n - X$ and $\pi_2(X) \neq 0$ (even though $\pi_1(X) \approx \pi_1(S^1) \approx Z$).

**Theorem 3.** Let $X$ be a globally 1-alg continuum in $S^n$ ($n \geq 6$), having the shape of a simply connected, finite complex $\mathcal{K}$, $\dim \mathcal{K} \leq n - 3$. Then, $S^n - X$ has a collar at the end $e$, i.e. there is a PL-manifold neighborhood $W$ of $X$ in $S^n$ such that $W - X \approx \partial W \times [0, 1)$.

(Furthermore, $W$ and $K$ have the same homotopy type.)
Corollary 3. Let $X$ be a continuum in $S^n$ ($n \geq 6$) as in Theorem 3 above. If either $2 \dim k + 1 < n$ or $K$ is a closed PL-manifold, then $S^n - X \approx S^n - K_1$, where $K_1$ is the image of a PL-embedding of $K$ into $S^n$.

The following corollary follows from Irwin's embedding theorem [13] and Zeeman's unknotting theorem [20].

Corollary 4. Let $X$ and $Y$ be globally 1-alg continua in $S^n$ ($n \geq 6$), which have the shape of a simply connected codimension $\geq 3$, closed, $(2m - n + 1)$-connected PL-manifold $M^m$. Then, $S^n - X \approx S^n - Y$.

Remark 3. For $n \geq 6$, Corollary 4 generalizes the weakly flat $k$-spheres theorem of Duvall [9, Theorem 2.1], and it also generalizes Rushing [16, Theorem 3] for $2 \leq k \leq n - 3$ and $n \geq 6$.

4. Details of the proof. Let $X$ be a continuum in $S^n$. Suppose that $X$ has the shape of a finite complex $K^k$. By Mardešić and Segal's definition of shape [15] (as observed [5]), we can find a cofinal sequence $\{V_i\}_{i=1}^{\infty}$ of connected neighborhoods of $X$ in $S^n$ with $V_{i+1} \subset V_i$, for each $i$, and maps $f_i:K \to V_i$, $g_i:V_i \to K$ such that if $\beta^j$ denotes the inclusion map of $V_i$ into $V_j$ for $i \geq j$, then

1. $f_i$ is a PL embedding for each $i$, if $2k + 1 < n$,
2. $f_i g_i \approx \beta^j$ if $i > j$,
3. $g_i \beta^j f_i \approx 1_K$, and
4. $\beta^j f_i \approx f_j$.

Lemma 1. The following sequence is constant:

$$\cdots \longrightarrow H_1(V_{j+1}) \xrightarrow{(\beta^{j+1},1)_\#} H_1(V_j) \xrightarrow{(\beta^{j},1-1)_\#} H_1(V_{j-1}) \longrightarrow \cdots$$

and $\lim H_1(V_j) \approx H_1(K)$.

Proof. The following commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
H_1(V_{j+1}) & \xrightarrow{(\beta^{j+1},1)_\#} & H_1(V_j) & \xrightarrow{(\beta^{j},1-1)_\#} & H_1(V_{j-1}) \\
(f_{j+1})_\# & (g_{j+1})_\# & (f_{j})_\# & (g_{j})_\# & (f_{j-1})_\# \\
H_1(K) & \xrightarrow{1_\#} & H_1(K) & \xrightarrow{1_\#} & H_1(K)
\end{array}
\end{array}
$$

gives

1. $(g_{j+1})_\#, (g_{j})_\#$ are onto,
2. $(f_{j})_\#, (f_{j-1})_\#$ are 1-1.

We have
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\[
\text{Im}(\beta^{j+1,i})_\# = \text{Im} \left[ (f_j)_\# \circ (g_{j+1})_\# \right] = \text{Im}(f_j)_\#
\]
\[
\approx (g_{j+1})_\# (\text{Im}(f_j)_\#) = \text{Im}(f_j)_\# (= H_1(K))
\]
\[
\quad \text{(since } (g_{j+1})_\# (f_j)_\# = 1\#)\]
\[
\approx (f_{j-1})_\# (\text{Im}(g_{j+1})_\#) \quad \text{since } (f_{j-1})_\# \text{ is 1-1}
\]
\[
= \text{Im}(\beta^{j-1,i})_\#.
\]

Therefore, $(\beta^{j,i})_\#: \text{Im}(\beta^{j+1,i})_\# \rightarrow \text{Im}(\beta^{j-1,i})_\#$ is an isomorphism. That means the given sequence is constant. It is also clear that $H_1(V_j) \approx H_1(K)$.

**Lemma 2.** Let $X$ be a continuum in $S^n$ such that $\text{Sh}(X) = \text{Sh}(K)$, where $K$ is a finite complex. Then $H_q(V, V - X) = 0$ for all $q \leq n - \text{dim } K - 1$, and neighborhoods $V$ of $X$ in $S^n$.

**Proof.** Indeed, we have
\[
H_q(V, V - X) \approx \tilde{H}^{n-q}(X) \quad [19, \text{Theorem 6.2.17}]
\]
\[
\approx \tilde{H}^{n-q}(X) \quad [19, \text{Corollary 6.8.8}]
\]
\[
\approx \tilde{H}^{n-q}(K) \quad [15, \text{Theorem 16}]
\]
\[
= 0 \quad \text{if } n - q > \text{dim } K \quad \text{(i.e., } q < n - \text{dim } K - 1).\]

**Lemma 3.** Let $X$ be a continuum in $S^n$ having the shape of a finite complex $K$, $\text{dim } K \leq n - 3$, then the following sequence is constant.

\[
\cdots \rightarrow H_1(V_{j+1} - X) \quad (i_j)_\# \rightarrow H_1(V_j - X) \rightarrow \cdots
\]

and $H_1(\epsilon) \equiv \lim H_1(V_j - X) \approx H_1(K)$.

**Proof.** By Lemma 2, we have the following commutative diagram:

\[
\begin{array}{ccc}
0 \rightarrow H_1(V_{j+1} - X) & \approx & H_1(V_{j+1}) \rightarrow 0 \\
\downarrow (i_j)_\# & & \downarrow (\beta^{j+1,i})_\# \\
0 \rightarrow H_1(V_j - X) & \approx & H_1(V_j) \rightarrow 0
\end{array}
\]

The lemma follows easily from Lemma 1.

**Lemma 4.** Let $X$ be a continuum in $S^n$ having the shape of a finite complex $K$, with $\text{dim } K \leq n - 3$. If $X$ is globally $1$-alg in $S^n$, then the end $\epsilon$ of $S^n - X$ is stable and $\pi_1(\epsilon) \approx H_1(K)$.

**Proof.** (The proof of this lemma is similar to the last part of the proof of Lemma 1 in [7].)

We can choose a subsequence $\{V_{j_p}\}$ of $\{V_j\}$ such that every loop in $(V_{j_{p+1}} - X)$ which is null-homologous in $(V_{j_{p+1}} - X)$ is null-homotopic in
Thus, we may assume the sequence \( \{ V_j \} \) has this property.

Using the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(V_{j+1} - X) & \xrightarrow{\varphi_{j+1}} & H_1(V_{j+1} - X) \\
| & | & | \\
(i_j)_* & \downarrow & (i_j)_# \\
\pi_1(V_j - X) & \xrightarrow{\varphi_j} & H_1(V_j - X) \\
| & | & | \\
(i_{j-1})_* & \downarrow & (i_{j-1})_# \\
\pi_1(V_{j-1} - X) & \xrightarrow{\varphi_{j-1}} & H_1(V_{j-1} - X)
\end{array}
\]

where \( \varphi_j \)'s are Hurewicz's homomorphisms, the globally 1-alg property implies that

\[
\varphi_j|\operatorname{Im}(i_j)_* : \operatorname{Im}(i_j)_* \rightarrow \operatorname{Im}(i_{j-1})_#
\]

is an isomorphism, for each \( j \), by the diagram chasing argument.

Therefore,

\[
(i_j)_*|\operatorname{Im}(i_j)_* : \operatorname{Im}(i_j)_* \rightarrow \operatorname{Im}(i_{j-1})_#
\]

is an isomorphism and \( \operatorname{Im}(i_j)_* \approx H_1(K) \).

Thus, the open PL-manifold \( S^n - X \) has a stable isolated end \( e \) with \( \pi_1(e) \approx H_1(K) \) being finitely presented. By Siebenmann [17, Theorem 3.10], there exist arbitrarily small 1-neighborhoods of \( e \), if \( n \geq 5 \); i.e., for every compact subset \( C \) of \( S^n - X \), there is a neighborhood \( V \) of \( X \) in \( S^n \) such that

1. \( V \cap C = \emptyset \),
2. the natural map \( \pi_1(e) \rightarrow \pi_1(V - X) \) is an isomorphism,
3. the inclusion map \( \partial V \subset V - X \) gives an isomorphism \( \pi_1(\partial V) \rightarrow \pi_1(V - X) \),
4. \( \partial V \) and \( V - X \) are connected.

**Lemma 5.** Let \( X \) be a continuum in \( S^n \) \((n \geq 5)\) having the shape of a finite complex \( K \), with \( \pi_1(K) \) abelian and \( \dim K \leq n - 3 \). If \( X \) is globally 1-alg in \( S^n \), then given any neighborhood \( U \) of \( X \) in \( S^n \), there is a neighborhood \( V \) of \( X(V \subset U) \) such that

1. \( V - X \) is a 1-neighborhood of the end, and
2. \( i_* : \pi_1(V - X) \rightarrow \pi_1(V) \) is an isomorphism.

**Proof.** Let \( W \) be a neighborhood of \( X \) such that

1. \( W \subset U \), and
2. \( W - X \) is a 1-neighborhood of the end.

Let \( i > j \) be two integers and \( V \) a neighborhood of \( X \) such that

1. \( V \subset V_i \subset V_j \subset W \subset U \), and
2. \( \pi_1(V_j - X) \approx H_1(K) \).
3. the natural map \( \pi_1(e) \rightarrow \pi_1(V - X) \) is an isomorphism,
4. the inclusion map \( \partial V \subset V - X \) gives an isomorphism \( \pi_1(\partial V) \rightarrow \pi_1(V - X) \),
5. \( \partial V \) and \( V - X \) are connected.
(2) $V - X$ is a 1-neighborhood of the end.

($V_p, V_j$ satisfies conditions (2), (3), (4) preceding Lemma 1.)

First, by Lemmas 2 and 4, and the following commutative diagram, we have $i_* : \pi_1(V - X) \rightarrow \pi_1(V)$ is 1-1, since $V - X$ is a 1-neighborhood of the end.

\[
\pi_1(V - X) \rightarrow \pi_1(V) \\
\approx \\
H_1(V - X) \approx H_1(V)
\]

Then, $\beta^w_* i_* : \pi_1(V - X) \rightarrow \pi_1(W)(\beta^w_* : V \subset W)$ is also 1-1, by the following commutative diagram.

\[
\pi_1(V - X) \rightarrow \pi_1(V) \\
\approx \\
\pi_1(W - X) \rightarrow \pi_1(W)
\]

Hence $(f_j)_* (g_i | V)_* i_*$ is one-to-one, since $\beta^w_* = (f_j)_* (g_i | V)_*$ by the following commutative diagram.

We claim that $(g_i | V)_* i_*$ is one-to-one. Hence, it will follow that $\pi_1(V)$ is abelian, since $\pi_1(K)$ is abelian. Thus, $i_*$ is an isomorphism by the following commutative diagram.

\[
\pi_1(V) \rightarrow \pi_1(V_j) \rightarrow \pi_1(W) \\
\beta^w_*
\]

We now prove the claim. Let $\varphi: \partial \Delta^2 \rightarrow V$ be a loop representing an element of $\ker(g_i | V)_*$. Then $[\varphi] \in \ker \beta^w_*$ since $\beta^w_* = (f_j)_* (g_i | V)_*$; i.e., $\varphi \approx 0$ in $W$.

Let $\varphi: \Delta^2 \rightarrow W$ be an extension of $\varphi$ over $\Delta^2$ into $W$. We may assume that $\varphi(\partial \Delta^2) \cap \partial V = \emptyset$ and $\varphi^{-1}(\partial V)$ is a family of disjoint simple closed curves in $\text{Int} \Delta^2$.

Let $\Gamma_1, \ldots, \Gamma_s$ be outermost loops in this family, then each $\varphi(\Gamma_j)$ bounds a disk in $W$; hence $\varphi(\Gamma_j) \approx 0$ in $W$, $j = 1, \ldots, s$. Thus, $\varphi(\Gamma_j) \approx 0$ in $V - X$. 
for \( j = 1, \ldots, s \), since \( \overline{\varphi}(\Gamma_j) \subset V - X \) and \( \rho_*^{V_W} \circ i_* : \pi_1(V - X) \to \pi_1(W) \) is one-to-one.

Therefore, by changing the value of \( \overline{\varphi} \) inside \( \Gamma_j \), for \( j = 1, \ldots, s \), we can define \( \overline{\varphi} \) to obtain an extension of \( \varphi \) in \( V \) over \( \Delta^2 \). In other words, \( \varphi \approx 0 \) in \( V \).

**Lemma 6.** Let \( X \) be a continuum in \( S^n \), \( n \geq 5 \), such that

1. \( \text{Sh}(X) = \text{Sh}(K) \), where \( K \) is a finite complex with \( \pi_1(K) \) abelian and \( \dim K \leq n - 3 \),
2. \( X \) is globally 1-alg in \( S^n \),
3. Either \( \pi_1(K) = 0 \) or \( \pi_2(K) = 0 \).

Then, given a neighborhood \( U \) of \( X \), there is a neighborhood \( V \) of \( X(V \subset U) \) such that \( \pi_i(V, V - X) = 0 \), for \( i = 0, 1, 2 \).

**Proof.** Let \( V \) be a neighborhood of \( X \) as in Lemma 5.

(i) 1-connectedness is trivial, since

\[
H_0(V, V - X) \cong \pi_1(V) \approx 0, \quad \text{and}
\]

\[
\pi_1(V - X) \approx \pi_1(V) \to \pi_1(V, V - X) \to 0.
\]

(ii) To show \( \pi_2(V, V - X) = 0 \).

Case 1. \( \pi_1(K) = 0 \). We have

\[
\pi_1(V) \approx \pi_1(V - X) \approx \pi_1(e) \approx H_1(K) = 0.
\]

Hence, the relative Hurewicz isomorphism theorem [19, Theorem 7.5.4] gives

\[
\pi_2(V, V - X) \approx H_2(V, V - X) = 0.
\]

Case 2. \( \pi_2(K) = 0 \). We choose \( V' \) to be a small neighborhood of \( X \) such that

1. \( V' \subset \text{Int} \, V \),
2. \( V' - X \) is a 1-neighborhood of the end satisfying Lemma 5, and
3. \( \iota_* : \pi_2(V') \to \pi_2(V) \) is trivial. (From the fact that \( \pi_2(K) = 0 \) and \( \text{Sh}(X) = \text{Sh}(K) \).)

Now, let \( \varphi : (\Delta^2, \partial \Delta^2) \to (V, V - X) \) be a map representing an element of \( \pi_2(V, V - X) \). We may assume \( \varphi(\partial \Delta^2) \subset \partial V \).

Let \( T \) be a fine subdivision of \( \Delta^2 \) so that for every \( \sigma \in T \) we have \( \varphi(\sigma) \subset \text{Int} \, V' \), if \( \varphi(\sigma) \cap X \neq \emptyset \).

We are through if we have a map \( G : \Delta^2 \times [0, 1] \to V \) satisfying

1. \( G|\Delta^2 \times 0 = \varphi \),
2. \( G(\Delta^2 \times 1) \subset V - X \),
3. \( G(\partial \Delta^2 \times 1) \subset V - X \).

First, we define \( G : (T \times 0) \cup (T^{(1)} \times [0, 1]) \to V \) as follows (\( T^{(1)} \) is the 1-skeleton of \( T \)).

(i) \( G|T \times 0 = \varphi \).
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(ii) Let $v$ be a vertex of $T$. Then

(a) $G(v \times [0, 1]) = \varphi(v)$, if $\varphi(v) \not\in X$.

(b) $G(v \times [0, 1])$ is an arc $\alpha_v$ in $V'$ joining $\varphi(v)$ and $G(v \times 1) \in V' - X$, if $\varphi(v) \in X$.

(iii) Let $\langle uv \rangle$ be a 1-simplex of $T$. Then

(a) $G(x, t) = \varphi(x)$ for every $x \in \langle uv \rangle$, $t \in [0, 1]$, if $\varphi(\langle uv \rangle) \cap X = \emptyset$.

(b) If $\varphi(\langle uv \rangle) \cap X \neq \emptyset$, then $G(\langle uv \rangle \times 1)$ will be an arc $\alpha_{uv}$ in $V' - X$ joining $G(u \times 1)$ and $G(v \times 1)$ such that the loop $G(\langle uv \rangle \times 0) \cup \alpha_u \cup \alpha_{uv} \cup \alpha_v$ is null-homotopic in $V'$ (first we join $G(u \times 1)$ and $G(v \times 1)$ by an arc in $V' - X$, then we use the fact that $\pi_1(V' - X) \cong \pi_1(V')$). Therefore, we can extend $G$ over $\langle uv \rangle \times [0, 1]$ into $V'$.

Similarly for all 1-simplexes of $T$.

Secondly, we define $G : T \times [0, 1] \to V$ as follows. Let $\sigma$ be a 2-simplex of $T$.

(a) If $\varphi(\sigma) \cap X = \emptyset$, $G(x, t) = \varphi(x)$, for every $x \in \sigma$ and $t \in [0, 1]$.

(b) If $\varphi(\sigma) \cap X \neq \emptyset$, then $G$ has been already defined on $(\sigma \times 0) \cup (\partial \sigma \times [0, 1])$ into $V'$ with $G(\partial \sigma \times 1) \subset V' - X$. It is clear that $G|\partial \sigma \times 1 \cong 0$ in $V'$, hence $G|\partial \sigma \times 1 \cong 0$ in $V' - X$. Therefore, we can extend $G$ over $\sigma \times 1$ into $V' - X$. Thus, $G(\partial(\sigma \times [0, 1])) \subset V'$. Now $G$ can be extended over $\sigma \times [0, 1]$ into $V$ by the choice of $V'$.

Similarly for all 2-simplexes of $T$, we can define a map $G : T \times [0, 1] \to V$ such that $G|T \times 0 = \varphi$ and $G(T \times 1) \subset V - X$ as we desired.

**Proof of Theorem 1.** The proof follows by combining Lemma 6 with the following result, the proof of which is intrinsic in [5].

**Lemma 7 (Coram, Daverman, Duvall).** Let $X$ be a compactum in $S^n$, $n \geq 5$, which has the shape of a finite complex in the trivial range. Suppose that given a neighborhood $U$ of $X$ in $S^n$, there exists a neighborhood $V$ such that $X \subset V \subset U$ and $\pi_i(V, V - X) = 0$ for $i = 0, 1, 2$. Then, $X$ has a neighborhood $N$, which is a regular neighborhood of a copy $K_1$ of $K$ in $S^n$, such that $N - X \approx \partial N \times [0, 1] (\approx N - K_1)$.

**Proof of Theorem 2.** Since $S^n - X \approx S^n - K_1$ and $S^n - Y \approx S^n - L_1$ by Theorem 1, the conclusion of Theorem 2 is equivalent to saying that $K \approx L$ if and only if $S^n - K \approx S^n - L$. The “only if” part is a special case of Theorem 1.

On the other hand, since $K$ unknots in $S^n$, we may assume $K \cap L = \emptyset$.

Again, since $K \cup L$ unknots in $S^n$, we may assume $K \cup L$ lies in $S^{n-1}$ (the standard $(n-1)$-sphere). The “if” part of the theorem now follows from Lemma 4.3 of [10].

**Remark 4.** Combining Theorem 2.4 in [5] and the proof of Theorem 2 above, we can state the following result.
Let $X$, $Y$ be continua in $S^n$, $n \geq 5$, having the shape of finite complexes in the trivial range, satisfying SLC (definition in [5]) and $\max(\dim X, \dim Y) \leq n - 3$. Then $\text{Sh}(X) = \text{Sh}(Y)$ if and only if $S^n - X \approx S^n - Y$.

**Proof of Theorem 3.** We have $\tilde{H}^*(X) \approx \tilde{H}^*(K)$, since $\text{Sh}(X) = \text{Sh}(K)$ [15, Theorem 16]. Hence, we can prove that $H_\ast(S^n - X)$ is finitely generated by using Alexander duality and the fact that $H^\ast(K)$ is finitely generated.

Furthermore, the end $e$ is stable and $\pi_1(e) \approx H_1(K) = 0$ (Lemma 4). We have $S^n - X$ is PL-homeomorphic to the interior of a compact PL $n$-manifold $M$ [17, Theorem 5.9]. Then, we may assume that $M$ is contained in $S^n - X$.

Let $W = S^n - \text{Int} M$. It is clear that $W - X$ is PL-homeomorphic to $\partial W \times [0, 1)$.

Now, it is easy to see that $X$ has a nested sequence $\{W_j\}$ of PL $n$-manifold neighborhoods such that $W_1 = W$ and the inclusion $W_{j+1} \hookrightarrow W_j$ is a homotopy equivalence for every $j \in \mathbb{N}$. By terminology of [15], we can say that $X$ is associated with the ANR-sequence $X = \{W_j, \iota_{j+1,j}, N\}$. Then, by Theorem 6 and Theorem 5 of [15], $\text{Sh}(X) = \text{Sh}(W)$. Hence $W \approx K$ by Theorem 4 in [15] and 8.6 of [1].

**Proof of Corollary 3.** Let $f: K \rightarrow \text{Int} W$ ($W$ in the previous theorem) by a map that defines the homotopy equivalence between $K$ and $\text{Int} W$. We may assume that $f$ is a PL-embedding by the following observation. If $2 \dim K + 1 \leq n$, $f$ is homotopic to a PL-embedding in $\text{Int} W$ by PL-approximation and general position theorem. If $K$ is a closed PL-manifold and $\dim K \leq n - 3$, $f$ is homotopic to a PL-embedding in $\text{Int} W$ by Corollary 11.3.4 [21]. Therefore, in either case, $f$ is homotopic to a PL-embedding in $\text{Int} W$, say $f'$. It is clear that $f'$ is also a homotopy equivalence. That proves the claim.

Let $K_1 = f(K)$. Then $K_1 \hookrightarrow \text{Int} W$ is also a homotopy equivalence.

We can now apply Theorem 2.1 of [18] to conclude that $\text{Int} W$ is PL-homeomorphic to the interior of a regular neighborhood $N$ of $K_1$ in $S^n$, fixing $K_1$ ($\pi_1(\partial W) = 0 = \pi_1(\text{Int} W)$).

Now, it can be shown that the PL $n$-manifold $\overline{W - N_1}$ is a $H$-cobordism whose boundary is $\partial W \cup \partial N_1$ with $\pi_1(\partial W) = 0$ and $\pi_1(\partial N_1) = 0$ where $N_1$ is a regular neighborhood of $K_1$ in $\text{Int} W$. From $H$-cobordism theorem, we infer that $W - K_1 \approx \partial W \times [0, 1) \approx W - X$, and the corollary follows.

**References**

MR 39 #7604.
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