MEAN CONVERGENCE OF FOURIER SERIES
ON COMPACT LIE GROUPS

BY

ROBERT J. STANTON

ABSTRACT. The main result is an L^p mean convergence theorem for
the partial sums of the Fourier series of a class function on a compact semi-
simple Lie group. A central element in the proof is a Lie group-Lie algebra
analog of the theorems in classical Fourier analysis that allow one to pass back
and forth between multiplier operators for Fourier series in several variables
and multiplier operators for the Fourier transform in Euclidean space. To ob-
tain the L^p mean convergence theorem, the theory of the Hilbert transform
with weight function is needed.

Introduction. A theorem of M. Riesz says that if f is in L^p of the circle, $1 < p < \infty$, and if $S_N f(x) = \Sigma_{k=-N}^{N} a_k e^{ikx}$ is the Nth partial sum of the Fourier series of f, then $S_N f$ converges to f in the L^p norm as $N \rightarrow \infty$. Pollard [15] proved a similar result for Jacobi polynomials on the interval $[-1, 1]$. If f is in $L^p([-1, 1]; (1 - x)^\alpha (1 - x)^\beta \, dx)$ and if

$$S_N f(x) = \sum_{k=0}^{N} d_k^{\alpha, \beta} a_k R_k^{\alpha, \beta}(x)$$

is the Nth partial sum of the Jacobi series of f, then $S_N f$ converges to f in the L^p norm provided

$$4 \max\left\{\frac{\alpha + 1}{2\alpha + 3}, \frac{\beta + 1}{2\beta + 3}\right\} < p < 4 \min\left\{\frac{\alpha + 1}{2\alpha + 1}, \frac{\beta + 1}{2\beta + 1}\right\}.$$

Here $R_k^{\alpha, \beta}$ is a normalized Jacobi polynomial and $d_k^{\alpha, \beta}$ is an appropriate constant.

It is well known that for suitable choices of α, β the $\{R_k^{\alpha, \beta}\}$ are the elementary spherical functions for the rank 1 symmetric spaces of compact type. Cast in this setting Pollard's theorem is an L^p mean convergence result for bi-K invariant functions on the rank 1 compact symmetric space U/K. In this paper we investigate extending this result to higher rank symmetric spaces.

Even in the abelian case of the n-torus T^n the result depends drastically on how the multiple series is summed. Consider for example f in $L^p(T^n)$ with

Fourier series $\Sigma_{k=(k_1,\ldots,k_n)} a_k e^{ik\cdot x}$ and form the spherical partial sums

$$S_R f(x) = \sum_{k_1^2 + \ldots + k_n^2 < R^2} a_k e^{ik\cdot x}.$$

Fefferman [6] has proved that $S_R f$ converges to f in the L^p norm only if $p = 2$. However if one uses square partial sums

$$S_N f(x) = \sum_{|k_i| < N} a_k e^{ik\cdot x}$$

the problem essentially factors into n iterates of the dimension 1 case and L^p mean convergence holds for $1 < p < \infty$.

Our main result is an L^p mean convergence theorem for Fourier series of class functions on a compact semisimple Lie group in which the partial sums are analogous to the square ones for the torus. The method of proof closely parallels the abelian case. The technique is to transfer spherical multiplier problems on a general compact symmetric space (of which a compact semisimple Lie group is the the special case $G \times G/\Delta$ with Δ the diagonal subgroup) to equivalent Fourier multiplier problems on the tangent space at the identity coset of the symmetric space. For symmetric spaces that are Lie groups this equivalence is essentially established; however, for a general compact symmetric space we prove only one direction. The Fourier multiplier problems arising from a convex polyhedra method of summing the Fourier series of class functions turn out to reduce to problems about iterates of the Hilbert transform in one variable with weight functions. Results in [12] on the Hilbert transform with weight then provide the tool to prove these are bounded operators.

In §1 we summarize notation and review basic aspects of harmonic analysis on compact symmetric spaces and of Fourier analysis on the tangent space at the identity coset.

In §2 passage from spherical multipliers on the symmetric space to Fourier multipliers on the tangent space is obtained.

From §3 on, we restrict attention to compact semisimple Lie groups. In §3 we prove a multiplier restriction theorem from Fourier multipliers on invariant functions to spherical multipliers on class functions. Then for these multipliers the problem on the group and the problem on the Lie algebra are equivalent. An observation further reduces Fourier multiplier problems on invariant functions on the Lie algebra to classical Fourier multipliers on Euclidean space with weight functions.

In §4 we apply the results of the preceding sections to prove the mean convergence theorem for Fourier series of class functions. In the converse direction, we establish continuity properties of the Fourier transform of invariant functions on the Lie algebra that yield negative results for multiplier problems.

I would like to thank my thesis advisor, A. W. Knapp, for his invaluable

advice and encouragement while this research was being done. It is also a pleasure to thank R. Coifman, C. Fefferman, E. M. Stein, R. Strichartz and P. Tomas for their suggestions.

1. **Fourier transforms, multipliers, and Bessel functions.** Let U be a compact simply connected semisimple Lie group. Let $\widetilde{\theta}$ be an involution on U and K its fixed point set. K is a connected compact subgroup and U/K is a Riemannian globally symmetric space of compact type which is also simply connected. Let $\mathfrak{u}$ and $\mathfrak{k}$ be the respective Lie algebras. Then $\widetilde{\theta}$ induces an involution automorphism θ of $\mathfrak{u}$ and a direct sum decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}_*$ where $\mathfrak{k}$ (resp. $\mathfrak{p}_*$) is the $+1$ (resp. -1) eigenspace of θ. The concept of roots and restricted roots can be defined intrinsically for U and U/K; however, we shall define them by means of the noncompact duals since we shall have to introduce these duals later anyway.

Let $\mathfrak{g}^{\mathbf{C}}$ be the complexification of $\mathfrak{u}$ and extend θ to $\mathfrak{g}^{\mathbf{C}}$. Set $\mathfrak{p} = i\mathfrak{p}_*$ and $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{g}_0$ is a noncompact semisimple Lie algebra with Cartan involution θ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{h}$ a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{a}$. Then $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{a}$ where $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$. The complexification $\mathfrak{h}^{\mathbf{C}}$ of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$, and we denote the set of roots of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{h}^{\mathbf{C}}$ by Δ. We fix compatible orderings on the real duals of $\mathfrak{a}$ and $\mathfrak{h}_R = \mathfrak{a} \oplus i\mathfrak{h}_{\mathfrak{k}}$. Since the roots take real values in $\mathfrak{h}_R$ we get an order on Δ. Denote by Δ^+ the set of positive roots and $\mathfrak{h}_R^+$ the corresponding positive Weyl chamber.

The Killing form $B(\cdot, \cdot)$ of $\mathfrak{g}^{\mathbf{C}}$ restricted to $\mathfrak{h}_R \times \mathfrak{h}_R$ is nondegenerate and we use B to identify $\mathfrak{h}_R$ and $\mathfrak{h}_R'$ in the usual way. For λ, μ in $\mathfrak{h}_R'$ we denote by $\langle \lambda, \mu \rangle$ the bilinear form induced by B. An element λ in $\mathfrak{h}_R'$ is called integral if for all $\alpha \in \Delta$, $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbf{Z}$ and is called dominant if for all $\alpha \in \Delta^+$, $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \geqslant 0$. Let $L \subset \mathfrak{h}_R'$ denote the semilattice of dominant integral linear functionals. The set of equivalence classes of irreducible unitary representations of U is in 1-1 correspondence with L. The semilattice in $\mathfrak{h}_R$ dual to L by means of B is contained in the closure of $\mathfrak{h}_R^+$.

Set $P_+ = \{\alpha \in \Delta^+ \mid \alpha(H) \neq \alpha(\theta H) \text{ for some } H \in \mathfrak{h}_R\}$ and set $\mathfrak{a}^+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in P_+\}$. Let M' be the normalizer of $\mathfrak{a}$ in K and M the centralizer of $\mathfrak{a}$ in K. The Weyl group $W = M'/M$ leaves $\mathfrak{a}$ invariant and has $\mathfrak{a}^+$ for a fundamental domain.

We now review some aspects of analysis on U/K, most of which can be found in Helgason [9] or Coifman and Weiss [3]. Let du and dk denote Haar measures on U and K normalized to have total mass 1 and let $d\ddot{u}$ denote the U invariant measure on U/K with total mass 1. Let $L^2(U/K)$ be the Hilbert space of square integrable measurable functions on U/K.

The left regular representation of U on $L^2(U/K)$ provides a decomposition of $L^2(U/K)$ into an orthogonal direct sum of minimal invariant subspaces. The irreducible subrepresentations are precisely the class 1 representations of (U, K), those with a one-dimensional subspace of K-fixed vectors, and occur with multiplicity 1. Let $\Lambda \subset L$ be the set of class 1 highest weights. For λ in Λ let V_λ be the subspace of $L^2(U/K)$ of λ type, let $d_\lambda = \dim V_\lambda$ and let φ_λ in V_λ be the corresponding elementary spherical function. The dual of Λ is contained in $\mathfrak{a}^+$.

The projection operator $P_\lambda: L^2(U/K) \longrightarrow V_\lambda$ may be realized as an integral operator in the following way. If f is in $L^2(U/K)$ and the projection $U \longrightarrow U/K$ is denoted by $u \longrightarrow \dot{u}$ then

$$P_\lambda f(x) = d_\lambda \int_U f(\dot{u}) \varphi_\lambda(u^{-1}x)\, du, \quad x \in U/K.$$

We denote this integral by $P_\lambda f = d_\lambda f * \varphi_\lambda$.

For f in $L^2(U/K)$ we call the expression $\Sigma_{\lambda \in \Lambda}\, d_\lambda f * \varphi_\lambda$ the Fourier series of f. If f is in $L^2(K\backslash U/K)$ the Fourier series of f has the simpler form $\Sigma_{\lambda \in \Lambda}\, d_\lambda \hat{f}(\lambda) \varphi_\lambda$ where $\hat{f}(\lambda) = \int_{U/K} f(\dot{u}) \overline{\varphi_\lambda(\dot{u})}\, d\dot{u}$.

A bounded linear operator T on $L^p(U/K)$ is called a spherical multiplier operator if there exists a sequence $\{m(\lambda)\}_{\lambda \in \Lambda}$ such that for f in $L^2(U/K) \cap L^p(U/K)$, $Tf = \Sigma_{\lambda \in \Lambda}\, m(\lambda) d_\lambda f * \varphi_\lambda$. Spherical multiplier operators on $L^p(K\backslash U/K)$ are defined similarly. In either case the sequence is called a spherical multiplier. Spherical multiplier operators on $L^p(U/K)$ can be characterized as bounded linear operators commuting with left translation by K.

The Riemannian structure on U/K that we have implicitly assumed is the restriction of B to $\mathfrak{p}_* \times \mathfrak{p}_*$ and then translated by U. The fact that B restricted to $\mathfrak{p}_* \times \mathfrak{p}_*$ is negative definite will now be exploited to study Fourier analysis on $\mathfrak{p}_*$.

We view $\mathfrak{p}_*$ as a locally compact abelian group self-dual by means of B. Lebesgue measure dX on $\mathfrak{p}_*$ is normalized to be self-dual. For f, g in $C_c(\mathfrak{p}_*)$ the Fourier transform is given by

$$\hat{f}(Y) = \int_{\mathfrak{p}_*} f(X) e^{-iB(X,Y)}\, dX, \quad Y \in \mathfrak{p}_*,$$

and the Fourier inversion formula by

$$\check{g}(X) = \int_{\mathfrak{p}_*} g(Y) e^{iB(X,Y)}\, dY.$$

A bounded linear operator T on $L^p(\mathfrak{p}_*)$ is called a multiplier operator if there exists a measurable function m so that $Tf = [m\hat{f}]^\vee$ for f in $L^2(\mathfrak{p}_*) \cap L^p(\mathfrak{p}_*)$.

The group K acts on $\mathfrak{p}_*$ via the adjoint map. Set $\mathfrak{a}_*^+ = -i\mathfrak{a}^+$. The map $\Phi: K/M \times \mathfrak{a}_*^+ \longrightarrow \mathfrak{p}_*$ given by $\Phi(kM, H) = \operatorname{Ad} k(H)$ is a diffeomorphism onto a dense open subset. A measurable function f on $\mathfrak{p}_*$ is called Ad K *invariant* if for

all k in K and almost all X in $\mathfrak{p}_*$, $f(\text{Ad } k(X)) = f(X)$. By the above diffeomorphism such a function is specified a.e. by its values on $\mathfrak{a}_*^+$. Corresponding to Φ there is an integration formula:

$$\int_{\mathfrak{p}_*} f(X)\,dX = \int_{\mathfrak{a}_*^+} \int_K f(\text{Ad } k(H))\,dk \left| \prod_{\alpha \in P_+} \alpha(H) \right| dH, \quad f \in C_c(\mathfrak{p}_*).$$

Now suppose f is in $C_c(\mathfrak{p}_*)$ and is Ad K invariant. Then

$$\hat{f}(Y) = \int_{\mathfrak{p}_*} f(X) e^{-iB(X,Y)}\,dX$$

$$= \int_{\mathfrak{a}_*^+} \int_K f(\text{Ad } k(H)) e^{-iB(\text{Ad } k(H),Y)} \left| \prod_{\alpha \in P_+} \alpha(H) \right| dk\,dH$$

$$= \int_{\mathfrak{a}_*^+} f(H) \left[\int_K e^{-iB(\text{Ad } k(H),Y)}\,dk \right] \left| \prod_{\alpha \in P_+} \alpha(H) \right| dH.$$

From this we see that $\hat{f}$ is also Ad K invariant. We define a function J on $\overline{\mathfrak{a}_*^+} \times \overline{\mathfrak{a}_*^+}$ by

$$J(H_1, H_2) = \int_K e^{iB(\text{Ad } k(H_1),H_2)}\,dk.$$

J is W invariant and so extends to an Ad K invariant function on $\mathfrak{p}_* \times \mathfrak{p}_*$. We call J a Bessel function since for rank $U/K = 1$ these are classical Bessel functions $J_k(x)/|x|^k$. These functions have a very rich structure and many properties in common with the classical Bessel functions. The proofs of the properties are group theoretic or geometric in nature. In the next section one of these properties is developed that has applications to multiplier problems.

2. **A multiplier theorem from confluence.** In this section we derive a relationship between spherical functions on the symmetric space and Bessel functions on $\mathfrak{p}_*$ considered as the tangent space at the identity coset of the symmetric space. This is a generalization of the classical Mehler-Heine theorem relating Jacobi polynomials and Bessel functions. We then take the first step toward transferring spherical multiplier problems on U/K to equivalent Fourier multiplier problems on $\mathfrak{p}_*$. We prove that an Ad K invariant function is a Fourier multiplier on $\mathfrak{p}_*$ if a certain family of sequences obtained from this function are uniformly bounded spherical multipliers on U/K.

Let G^C be the complexification of U and G_0 the real analytic subgroup of G^C with Lie algebra $\mathfrak{g}_0$. Corresponding to $\mathfrak{a}$ we have a local Iwasawa decomposition $\mathfrak{g}_0 = k \oplus \mathfrak{a} \oplus \mathfrak{n}$ and a global Iwasawa decomposition $G_0 = KAN$.

LEMMA 2.1. (a) *There exist neighborhoods N_0 of 0 in $\mathfrak{g}^C$ and N_e of e in G^C such that* $\exp: N_0 \longrightarrow N_e$ *is a complex biholomorphic map.*

(b) *Suppose $\mathfrak{g}^C = L_1 \oplus L_2 \oplus L_3$ with the L_i complex subspaces. Let V be an open neighborhood of 0 in $\mathfrak{g}^C$ such that if Y is in V and $Y = X_1 + X_2 +$*

X_3 with $X_i \in L_i$, then $\exp X_1 \exp X_2 \exp X_3$ is in N_e. *Then the map* $\varphi: V \longrightarrow$ *N_0 given by* $\varphi(Y) = \log(\exp X_1 \exp X_2 \exp X_3)$ *has* $d\varphi(0)$ *equal to the identity and is a complex biholomorphic map of an open neighborhood of 0 in* $\mathfrak{g}^C$ *onto another such neighborhood.*

PROOF. This is essentially the existence of holomorphic canonical coordinates of the 1st and 2nd kind and the proof in Hochschild [11] for real Lie groups can be repeated for the complex case.

We apply Lemma 2.1 to the complexified local Iwasawa decomposition. Write $\mathfrak{g}^C = \mathfrak{k}^C \oplus \mathfrak{a}^C \oplus \mathfrak{n}^C$ and for Z in $\mathfrak{g}^C$ let $\mathfrak{H}(Z)$ be the projection of Z onto $\mathfrak{a}^C$. For g in G_0 let $H(g)$ in $\mathfrak{a}$ be defined by $g = k \exp H(g)n$. We want to continue H analytically to a neighborhood of e in G^C. Let V_1 and V_2 be the neighborhoods of 0 given by Lemma 2.1(b). The map $\mathfrak{H} \circ \varphi^{-1} \circ \exp^{-1}$ is holomorphic from $\exp V_1$ to $\mathfrak{a}^C$ and agrees with H on $\exp V_1 \cap G_0$. We denote this extension of H also by H.

LEMMA 2.2. *There exists a neighborhood V of 0 in* $\mathfrak{g}^C$ *and a positive number ϵ such that*

$$H(\exp tZ) = t\mathfrak{H}(Z) + O(t^2)$$

for Z in V and $t \in \mathbf{C}$ with $|t| < \epsilon$.

PROOF. We use a series expansion for H and Lemma 2.1. Let τ denote conjugation on $\mathfrak{g}^C$ with respect to $\mathfrak{u}$. Then $B_\tau(X, Y) = -B(X, \tau Y)$ is a positive definite hermitian inner product on $\mathfrak{g}^C$. For Z in $\mathfrak{g}^C$ set $\|Z\|^2 = B_\tau(Z, Z)$. Now choose V_1 as above. As H is holomorphic on V_1 for some $\epsilon > 0$, the ball $B_\epsilon = \{Z \in \mathfrak{g}^C \mid \|Z\| < \epsilon\}$ is contained in V_1 and H has a series expansion at e valid on $\exp B_\epsilon$. So for Z in B_ϵ and $t \in \mathbf{C}$ with $|t| < 1$, tZ is in B_ϵ and

$$H(\exp tZ) = H(e) + dH(e)(tZ) + O(t^2).$$

We shall denote the point at which the differential is computed by a subscript on the differential. $H(e) = 0$ and on B_ϵ exp defines a chart, so that

$$H(\exp tZ) = d(H \circ \exp)_0(tZ) + O(t^2).$$

Since $H \circ \exp = \mathfrak{H} \circ \varphi^{-1}$, $d(H \circ \exp)_0 = d\mathfrak{H}_{\varphi^{-1}(0)} \circ d\varphi_0^{-1}$. By Lemma 2.1 $d\varphi_0^{-1}$ is equal to the identity; $\mathfrak{H}$ linear means that $d\mathfrak{H} = \mathfrak{H}$. Then

$$d(H \circ \exp)_0(tZ) = t\mathfrak{H}(Z),$$

and we have the desired result.

Harish-Chandra [8] obtained an integral formula for spherical functions corresponding to finite-dimensional class 1 representations of G_0: If λ is in Λ, then

$$\varphi_\lambda(x) = \int_K e^{\lambda(H(xk))}\, dk \quad \text{for } x \in G_0.$$

The analytic continuation of the H function allows us to obtain a local integral formula for φ_λ on U. In fact,

$$H(\exp Xk) = H(kk^{-1}\exp Xk) = H(\exp Ad\, k^{-1}(X)) \quad \text{for } X \in \mathfrak{g}_0,$$

and B_ϵ is invariant under $Ad\, K$; thus the integrand in Harich-Chandra's formula for φ_λ is defined for $x = \exp Z$ with Z in B_ϵ. Now φ_λ is a matrix coefficient of a finite-dimensional representation of G^C, hence is holomorphic on all G^C. On $\exp B_\epsilon$, φ_λ and the analytically continued integral are holomorphic and agree on $\exp(B_\epsilon) \cap G_0$, hence are equal on $\exp B_\epsilon$. Summarizing, we have

PROPOSITION 2.3. *If X is in $\mathfrak{u}$ and $\|X\| < \epsilon$, then*

$$\varphi_\lambda(\exp X) = \int_K e^{\lambda(H(\exp Ad\, k(X)))}\, dk.$$

REMARK. For special U, e.g., rank $U/K = 1$, the ϵ can be determined and $\exp B_\epsilon$ fills a dense open subset of U. The integral has been called a "Laplace type" integral because for $SU(2)/SO(2)$ it gives the Laplace integral for Legendre polynomials.

PROPOSITION 2.4. *For $\lambda \in \Lambda$ and n a positive integer*

$$\lim_{n\to\infty} \varphi_{n\lambda}(\exp X/n) = J(H_\lambda^*, X), \quad X \in \mathfrak{u},$$

with the convergence uniform on compact subsets of $\mathfrak{u}$.

PROOF. Recall from §1 that the element dual to λ is in $\overline{\mathfrak{a}^+}$. If H_λ is this element, we set $H_\lambda^* = -iH_\lambda$. Then H_λ^* is in $\overline{\mathfrak{a}_*^+}$. Let $L \subset \mathfrak{u}$ be a compact set. Then L is contained in B_N, a ball of radius N, for N large enough. Let ϵ be as in Proposition 2.3. For all $n > N/\epsilon$ and for all X in L, X/n is in B_ϵ. Then

$$\varphi_{n\lambda}\left(\exp\frac{X}{n}\right) = \int_K e^{n\lambda(H(\exp Ad\, k(X/n)))}\, dk.$$

By Lemma 2.2 we have

$$\varphi_{n\lambda}\left(\exp\frac{X}{n}\right) = \int_K e^{n\lambda(\Phi(Ad\, k(X/n)))} e^{n\lambda(O(1/n^2))}\, dk$$

$$= \int_K e^{\lambda(\Phi(Ad\, k(X)))} e^{n\lambda(O(1/n^2))}\, dk.$$

As $n \to \infty$, $n\lambda(O(1/n^2)) \to 0$, so that $e^{n\lambda(O(1/n^2))} \to 1$. Then

$$\lim_{n\to\infty} \varphi_{n\lambda}\left(\exp\frac{X}{n}\right) = \int_K e^{\lambda(\Phi(Ad\, k(X)))}\, dk.$$

Recall that

$$J(H_\lambda^*, X) = \int_K e^{iB(H_\lambda^*,\, Ad\, k(X))}\, dk.$$

To complete the proof we need only observe that

$$B(H_\lambda, \text{Ad } k(X)) = B(H_\lambda, \mathfrak{H}(\text{Ad } k(X))) = \lambda(\mathfrak{H}(\text{Ad } k(X))).$$

To see the first equality suppose Z is in $\mathfrak{z}^C$ and $Z = X_1 + X_2 + X_3$ with $X_1 \in \mathfrak{k}^C$, $X_2 = \mathfrak{H}(Z)$, $X_3 \in \mathfrak{n}^C$. Then $\mathfrak{k} \perp \mathfrak{p}$ implies that $B(H_\lambda, X_1) = 0$. Lemma VI, 3.5 in Helgason [9] shows the existence of a basis of $\mathfrak{g}^C$ with respect to which ad X_3 is strictly lower triangular and ad H_λ is diagonal. Hence $B(H_\lambda, X_3) = 0$.

THEOREM 2.5.   *Let m be a bounded* Ad K *invariant function on $\mathfrak{p}_*$ that is continuous a.e. Suppose that for each $r > 0$ the sequence $\{m(H_\lambda^*/r)\}_{\lambda \in \Lambda}$ is a spherical multiplier on $L^p(U/K)$ and the norms of the corresponding operators are uniformly bounded. Then m is a multiplier on $L^p(\mathfrak{p}_*)$ $(1 < p < \infty)$.*

REMARK.   This result was proved by Igari [13] and Clerc [2] independently when rank $U/K = 1$. Strichartz [18] then extended it to symmetric spaces that are compact Lie groups. Proposition 2.4 allows us to push Strichartz's proof through for arbitrary compact U/K. Clerc has independently found Proposition 2.4 with essentially the same proof. We include the proof then only for completeness.

PROOF.   Let q be the index conjugate to p. We first assume m decreases rapidly, say $|m(X)| \leqslant c_1 \exp(-c_2 \|X\|^2)$. Let f and g be in $C_c^\infty(\mathfrak{p}_*)$. It suffices to show that $I = \int_{\mathfrak{p}_*} Tf(X)g(X)\, dX$ is bounded in absolute value by $C\|f\|_p\|g\|_q$ with C independent of f and g. The idea is to obtain a Riemann sum approximation to I by using Proposition 2.4 and then to use the uniform boundedness of the operators corresponding to the dilates of m.

$$I = \int_{\mathfrak{p}_*} Tf(X)g(X)\, dX = \int_{\mathfrak{p}_*} \int_{\mathfrak{p}_*} m(Z)\hat{f}(Z)e^{iB(X,Z)}\, dZ\, g(X)\, dX$$

$$= \int_{\mathfrak{p}_*} \int_{\mathfrak{p}_*} \int_{\mathfrak{p}_*} m(z)e^{iB(X,Z)}f(Y)e^{-iB(Y,Z)}g(X)\, dY\, dZ\, dX.$$

As m decreases rapidly, we may apply the Fubini theorem to the two inside integrals and then use the integration formula for Ad K invariant functions on $\mathfrak{p}_*$ to obtain

$$I = \int_{\mathfrak{p}_*} \int_{\mathfrak{p}_*} \int_{\mathfrak{a}_+^*} m(H)J(H, X - Y)\left|\prod_{\alpha \in P_+} \alpha(H)\right| dH\, f(Y)g(X)\, dY\, dX.$$

Now choose R large enough so that exp is a chart on $R^{-1}\text{supp } f$ and $R^{-1}\text{supp } g$. Define functions f_R and g_R on U/K by $f_R(\exp X/R) = f(X)$ and $g_R(\exp X/R) = g(X)$. Let T_R be the operator corresponding to $\{m(H_\lambda^*/R)\}_{\lambda \in \Lambda}$ and let $I_R = \int_{U/K} T_R f_R(\dot{u})g_R(\dot{u})\, d\dot{u}$. By hypothesis there exists $A < \infty$ independent of R, f, g such that $|I_R| \leqslant A\|f_R\|_p\|g_R\|_q$. We claim there is a constant C depend-

ing only on U such that $R^N I_R \longrightarrow CI$ as $R \longrightarrow \infty$ where $N = \dim U/K$. In fact

$$R^N I_R = R^N \int_{U/K} \int_{U/K} \sum_{\lambda \in \Lambda} m\left(\frac{H_\lambda^*}{R}\right) d_\lambda \varphi_\lambda(y^{-1}x) f_R(y) g_R(x) \, d\dot{u}(y) \, d\dot{u}(x),$$

$$R^N I_R = R^N \int_{\mathfrak{p}_*} \int_{\mathfrak{p}_*} \sum_{\lambda \in \Lambda} m\left(\frac{H_\lambda^*}{R}\right) d_\lambda \varphi_\lambda(\exp - Y \exp X)$$

$$\times f_R(\exp Y) g_R(\exp X) \Delta_0(Y) \Delta_0(X) \, dY \, dX,$$

where we have used exp as a chart and if $Y = \mathrm{Ad}\, kH$,

$$\Delta_0(Y) = \text{Jacobian } \exp_{eK}(Y) = \Delta_0(\mathrm{Ad}\, kH) = \prod_{\alpha \in P_+} \frac{\sin \alpha(iH)}{\alpha(iH)}.$$

Now we make the change of variables $Y = Y/R$, $X = X/R$ to obtain

$$R^N I_R = R^N \cdot R^{-2N} \int_{\mathfrak{p}_*} \int_{\mathfrak{p}_*} \sum_{\lambda \in \Lambda} m\left(\frac{H_\lambda^*}{R}\right) d_\lambda \varphi_\lambda(\exp - Y/R \exp X/R)$$

$$\times f(Y) f(X) \Delta_0(Y/R) \Delta_0(X/R) \, dY \, dX.$$

Consider the inside sum

$$R^{-N} \sum_{\lambda \in \Lambda} m(H_\lambda^*/R) d_\lambda \varphi_\lambda(\exp - Y/R \exp X/R) \Delta_0(Y/R) \Delta_0(X/R).$$

As $R \longrightarrow \infty$, $\Delta_0(Y/R)$ and $\Delta_0(X/R)$ tend to 1 and will be no problem. Using the Campbell-Hausdorff formula we see that $\exp - Y/R \exp X/R = \exp Z_R/R$ with $Z_R \longrightarrow X - Y$ as $R \longrightarrow \infty$. Make the change of variable $\lambda = \lambda/R$ so the expression becomes

$$R^{-N} \sum_{\lambda \in \Lambda/R} m(H_\lambda^*) d_{\lambda R} \varphi_{R\lambda}(\exp Z_R/R) \Delta_0(Y/R) \Delta_0(X/R).$$

Weyl's formula for the degree of a representation is

$$d_{\lambda R} = \prod_{\alpha \in \Delta^+} \frac{\langle \alpha, \lambda R + \rho \rangle}{\langle \alpha, \rho \rangle}.$$

As λR is class 1 we have $\langle \alpha, \lambda R \rangle = 0$ for $\alpha \notin P_+$. If $|P_+|$ is the cardinality of P_+, then

$$d_{\lambda R} = R^{|P_+|} \prod_{\alpha \in \Delta^+} \frac{\langle \alpha, \lambda + \rho/R \rangle}{\langle \alpha, \rho \rangle}$$

and as $R \longrightarrow \infty$,

$$\frac{d_{\lambda R}}{R^{|P_+|}} \longrightarrow \frac{\Pi_{\alpha \in P_+} \langle \alpha, \lambda \rangle}{\Pi_{\alpha \in \Delta^+} \langle \alpha, \rho \rangle}.$$

Since $N = \dim \mathfrak{a}_* + |P_+|$, the expression is

$$\frac{1}{R^{\dim \mathfrak{a}_*}} \sum_{\lambda \in \Lambda/R} m(H_\lambda^*) \frac{d_{\lambda R}}{R^{|P_+|}} \varphi_{R\lambda} \left(\frac{\exp Z_R}{R}\right) \Delta_0\left(\frac{Y}{R}\right) \Delta_0\left(\frac{X}{R}\right).$$

Using Proposition 2.4 and the continuity a.e. of m we see that the above is a Riemann sum approximation to

$$\frac{1}{\Pi_{\alpha \in \Delta^+} \langle \alpha, \rho \rangle} \int_{\mathfrak{a}_*^+} m(H) J(H, X - Y) \prod_{\alpha \in P_+} \langle \alpha, H \rangle \, dH.$$

Then

$$\lim_{R \to \infty} R^N I_R = \frac{1}{\Pi_{\alpha \in \Delta^+} \langle \alpha, \rho \rangle} I,$$

which is the claim.

Finally,

$$|I| = C \lim_{R \to \infty} R^N \left| \int_{U/K} T_R f_R(\dot{u}) g_R(\dot{u}) \, d\dot{u} \right|,$$

$$|I| \leqslant C \lim_{R \to \infty} R^N \cdot A \|f_R\|_p \|g_R\|_q$$

$$= C \cdot A \cdot \lim_{R \to \infty} R^{N/p} \|f_R\|_p \, R^{N/q} \|g_R\|_q$$

$$= C \cdot A \|f\|_p \|g\|_q.$$

And so m is a multiplier on $L^p(\mathfrak{p}_*)$. To remove the hypothesis that m decreases rapidly, define for $\epsilon > 0$ a new multiplier $m^\epsilon(X) \exp(-\epsilon \|X\|^2)$ and observe that the constant A can be chosen independent of ϵ. Letting $\epsilon \to 0$, we have the desired result.

REMARK 2.6. Theorem 2.5 is true for multipliers on Ad K invariant functions on $\mathfrak{p}_*$ if the dilated multipliers are uniformly bounded operators on bi-K invariant functions on U/K. The proof is exactly the same. In fact if our original f and g were Ad K invariant, the only fact one needs to check is that f_R and g_R are bi-K invariant. But this is obvious.

REMARK 2.7. One can formulate Theorem 2.5 for weak type boundedness of multipliers rather than strong type. For rank 1, Connett and Schwartz [5] have proved such a result. Proposition 2.4 allows one to carry their proof over to arbitrary rank.

3. **Restriction and transplantation of multipliers.** When the rank of U/K is 1, one can use Theorem 2.5 to obtain many well-known multiplier theorems on $\mathbf{R}^n$ from the corresponding results for Jacobi polynomials (see Igari [13]). However, for higher rank little is known about analysis on U/K, whereas multiplier theorems on $\mathbf{R}^n$ have been deeply investigated. So it would be much more useful to have a "restriction" theorem for multipliers from the tangent space to the symmetric space.

Let G be a compact simply connected semisimple Lie group and let $\mathfrak{g}$ be

its Lie algebra. In this section we shall prove a theorem concerning restrictions of multipliers on Ad G invariant functions on $\mathfrak{g}$ to central multipliers on class functions on G. The analogous abelian theorem is due to de Leeuw [14], and our proof parallels the proof of de Leeuw's theorem found in Stein and Weiss [17]. Afterward we make an observation that allows us to transplant Fourier multiplier questions on Ad G invariant functions to classical Fourier multipliers on Euclidean space with weight functions. Coifman and Weiss [4] also have transplantation type results. The essential difference between their approach and ours is they use the weight function to modify the multiplier whereas we keep the multiplier fixed and face handling multiplier problems with weight.

The group G with the Riemannian structure induced from the Killing form on $\mathfrak{g}$ is isometric to the symmetric space U/K where $U = G \times G$ and $K = \Delta$, the diagonal subgroup. By means of this diffeomorphism, the bi-K invariant functions on U corresponds to class functions on G and the spherical functions on U/K correspond to characters of irreducible unitary representations of G divided by the degree of the representation. In this setting the restriction theorem in this section is a partial converse to Theorem 2.5.

Since G is a symmetric space the notation introduced in §1 could still be used; however results in the literature that we need are expressed in more traditional notation. For the purpose of compatibility with the literature we employ the traditional notation.

Let G be a compact simply connected semisimple Lie group and T a maximal torus in G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras and $\mathfrak{g}^C$ and $\mathfrak{h}^C$ the respective complexifications. Let Δ be the set of roots of $(\mathfrak{g}^C, \mathfrak{h}^C)$; Δ^+ the positive roots for some order; and W the Weyl group. Let ρ be half the sum of the positive roots. For H in $\mathfrak{h}^C$ set

$$D(H) = \prod_{\alpha \in \Delta^+} [e^{\alpha(H)/2} - e^{-\alpha(H)/2}],$$

$$P(H) = \prod_{\alpha \in \Delta^+} \alpha(H) \quad \text{and} \quad D_0(H) = D(H)/P(H).$$

The positive roots determine a positive Weyl chamber $\mathfrak{a} \subseteq i\mathfrak{h}$. Set $\mathfrak{h}^+ = -i\mathfrak{a}$. The Killing form on $\mathfrak{g}^C \times \mathfrak{g}^C$ will be denoted by $B(\cdot, \cdot)$. When restricted to $\mathfrak{g} \times \mathfrak{g}$ it agrees with the Killing form of $\mathfrak{g}$. The set of equivalence classes of irreducible unitary representations of G are in 1-1 correspondence with the semi lattice L of dominant integral forms. If λ is in L, let H_λ be its dual element in $i\mathfrak{h}$ and set $H_\lambda^* = -iH_\lambda$, let χ_λ be the character and d_λ the degree of the associated representation. Normalization of measures is as in §1. If Γ denotes $\{H \in \mathfrak{h} \mid |\alpha(H)| < 2\pi \text{ for all } \alpha \in \Delta\}$, then the Weyl integration formula for a class function f is

$$\int_G f(g)\, dg = \frac{1}{|W|} \int_\Gamma f(\exp H) |D(H)|^2 \, dH.$$

The ability to transport analysis problems between group and algebra is provided by the following theorem of Harish-Chandra [7].

PROPOSITION 3.1. *For all* H_1, H_2 *in* $\mathfrak{h}^C$

$$P(H_1)P(H_2)\int_G e^{B(\mathrm{Ad}\, g(H_1), H_2)}\, dg = P(H_\rho^*) \sum_{s \in W} \epsilon(s) e^{B(sH_1, H_2)}.$$

We set

$$J(H_1, H_2) = \int_G e^{iB(\mathrm{Ad}\, g(H_1), H_2)}\, dg$$

for H_1, H_2 in $\mathfrak{h}$. This function will be called a Bessel function. As it is W invariant it has an extension to $\mathfrak{g} \times \mathfrak{g}$. The following properties of J are easily verified.

(1) J is Ad G invariant in each variable, hence is specified by its values on $\overline{\mathfrak{h}^+} \times \overline{\mathfrak{h}^+}$.

(2) $J(H_1, H_2) = J(H_2, H_1); \overline{J(H_1, H_2)} = J(-H_1, H_2) = J(H_1, -H_2)$.

(3) $|J(H_1, H_2)| \leqslant 1$ and $J(0, H_2) = 1 = J(H_1, 0)$.

(4) For H_1, H_2 in $\mathfrak{h}^+$,

$$J(H_1, H_2) = \frac{P(H_\rho^*) \sum \epsilon(w) e^{iB(wH_1, H_2)}}{P(H_1)P(H_2)}.$$

PROPOSITION 3.2. *For* λ *in* L *and* H *in* $\mathfrak{h}^+$

$$J(H_{\lambda+\rho}^*, H) = D_0(H)\chi_\lambda(\exp H)/d_\lambda.$$

PROOF. Use the Weyl character formula, the Weyl dimension formula, and (4) above.

The Gauss kernel on $\mathfrak{h}$ is given by

$$g_s(H) = c \exp(-\|H\|^2/s)/s^{l/2} \quad \text{for } s > 0 \text{ and } l = \dim \mathfrak{h}.$$

LEMMA 3.3. *Let* λ *be in* L, *regard* D_0 *and* g_s *as extended to functions on* $\mathfrak{g}$ *and pull back* χ_λ *to an invariant function on* $\mathfrak{g}$ *via the exponential. Then for* $H_0 \in \mathfrak{h}^+$

$$\left[D_0 \frac{\chi_\lambda}{d_\lambda} g_s \right]^{\widehat{\ }}(H_0)$$

$$= \frac{P(H_\rho^*)}{|W| d_\lambda P(H_0)} \sum_{W, W} \epsilon(\sigma)\epsilon(\tau)\exp(-s\|\tau H_0 - \sigma H_{\lambda+\rho}^*\|^2).$$

PROOF.

$$\left[D_0 \frac{\chi_\lambda}{d_\lambda} g_s \right] (H_0) = \frac{1}{|W|} \int_{\mathfrak{h}} D_0(H) \frac{\chi_\lambda(\exp H)}{d_\lambda} g_s(H) \overline{J(H, H_0)} |P(H)|^2 \, dH.$$

By Proposition 3.2 and the Weyl character formula the right side is

$$= \frac{1}{|W|} \int_{\mathfrak{h}} \frac{D(H)}{P(H)} \frac{\Sigma \epsilon(\sigma) e^{iB(\sigma H^*_{\lambda+\rho}, H)}}{d_\lambda D(H)}$$

$$\times \frac{P(H^*_\rho) \Sigma \epsilon(\tau) e^{-iB(\tau H_0, H)}}{\overline{P(H_0)P(H)}} g_s(H) |P(H)|^2 \, dH$$

$$= \frac{P(H^*_\rho)}{|W| d_\lambda P(H_0)} \sum_{W, W} \epsilon(\sigma)\epsilon(\tau) \int_{\mathfrak{h}} g_s(H) e^{-iB(\tau H_0 - \sigma H^*_{\sigma+\rho}, H)} \, dH$$

$$= \frac{P(H^*_\rho)}{|W| d_\lambda P(H_0)} \sum_{W, W} \epsilon(\sigma)\epsilon(\tau) \exp(-s \| \tau H_0 - \sigma H^*_{\lambda+\rho} \|^2).$$

Next we shall develop a series of lemmas paralleling §3 of Chapter 7 of Stein and Weiss [17].

LEMMA 3.4. *Let f be a continuous class function. Then*

$$\lim_{s \to \infty} \int_{\mathfrak{h}} g_s(H) f(\exp H) |D(H)|^2 \, dH = \int_{\Gamma} f(\exp H) |D(H)|^2 \, dH.$$

PROOF. This is a well-known fact in abelian harmonic analysis. A proof can be found in, e.g., [17].

Now suppose m is a bounded measurable Ad G invariant function on $\mathfrak{g}$. Let K be the multiplier operator acting on Ad G invariant functions corresponding to m. If, in addition, m is continuous at $H^*_{\lambda+\rho}$ for each $\lambda \in L$, we define a multiplier on G by the sequence $m^* = \{m(H^*_{\lambda+\rho})\}_{\lambda \in L}$. We denote by K^* the corresponding operator on class functions on G. For the remainder of this section these hypotheses and notations will remain in effect.

LEMMA 3.5. *For any characters χ_λ and χ_μ*

$$\lim_{s \to \infty} s^{1/2} \int_{\mathfrak{h}} K\left(D_0 \frac{\chi_\lambda}{d_\lambda} g_s \right) (H) \overline{D_0(H)} \frac{\overline{\chi_\mu(\exp H)}}{d_\mu} g_s(H) |P(H)|^2 \, dH$$

$$= P(H^*_\rho)^2 \int_{\Gamma} K^*\left(\frac{\chi_\lambda}{d_\lambda} \right) (\exp H) \frac{\overline{\chi_\mu(\exp H)}}{d_\mu} |D(H)|^2 \, dH.$$

PROOF.

$$s^{l/2} \int_{\mathfrak{H}} K\left(D_0 \frac{\chi_\lambda}{d_\lambda} g_s\right)(H) \overline{D_0(H)} \frac{\overline{\chi_\mu(\exp H)}}{d_\mu} g_s(H) |P(H)|^2 \, dH$$

$$= s^{l/2} \int_{\mathfrak{H}} m(H) \left[D_0 \frac{\chi_\lambda}{d_\lambda} g_s\right]^{\wedge}(H) \overline{\left[D_0 \frac{\chi_\mu}{d_\mu} g_s\right]^{\wedge}(H)} |P(H)|^2 \, dH$$

$$= \frac{s^{l/2} P(H_\rho^*)^2}{d_\lambda d_\mu} \int_{\mathfrak{H}} m(H) \frac{\Sigma \, \epsilon(w_1) \exp(-s\|H - w_1 H_{\lambda+\rho}^*\|^2)}{P(H)}$$

$$\times \frac{\Sigma \, \epsilon(w_2) \exp(-s\|H - w_2 H_{\mu+\rho}^*\|^2)}{\overline{P(H)}} |P(H)|^2 \, dH$$

$$= \frac{P(H_\rho^*)^2}{d_\lambda d_\mu} s^{l/2} \sum_{W, W} \epsilon(w_1)\epsilon(w_2) \int m(H) \exp(-s\|H - w_1 H_{\lambda+\rho}^*\|^2)$$

$$\times \exp(-s\|H - w_2 H_{\mu+\rho}\|^2).$$

If $\lambda \neq \mu$, the sets $\{\sigma(\lambda + \rho)\}$ and $\{\tau(\mu + \rho)\}$ are disjoint. So we choose δ small enough so that for all w_1, w_2 in W, $\|w_1 H_{\lambda+\rho}^* - w_2 H_{\mu+\rho}^*\|^2 > 2\delta$. Then

$$s^{l/2} \left| \int_{\mathfrak{H}} m(H) \exp(-s\|H - w_1 H_{\lambda+\rho}\|^2) \exp(-s\|H - w_2 H_{\mu+\rho}^*\|^2) dH \right|$$

$$\leqslant s^{l/2} \int_{\|H - w_1 H_{\lambda+\rho}^*\|^2 > \delta} |m(H)| \exp(-s\|H - w_2 H_{\mu+\rho}^*\|^2) e^{-\delta s} \, dH$$

$$+ s^{l/2} \int_{\|H - w_2 H_{\mu+\rho}^*\|^2 > \delta} \exp(-s\|H - w_1 H_{\lambda+\rho}^*\|^2) e^{-\delta s} |m(H)| \, dH.$$

And as $s \to \infty$ both terms on the right $\to 0$. Next we compute the integral on the group.

$$\int_\Gamma K^*\left(\frac{\chi_\lambda}{d_\lambda}\right)(\exp H) \frac{\overline{\chi_\mu(\exp H)}}{d_\mu} |D(H)|^2 \, dH$$

$$= \frac{m(H_{\lambda+\rho}^*)}{d_\lambda d_\mu} \int_\Gamma \chi_\lambda(\exp H) \overline{\chi_\mu(\exp H)} |D(H)|^2 \, dH = 0.$$

Suppose then that $\lambda = \mu$. The argument above works for w_1, w_2 in W with $w_1 \neq w_2$ to show the integral on $\mathfrak{H}$ is zero. If $w_1 = w_2$,

$$s^{l/2} \frac{P(H_P^*)^2}{d_\lambda^2} \sum_W \int_{\mathfrak{H}} m(H) \exp(-2s\|H - w H_{\lambda+\rho}^*\|^2) dH$$

is the classical Gauss convolution on $\mathbf{R}^l$ and since m is continuous at $H_{\lambda+\rho}^*$ it

equals, in the limit $s \longrightarrow \infty$, $P(H_\rho^*)^2 |W| m(H_{\lambda+\rho}^*)/d_\lambda^2$. Again we compute the integral on the group and by orthonormality of characters we have

$$\int_\Gamma K^*\left(\frac{\chi_\lambda}{d_\lambda}\right)(\exp H)\frac{\chi_\lambda(\exp H)}{d_\lambda}|D(H)|^2 \, dH = |W|\frac{m(H_{\lambda+\rho}^*)}{d_\lambda^2}.$$

REMARK 3.6. The preceding lemma is true for trigonometric polynomials.

THEOREM 3.7. *Suppose m is an* Ad *G invariant measurable function on* $\mathfrak{g}$ *that is continuous at* $\{H_{\lambda+\rho}^* | \lambda \in L\}$. *If there is a* $C < \infty$ *such that for f in* $L_{\text{inv}}^p(\mathfrak{g})$

$$\int_{\mathfrak{h}} |Kf(H)|^p |D_0(H)|^{2-p}|P(H)|^2 \, dH \leqslant C \int_{\mathfrak{h}} |f(H)|^p |D_0(H)|^{2-p}|P(H)|^2 \, dH,$$

then $m^* = \{m(H_{\lambda+\rho}^*)\}_{\lambda \in L}$ *is a multiplier on* $L_{\text{inv}}^p(G)$ *with operator norm* $\leqslant AC$ *where A depends only on* $\mathfrak{g}$.

PROOF. We prove the theorem for $1 < p < \infty$ and trigonometric polynomials. Suppose $f = \Sigma a_\lambda d_\lambda \chi_\lambda$ and $h = \Sigma b_\lambda d_\lambda \chi_\lambda$ are trigonometric polynomials. As usual it suffices to find a constant A independent of f, h with

$$\left|\int_\Gamma K^*f(\exp H)\overline{h(\exp H)}|D(H)|^2 \, dH\right| \leqslant A \|f\|_p \|h\|_q.$$

By Lemma 3.5,

$$I = P(H_\rho^*)^2 \int_\Gamma K^*f(\exp H)\overline{h(\exp H)}|D(H)|^2 \, dH$$

$$= \lim_{s \to \infty} s^{l/2} \int_{\mathfrak{h}} K(D_0 g_s f)(H)\overline{D_0(H)}\overline{h(\exp H)}g_s(H)|P(H)|^2 \, dH.$$

Then

$$|I| \leqslant \lim_{s \to \infty} s^{l/2} \int_{\mathfrak{h}} |K(D_0 g_s f)(H)| |D_0(H)h(\exp H)g_s(H)| |P(H)|^2 \, dH.$$

Writing $|D_0(H)| = |D_0(H)|^{1-2/q}|D_0(H)|^{2/q}$ and using Hölder's inequality we find that the right side

$$\leqslant \lim_{s \to \infty} s^{l/2}\left[\int_{\mathfrak{h}} |K(D_0 g_s f)(H)|^p |D_0(H)|^{2-p}|P(H)|^2 \, dH\right]^{1/p}$$

$$\cdot \left[\int_{\mathfrak{h}} |h(H)|^q |g_s(H)|^q |D_0(H)|^2 |P(H)|^2 \, dH\right]^{1/q}$$

$$\leqslant C \lim_{s \to \infty} s^{l/2}\left[\int_{\mathfrak{h}} |D_0(H)g_s(H)f(H)|^p |D_0(H)|^{2-p}|P(H)|^2 \, dH\right]^{1/p}$$

$$\cdot \left[\int_{\mathfrak{h}} |h(H)|^q |g_s(H)|^q |D_0(H)|^2 |P(H)|^2 \, dH\right]^{1/q}.$$

Now g_s is positive; we apply Hölder again to bound the right side by

$$\leqslant C \lim_{s \to \infty} s^{l/2} \|g_s\|_\infty^{(p-1)/p} \|g_s\|_\infty^{(q-1)/q}.$$

$$\left[\int_{\mathfrak{h}} |f(\exp H)|^p \, g_s(H) |D_0(H)|^2 |P(H)|^2 \, dH \right]^{1/p}$$

$$\times \left[\int_{\mathfrak{h}} |h(\exp H)|^q \, g_s(H) |D_0(H)|^2 |P(H)|^2 \, dH \right].$$

Now

$$s^{l/2} \|g_s\|_\infty^{1-1/p} \|g_s\|_\infty^{1-1/q} = s^{l/2} \|g_s\|_\infty = \text{constant}.$$

Applying Lemma 3.4 completes the proof.

Theorem 3.7 provides a method for constructing multipliers on G; however, the hypothesis may be difficult to verify directly. Next we shall see that this verification reduces to Fourier multiplier problems on $\mathfrak{h}$ and then often can be done easily. Thus we now develop a procedure for transplanting multiplier problems on $L^p_{\text{inv}}(\mathfrak{g})$ to Fourier multiplier problems on $L^p(\mathfrak{h})$. First we see how the Fourier transform is transplanted and then how multiplier operators are transplanted.

For f a continuous compactly supported invariant function on $\mathfrak{g}$, the Fourier transform of f was shown in [7, Theorem 3] to be given by

$$\hat{f}(H_2) = \frac{P(H_\rho^*)}{|W| \overline{P(H_2)}} \int_{\mathfrak{h}} P(H_1) f(H_1) \sum \epsilon(w) e^{-iB(wH_1, H_2)} \, dH_1.$$

Since f is Ad G invariant, it is W invariant. The function P, however, is odd with respect to W, i.e., $P(wH) = \epsilon(w)P(H)$. Then fp is odd, and, making a change of variables, we see that

$$\hat{f}(H_2) = \frac{P(H_\rho^*)}{P(H_2)} [fP]^{\tilde{}}(H_2)$$

where $\tilde{}$ denotes the Fourier transform on $\mathfrak{h}$.

A similar computation for inverting the Fourier transform gives

$$g^{\vee}(H) = \frac{P(H_\rho^*)}{P(H)} [\bar{P}g]^{\vee}(H) \quad \text{if } g = \hat{f}.$$

If K is a multiplier operator on $L^2_{\text{inv}}(\mathfrak{g})$ with $(Kf)^{\wedge} = m\hat{f}$, denote by K_0 the Fourier multiplier on $L^2(\mathfrak{h})$ with $(K_0 f)^{\tilde{}} = m\tilde{f}$. Then for $f \in L^2_{\text{inv}}(\mathfrak{g}) \cap L^p_{\text{inv}}(\mathfrak{g})$

$$Kf(H) = [m\hat{f}]^{\vee}(H) = \frac{P(H_\rho^*)}{P(H)} [\bar{P}m\hat{f}]^{\vee}(H) = \frac{P(H_\rho^*)^2}{P(H)} [mP(H_\rho^*)[Pf]^{\tilde{}}]^{\vee}(H)$$

$$= \frac{P(H_\rho^*)^2}{P(H)} [m[Pf]^{\tilde{}}]^{\vee}(H) = \frac{P(H_\rho^*)^2}{P(H)} K_0(Pf)(H).$$

PROPOSITION 3.8. *If m is a multiplier on $L^p(\mathfrak{h})$ with respect to the weight function $|D(H)|^{2-p}$ and has operator norm $\|m\|$, then m is a multiplier on $L_{inv}^p(\mathfrak{g})$ with weight function $|D_0(H)|^{2-p}$ and operator norm at most $P(H_\rho^*)^2 \|m\|$.*

PROOF.

$$\int_{\mathfrak{h}} |Kf(H)|^p |D_0(H)|^{2-p} |P(H)|^2 \, dH$$

$$= P(H_\rho^*)^{2p} \int_{\mathfrak{h}} |K_0(Pf)(H)|^p |D_0(H)|^{2-p} |P(H)|^{2-p} \, dH$$

$$\leqslant P(H_\rho^*)^{2p} \|m\|^p \int_{\mathfrak{h}} |Pf|^p |D(H)|^{2-p} \, dH$$

$$= P(H_\rho^*)^{2p} \|m\|^p \int_{\mathfrak{h}} |f(H)|^p |D_0(H)|^{2-p} |P(H)|^2 \, dH.$$

4. **Mean convergence.** We come to the main result, a theorem concerning convergence in L^p of the Fourier series expansion in characters of a central function on the compact semisimple Lie group G. Briefly this convergence question is handled as follows: convergence on the group reduces to uniform boundedness of the partial sum operators, which reduces by Theorem 3.7 to uniform boundedness of multiplier operators on $L_{inv}^p(\mathfrak{g})$ with a weight function, which in turn reduces by Proposition 3.8 to the uniform boundedness of the multiplier operators on Euclidean space corresponding to a family of convex polygons. These last operators are to be bounded with respect to a weight function. Following an observation of Bochner [10], we simplify this question to one concerning uniform boundedness of the Hilbert transform with respect to a family of weight functions. Finally a theorem due to Hunt, Muckenhoupt and Wheeden [12] and a lemma due to Coifman allow us to establish this uniform boundedness. I am indebted to R. Coifman for pointing out the applicability of his lemma.

Typically the range of p for which mean convergence holds is a small interval centered at $p = 2$. We show that the Fourier transform maps $L_{inv}^p(\mathfrak{g})$ into continuous functions for p less than a critical exponent. Taken with a theorem of Strichartz or Theorem 2.5 this provides negative results for a range of p. Unfortunately for rank $G > 1$, the mean convergence result together with this negative result do not exhaust all possible p and so complete information on mean convergence is not yet available. An example with rank $G = 2$ appears at the end of the section. I want to thank Peter Tomas for bringing my attention to Varopoulos's result on the Fourier transform on radial functions (see Stein and Weiss [17, p. 176]) which suggested this approach.

Before stating the main theorem, we shall specify the particular ways the partial sums of the Fourier series are formed.

Let $\omega_1, \ldots, \omega_l$ be the fundamental weights for G. The set $\{H_{\omega_1}^*, \ldots, H_{\omega_l}^*\}$ is a basis of $\mathfrak{h}$ and the positive Weyl chamber, $\mathfrak{h}^+$, is the set of all H in $\mathfrak{h}$

whose coordinates with respect to this basis are positive. Fix H_ν^* in $\overline{\mathfrak{h}^+}$ with $\|H_\nu^*\| = [-B(H_\nu^*, H_\nu^*)]^{1/2} = 1$ and $-B(H_{\omega_i}^*, H_\nu^*) > 0$ for $i = 1, \ldots, l$. Let ν in $(\mathfrak{h}^\mathbb{C})'$ be the element dual to $-iH_\nu^*$. If f is in $L_{\text{inv}}^2(G) \cap L_{\text{inv}}^p(G)$ with Fourier series $\Sigma_{\lambda \in L} a_\lambda d_\lambda \chi_\lambda$, for each $R > 0$ we define the partial sum operator

$$S_R f = \sum_{\langle \lambda + \rho, \nu \rangle \leqslant R} a_\lambda d_\lambda \chi_\lambda.$$

Let r equal the number of positive roots α such that $\alpha(H_\nu^*) \neq 0$. Of course both S_R and r depend on ν; however we shall suppress this in the notation.

THEOREM 4.1. *If f is in $L_{\text{inv}}^p(G)$ and $(2r + 1)/(r + 1) < p < (2r + 1)/r$, then $S_R f$ converges to f in the L^p norm as $R \longrightarrow \infty$.*

The theorem will be proved in a series of steps.

Step 1. The conclusion is certainly true for trigonometric polynomials and these are dense in $L_{\text{inv}}^p(G)$. By standard functional analysis it is enough to show the operators S_R are uniformly bounded:

Step 2. Reduction to a multiplier problem on $L_{\text{inv}}^p(\mathfrak{g})$. We construct an invariant function m on $\mathfrak{g}$ which when restricted to the lattice $\{H_{\lambda+\rho}^* | \lambda \in L\}$ gives the multiplier for S_R.

Set $\sigma = \{H \in \mathfrak{h} | -B(H, H_\nu^*) = 1\}$; σ is a hyperplane intersecting the positive Weyl chamber. Set $E_1 = \{H \in \mathfrak{h} | -B(H, H_\nu^*) \leqslant 1\}$. Then E_1 is a half space containing zero and with boundary σ. For each $w \in W$ set $E_w = wE_1$. As w acts by orthogonal transformations, $E_w = \{H \in \mathfrak{h} | -B(w^{-1}H, H_\nu^*) = -B(H, wH_\nu^*) \leqslant 1\}$ and so is a half space. Set $C = \bigcap_{w \in W} E_w$ and set m equal to the characteristic function of C. More generally, for each $R > 0$ set m_R equal to the characteristic function of RC.

LEMMA 4.2. (a) $RC \cap \mathfrak{h}^+ = RE_1 \cap \mathfrak{h}^+$.

(b) *The sequence $\{m_R(H_{\lambda+\rho}^*)\}$ is the multiplier corresponding to S_R.*

PROOF. (a) It suffices to prove (a) with $R = 1$. We always have $C \cap \mathfrak{h}^+$ contained in $E_1 \cap \mathfrak{h}^+$ so suppose H is in $E_1 \cap \mathfrak{h}^+$. Since both H and H_ν^* are in $\overline{\mathfrak{h}^+}$, $H = \Sigma h_i H_{\omega_i}^*$ with $h_i > 0$ and $H_\nu^* = \Sigma c_i H_{\omega_i}^*$ with $c_i \geqslant 0$. For $w \in W$, $wH_{\omega_i}^* = -iwH_{\omega_i} = -i[H_{\omega_i} - \Sigma_{k=1}^l a_i^k H_{\alpha_k}]$ with the a_k^i nonnegative integers and the α_k the simple roots. Then $wH_{\omega_i}^* = H_{\omega_i}^* - \Sigma_{k=1}^l a_k^i H_{\alpha_k}^*$. We shall compute $-B(H, wH_\nu^*)$ and show that H is in E_w.

$$-B(H, wH_\nu^*) = -B\left(\sum h_i H_{\omega_i}^*, w \sum c_j H_{\omega_j}^*\right) = -\sum_{i,j} h_i c_j B(H_{\omega_i}^*, wH_{\omega_j}^*)$$

$$= -\sum_{i,j} h_i c_j B(H_{\omega_i}^*, H_{\omega_j}^*) + \sum_{i,j} h_i c_j \sum_k a_k^i B(H_{\omega_i}^*, H_{\alpha_k}^*).$$

But $h_i > 0, c_j \geqslant 0, a_k^i \geqslant 0$ while $B(H_{\omega_i}^*, H_{\alpha_k}^*) \leqslant 0$, then

$$- B(H, wH_\nu^*) \leqslant - \sum_{i,j} h_i c_j B(H_{\omega_i}^*, H_{\omega_j}^*) = - B(H, H_\nu^*) \leqslant 1.$$

(b) If λ is in L, then $\langle \lambda + \rho, \nu \rangle \leqslant R$ if and only if $- B(H_{\lambda+\rho}^*, H_\nu^*) \leqslant R$ if and only if $H_{\lambda+\rho}^*$ is in $RE_1 \cap \mathfrak{h}^+$. By part (a) $H_{\lambda+\rho}^*$ is in $RC \cap \mathfrak{h}^+$ if and only if $H_{\lambda+\rho}^*$ is in $RE_1 \cap \mathfrak{h}^+$. Then we have the desired conclusion because the sequence $\{m_R(H_{\lambda+\rho}^*)\}$ contains precisely these lattice points.

In order to apply Theorem 3.7 to the multipliers m_R it is necessary that they be continuous at L. We can arrange matters so that this condition is satisfied. For each $R > 0$ the number of λ in L with $\langle \lambda + \rho, \nu \rangle \leqslant R$ is finite. Indeed if $\lambda = \Sigma\, l_i \omega_i$, $\langle \lambda + \rho, \nu \rangle = \Sigma\,(l_i + 1)\langle \omega_i, \nu \rangle$ and if this is $\leqslant R$ we must have $(l_i + 1)\langle \omega_i, \nu \rangle \leqslant R$ for each $i = 1, \ldots, l$. But ν was chosen so that $\langle \omega_i, \nu \rangle > 0$; then $(l_i + 1) \leqslant R/\langle \omega_i, \nu \rangle$ and only finitely many nonnegative integers l_i can satisfy this. Then it is clear that we can find an increasing sequence $R_0 < R_1 < R_2 \cdots$ of positive numbers such that m_{R_i} is continuous at L, $\langle \lambda + \rho, \nu \rangle \leqslant R_0$ if and only if $\lambda = 0$, for $R_i \leqslant R < R_{i+1}$ the sets $\{\lambda \in L\,|\langle\lambda + \rho, \nu \rangle \leqslant R_i\}$ and $\{\lambda \in L\,|\langle\lambda + \rho, \nu \rangle \leqslant R\}$ are equal, and the set $\{\lambda \in L\,|\langle\lambda + \rho, \nu \rangle \leqslant R_i\}$ is not equal to $\{\lambda \in L\,|\,\langle\lambda + \rho, \nu \rangle \leqslant R_{i+1}\}$. Then to prove Theorem 4.1 it suffices to show the operators S_{R_i} are uniformly bounded.

Let K_{R_i} be the multiplier operators on Ad G invariant functions on $\mathfrak{g}$ corresponding to m_{R_i}. By Theorem 3.7 the operators S_{R_i} are uniformly bounded on $L_{\text{inv}}^p(G)$ if the operators K_{R_i} are uniformly bounded on $L_{\text{inv}}^p(\mathfrak{g})$ with the weight function $|D_0(H)|^{2-p}$.

Step 3. Reduction to Fourier multipliers on $L^p(\mathfrak{h})$ with weight function. In fact by Proposition 3.8 the operators K_{R_i} have the required properties if the functions m_{R_i} are uniformly bounded Fourier multipliers on $L^p(\mathfrak{h})$ with respect to the weight function $|D(H)|^{2-p}$.

Step 4. Reduction to Hilbert transform.

For $w \in W$ let T_w be the Fourier multiplier operator on $\mathfrak{h}$ corresponding to the multiplier $\chi_w = $ characteristic function of E_w. Let K be the Fourier operator on $\mathfrak{h}$ corresponding to m. As $m = \Pi_{w \in W} \chi_w$, K is the composition of the T_w. A sufficient condition for K to be bounded on $L^p(\mathfrak{h})$ with respect to a weight function is that each of the T_w is so bounded.

It is useful to use different coordinates on $\mathfrak{h}$ for each of the T_w. Let σ_0 be the $l - 1$ dimensional subspace of $\mathfrak{h}$ orthogonal to RH_ν^* and let $\{H_1, \ldots, H_{l-1}\}$ be an orthonormal basis of σ_0. For notational convenience denote H_ν^* by H_l. For each w in W the set $\{wH_1, \ldots, wH_l\}$ is an orthonormal basis of $\mathfrak{h}$ and with these coordinates χ_w is the characteristic function of the half space

$\{(h_1, \ldots, h_l) | h_l \leqslant 1\}$. Following Stein [16],

$$T_w f(h_1, \ldots, h_l) = T_w f(\bar{h})$$

$$= \tfrac{1}{2} [f(\bar{h}) + ie^{iB(\bar{h}, wH_l)} \bar{H}(e^{-iB(\cdot, wH_l)}f)(\bar{h})]$$

where $\bar{H}$ is the classical Hilbert transform acting in the lth coordinate. Then T_w is bounded on $L^p(\mathfrak{h})$ with a weight function if this is true for $\bar{H}$. Actually more can be said; if for $R > 0$ one constructs the objects $T_w^R, \chi_w^R, m_R, K_R$ corresponding to RE_w, then K_R is bounded on $L^p(\mathfrak{h})$ with a weight function if each of the T_w^R is. But

$$T_w^R f(\bar{h}) = \tfrac{1}{2} [f(\bar{h}) + ie^{iB(\bar{h}, wRH_l)} \bar{H}(e^{-iB(\cdot, wRH_l)}f)(\bar{h})]$$

and is bounded provided $\bar{H}$ is. Summarizing we have

LEMMA 4.3 . *The functions m_{R_l} define uniformly bounded Fourier multiplier operators on $L^p(\mathfrak{h})$ with a weight function if the Hilbert transform $\bar{H}$ acting in the lth coordinate with respect to each of the bases $\{wH_1, \ldots, wH_l\}$ is bounded on $L^p(\mathfrak{h})$ with the same weight function.*

The weight function relevant to our problem is

$$|D(H)|^{2-p} = \prod_{\alpha \in \Delta^+} |\sin \alpha(iH)/2|^{2-p}.$$

This weight function is a product so when computing the L^p norm of $\bar{H}f$ only those factors that depend on the lth coordinate are required. For $w \in W$ set $\Delta^w = \{\alpha \in \Delta^+ | \alpha(wH_l) \neq 0\}$ and split $|D(H)|^{2-p}$ into two factors:

$$|D_w^1(H)|^{2-p} = \prod_{\alpha \in \Delta^w} \left| \sin \frac{\alpha}{2}(iH) \right|^{2-p} \quad \text{and}$$

$$|D_w^2(H)|^{2-p} = \prod_{\alpha \in \Delta^+ \setminus \Delta^w} \left| \sin \frac{\alpha}{2}(iH) \right|^{2-p}.$$

LEMMA 4.4. *The cardinality of Δ^w, $|\Delta^w|$, is independent of w.*

PROOF. For $\beta \in \Delta$ set $|\beta| = \beta$ if $\beta > 0$ and $-\beta$ if $\beta < 0$. We show that Δ^w is in 1-1 correspondence with Δ^1 where $\alpha \longrightarrow |w^{-1}\alpha|$ is the bijection. It is well known that this map is 1-1. If α is in Δ^w then $\alpha(wH_l) \neq 0$ but this is the same as $w^{-1}\alpha(H_l) \neq 0$. Then $|w^{-1}\alpha|$ is in Δ^1. Conversely if β is in $\Delta^1, \beta(H_l) \neq 0$ implies that $w\beta(wH_l) \neq 0$ so $|w\beta| \in \Delta^w$ and then the map is onto.

LEMMA 4.5. *If for almost all H_0 in $w\sigma_0, \bar{H}$ is bounded on $L^p(\mathbf{R}^1)$ with the weight function $|D_w^1(H_0, h_l)|^{2-p}$ with norm C independent of H_0, then $\bar{H}$ is bounded on $L^p(\mathfrak{h})$ with weight function $|D(H)|^{2-p}$.*

PROOF.

$$\int_{\mathfrak{H}} |\bar{H}f(H)|^p |D(H)|^{2-p}\, dH$$

$$= \int_{w\sigma_0} |D_w^2(H_0)|^{2-p} \left[\int_{\mathbf{R}^1} |\bar{H}f(H_0, h_l)|^p |D_w^1(H_0, h_l)|^{2-p}\, dh_l\right] dH_0$$

$$\leqslant C \int_{w\sigma_0} |D_w^2(H_0)|^{2-p} \left[\int_{\mathbf{R}^1} |f(H_0, h_l)|^p |D_w^1(H_0, h_l)|^{2-p}\, dh_l\right] dH_0$$

$$= C \int_{\mathfrak{H}} |f(H)|^p |D(H)|^{2-p}\, dH.$$

Fortunately the theory of the Hilbert transform with weight function has been completely worked out (see [12]). Suppose ω is a nonnegative function on $\mathbf{R}^1$. ω is said to satisfy condition A_p $(1 < p < \infty)$ if there is a constant C such that for every interval I,

$$\left[\int_I \omega(x)\, dx\right] \left[\int_I \omega(x)^{-1/(p-1)}\, dx\right]^{p-1} \leqslant C|I|^p.$$

The result is

PROPOSITION 4.6. *A necessary and sufficient condition for $\bar{H}$ to be bounded on $L^p(\mathbf{R}^1)$ with respect to ω is that ω satisfies condition A_p. The norm of $\bar{H}$ is dominated by the constant C in condition A_p times a function of p.*

Step 5. Verification of condition A_p.

Our problem now appears to be computational. By Lemma 4.5 and Proposition 4.6 it is enough to determine for which p, if any, the collection of weight functions $\{|D_w^1(H_0, h)|^{2-p}| w \in W\}$ satisfy A_p independent of H_0. For rank $G = 1$ this is the function $|\sin x|^{2-p}$ and it follows from a theorem of Hardy and Littlewood when the Hilbert transform is bounded for this weight.

We are grateful to R. Coifman for pointing out the next lemma which enables us to treat the general rank case.

LEMMA 4.7 (R. COIFMAN). *Suppose ω_1, ω_2 are nonnegative functions on $\mathbf{R}^1$ both of which satisfy A_p for some p with constants C_1, C_2. Then for $0 \leqslant a \leqslant 1$, $\omega_a = \omega_1^a \omega_2^{1-a}$ satisfies A_p with constant $C_1^a C_2^{1-a}$.*

PROOF. Let I be an interval and

$$* = \left[\int_I \omega_a(x)\, dx\right] \left[\int_I \omega_a(x)^{-1/(p-1)}\, dx\right]^{p-1}.$$

We use Hölder's inequality with $r = 1/a$ and $s = 1/(1-a)$.

$$\int_I \omega_a(x)\, dx \leqslant \left[\int_I \omega_1(x)\, dx\right]^a \left[\int_I \omega_2(x)\, dx\right]^{1-a}$$

and

$$\int_I \omega_a(x)^{-1/(p-1)}\, dx \leqslant \left[\int_I \omega_1(x)^{-1/(p-1)}\, dx\right]^a \left[\int_I \omega_2(x)^{-1/(p-1)}\, dx\right]^{1-a}.$$

Then

$$* \leqslant C_1^a C_2^{1-a} |I|^{ap} |I|^{(1-a)p} = C_1^a C_2^{1-a} |I|^p.$$

COROLLARY 4.7. *If $\omega_1, \ldots, \omega_s$ satisfy A_p and $a_1, \ldots, a_s$ are nonnegative numbers with $\Sigma a_i = 1$, then $\omega_1^{a_1} \cdots \omega_s^{a_s}$ satisfy A_p.*

We consider the weight function

$$|D_w^1(H_0, h)|^{2-p} = \prod_{\alpha \in \Delta^w} |\sin \alpha(iH)/2|^{2-p}.$$

By Lemma 4.3, $|\Delta^w| = |\Delta^1| = r$. Let $\omega_i(H) = |\sin \alpha_i(iH)/2|^{r(2-p)}$ for the $\alpha_1, \ldots, \alpha_r$ in Δ^w. If each ω_i satisfies A_p, then by Corollary 4.7 $|D_w^1(H)|^{2-p} = \omega(H) = \Pi_{i=1}^r \omega_i(H)^{1/r}$ also satisfies A_p. Using our coordinates $\{wH_1, \ldots, wH_l\}$, each of the ω_i is of the form $|\sin(c_i h + b_i)|^{r(2-p)}$. Condition A_p is independent of translation and the scalar multiple c_i is independent of H_0. Then $\omega_i(H_0, h) = |\sin \alpha_i(iH)/2|^{r(2-p)}$ satisfies A_p independent of H_0 if and only if $|\sin h|^{r(2-p)}$ does. It is known that the classical Hardy and Littlewood weight $|x|^a$ satisfies A_p if and only if $-1 < a < p - 1$. If $a = r(2 - p)$, solving for p we have $(2r + 1)/(r + 1) < p < (2r + 1)/r$. The following lemma will complete the proof of the theorem.

LEMMA 4.8. *If $(2r + 1)/(r + 1) < p < (2r + 1)/r$, $|\sin \alpha(H)/2|^{r(2-p)}$ satisfies A_p independent of H_0.*

PROOF. Using our convenient coordinates $|\sin \alpha(H)/2|^{r(2-p)} = |\sin(ch + b)|^{r(2-p)}$. Again the translation b and frequency c which is independent of H_0 will not affect the result, so we need only consider $|\sin h|^{r(2-p)}$. We verify A_p for various types of intervals. Set $a = r(2 - p)$.

(1) If valid for $|I| = \pi$, then valid for $|I| = k\pi$.

Proof. By periodicity

$$\left[\int_I |\sin x|^a \, dx \right] \left[\int_I |\sin x|^{-a/(p-1)} \, dx \right]^{p-1} \leqslant cK^p \pi^p.$$

(2) If valid for $|I| = \pi$ then valid for $k\pi \leqslant |I| \leqslant (k + 1)\pi$, $k \geqslant 1$.

Proof. $I = [a, b]$ and say $|I| = k\pi + \alpha$ with $0 \leqslant \alpha < \pi$.

$$\left[\int_I |\sin x|^a \, dx \right] \left[\int_I |\sin x|^{-a/(p-1)} \, dx \right]^{p-1}$$

$$\leqslant \left[\int_a^{b+\pi-\alpha} |\sin x|^a \, dx \right] \left[\int_a^{b+\pi-\alpha} |\sin x|^{-a/(p-1)} \, dx \right]^{p-1}$$

$$\leqslant c \left[(k + 1)\pi \right]^p.$$

As $k \geqslant 1$, $(k + 1)\pi \leqslant 2|I|$ so A_p is verified with a new constant $2^p c$.

(3) Fix $\epsilon > 0$. If A_p valid for $|I| = \pi$ then valid for $\pi \geqslant |I| > \epsilon$.

Proof. $I = [a, b]$,

$$\left[\int_I |\sin x|^a \, dx\right] \left[\int_I |\sin x|^{-a/(p-1)} \, dx\right]^{p-1}$$

$$\leqslant \left[\int_a^{b+\pi-|I|} |\sin x|^a \, dx\right] \left[\int_a^{b+\pi-|I|} |\sin x|^{-a/(p-1)}\right]^{p-1} \leqslant c\pi^p.$$

Now $\pi = |I| + \pi - |I| \leqslant |I| + \pi - \epsilon = |I| + (\pi - \epsilon)\epsilon/\epsilon$. Set $R_\epsilon = (\pi - \epsilon)/\epsilon$, then $\pi \leqslant |I| + R_\epsilon \epsilon \leqslant (1 + R_\epsilon)|I|$. So A_p is verified with new constant $c(1 + R_\epsilon)^p 2^p$.

(4) $|I| = \pi$. By periodicity it suffices to check for $[0, \pi]$. After the change of variable $\sin x = (1 - t)^{1/2}$, the integrals become

$$\left[\frac{1}{2}\int_0^1 (1 - t)^{(a-1)/2} \, t^{-1/2} \, dt\right] \left[\frac{1}{2}\int_0^1 (1 - t)^{-(a-p+1)/2(p-1)} \, t^{-1/2} \, dt\right]^{p-1}.$$

These integrals are classical beta functions and are finite provided $(a - 1)/2 > -1$ and $(-a - p + 1)/2(p - 1) > -1$. The first condition is always satisfied for $p < (2r + 1)/r$; while the second becomes on rearranging $a = r(2 - p) < p - 1$ or $p > (2r + 1)/(r + 1)$.

(5) $|I| < \epsilon$. If the distance of both endpoints of I from the set of integer mutliples of π is greater than ϵ, use the estimate $\epsilon/2 \leqslant |\sin x| \leqslant 1$. Condition A_p is satisfied with possibly the new constant $2/\epsilon$. Suppose the distance of either endpoint of I from $\{Z\pi\}$ is less than ϵ. We assume $3\epsilon < \pi$; then I is contained in an interval of the form $(k\pi - 2\epsilon, k\pi + 2\epsilon)$. Use the estimate $|x|/2 \leqslant |\sin x| \leqslant |x|$ and the integrals are bounded by

$$\left[\int_I |x|^a \, dx\right] \left[\int_I 2^{a/(p-1)}|x|^{-a/(p-1)} \, dx\right]^{p-1} \leqslant 2^a c|I|^p$$

provided $-1 < a = r(2 - p) < p - 1$.

We come to a type of converse to Theorem 4.1. To the Lie algebra $\mathfrak{g}$ we associate a number p_c with $1 < p_c < 2$ which we call the critical index. Its significance in analysis on $\mathfrak{g}$ is given by

PROPOSITION 4.9. *If f is in $L_{\text{inv}}^p(\mathfrak{g})$ with $1 \leqslant p < p_c$ then $\hat{f}$ is continuous on $\mathfrak{h}^+$.*

Let us defer the proof and see how this result yields negative results for mean convergence questions. We use the multipliers constructed at the beginning of this section. Suppose the operators S_R, $R > 0$, were uniformly bounded operators on $L_{\text{inv}}^p(G)$. The function m satisfies the hypotheses of Theorem 2.5 specialized to invariant functions or the hypotheses of Strichartz' theorem [18]. We would conclude then that m was a multiplier on $L_{\text{inv}}^p(\mathfrak{g})$. Applying Proposition 4.9, it is necessary that $p > p_c$.

The proof of Proposition 4.9 involves a closer study of Bessel functions. We need a refinement of Harish-Chandra's result (Proposition 3.2) and this

necessitates a digression into Lie algebra structure theory.

Let π = set of simple roots $\alpha_1, \ldots, \alpha_l$, and let S = set of nonempty subsets of π. If S is in $\mathcal{S}$, S determines a root system $\Delta_S \subseteq \Delta$, namely, take Δ_S to be the totality of roots in Δ spanned by the elements of S. The Weyl group W_S of Δ_S is generated by the reflections s_{α_i}, $\alpha_i \in S$, and is a subgroup of W. The relevance of subsets S is that they determine the walls and corners of the closed Weyl chamber, walls corresponding to one element set and corner S with more than one element. The origin corresponds to π.

Fix $\epsilon > 0$ and set $\mathfrak{h}_\epsilon^+ = \{ H \in \mathfrak{h}^+ \mid \alpha(H) \geqslant \epsilon \text{ for all } \alpha \in \Delta^+ \}$ and for each S in $\mathcal{S}$ set $\mathfrak{h}_{\epsilon,S}^+ = \{ H \in \mathfrak{h}^+ \mid \alpha(H) < \epsilon \text{ for all } \alpha \in S \text{ and } \beta(H) \geqslant \epsilon \text{ for all } \beta \text{ in } \pi \setminus S \}$. Then $\mathfrak{h}^+$ is the disjoint union of $\mathfrak{h}_\epsilon^+$ and $\bigcup_{S \in \mathcal{S}} \mathfrak{h}_{\epsilon,S}^+$.

Recall that by Proposition 3.2,

$$J(H_\rho^*, H) = D_0(H) \quad \text{and} \quad D_0(H) = c \prod_{\alpha \in \Delta^+} \frac{\sin \alpha(iH)/2}{\alpha(H)}.$$

On $\mathfrak{h}_{\epsilon,S}^+$, we have

$$D_0(H) = c \left[\prod_{\alpha \in \Delta_S^+} \frac{\sin \alpha(iH)/2}{\alpha(H)} \right] \left[\prod_{\alpha \in \Delta^+ \setminus \Delta_S^+} \frac{\sin \alpha(iH)/2}{\alpha(H)} \right]$$

and for ϵ small the first factor is essentially constant. We shall see that all the Bessel functions behave like this.

We fix S in $\mathcal{S}$ and examine the behavior of $J(H, H')$ on $\mathfrak{h}_{\epsilon,S}^+$ for H, H' in $\mathfrak{h}^+$ and H in $\mathfrak{h}_{\epsilon,S}^+$.

$$J(H, H') = \frac{P(H_\rho^*)}{P(H')P(H)} \sum_{w \in W} \epsilon(w) e^{iB(wH, H')}.$$

Define $P_S^1(H)$ and $P_S^2(H)$ by

$$P_S^1(H) = \prod_{\alpha \in \Delta_S^+} \alpha(H) \quad \text{and} \quad P_S^2(H) = \prod_{\alpha \in \Delta_S^+ \setminus \Delta_S^+} \alpha(H).$$

Since $P_S^2(H)J(H, H')$ is continuous on $\mathfrak{h}$, $(P(H_\rho^*)/P_S^1(H)) \sum_{w \in W} \epsilon(w) e^{iB(wH, H')}$ is continuous on $\mathfrak{h}$.

We shall use a coordinate system such that $H = H_1 + H_2$ with H_1 is in the span of $\{ H_\alpha^* \mid \alpha \in \Delta_S \}$ and H_2 is orthogonal to this space. Then

$$\frac{P(H_\rho^*)}{P_S^1(H)} \sum_{w \in W} \epsilon(w) e^{iB(wH, H')} = \frac{P(H_\rho^*)}{P_S^1(H_1)} \sum_{w} \epsilon(w) e^{iB(wH_1, H')} e^{iB(wH_2, H')}.$$

Since H' is in $\mathfrak{h}^+$, the function above is periodic in the H_2 variables. In fact, $H' = \Sigma h_i H_{\omega_i}^*$ with $h_i > 0$, so that if α is a simple root not in S,

$$B(wH_2 + w(2\pi(H_\alpha^*/h_\alpha)), H') = B(wH_2, H') + (2\pi/h_\alpha)B(wH_\alpha^*, H')$$

$$= B(wH_2, H') + (2\pi/h_\alpha)h_\alpha B(wH_\alpha^*, H_{\omega_\alpha}^*)$$

and this equals $B(wH_2, H') + 2\pi k$ for some integer k. Since on $\mathfrak{h}^+_{\epsilon, S}$, the H_1 variables range over a compact set we must have $|(P(H^*_\rho)/P^1_S(H)) \sum \epsilon(w) e^{iB(wH, H')}|$ bounded by a constant depending only on H'. Then on $\mathfrak{h}^+_{\epsilon, S}$, $|J(H, H')| \leqslant C(H')/P(H')P^2_S(H)$. Motivated by this asymptotic behavior, we make the following definition.

Set q_c equal to the infimum of $q \geqslant 2$ such that for each S in S,

$$\int_{\mathfrak{h}^+_{\epsilon, S}} \left| \frac{1}{P^2_S(H)} \right|^q |P(H)|^2 \, dH < \infty$$

and

$$\int_{\mathfrak{h}^+_\epsilon} \left| \frac{1}{P(H)} \right|^q |P(H)|^2 \, dH < \infty.$$

Each of the integrands is the reciprocal of a nowhere vanishing polynomial and so q_c is finite. We shall call the conjugate exponent p_c $(1/q_c + 1/p_c = 1)$ the *critical exponent*.

REMARK. We believe q_c to be the $\inf_{q \geqslant 2} \{q \,|\, \int_{\mathfrak{h}^+} |D_0(H)|^q |P(H)|^2 \, dH < \infty\}$; however we have been able to prove this only for a few cases.

We shall now prove Proposition 4.9.

PROOF. For H_1, H_2 in $\mathfrak{h}^+$,

$$\hat{f}(H_1) - \hat{f}(H_2) = \int_{\mathfrak{h}^+} f(H) \, [\overline{J(H, H_1)} - \overline{J(H, H_2)}] \, P(H)|^2 \, dH.$$

Applying Hölder's inequality, we obtain

$$|\hat{f}(H_1) - \hat{f}(H_2)| \leqslant \|f\|_p \|J(\cdot, H_1) - J(\cdot, H_2)\|_q.$$

We shall have proved the claim if we can justify applying the dominated convergence theorem to $\int_{\mathfrak{h}^+} |J(H, H_1) - J(H, H_2)|^q |P(H)|^2 \, dH$.

On each $\mathfrak{h}^+_{\epsilon, S}$ and on a fixed compact neighborhood of H_2 for all H in $\mathfrak{h}^+_{\epsilon, S}$ and H_1 in this compact neighborhood we have the estimate $|J(H, H_1)| \leqslant C(H_2)/|P(H_1)| \, |P^2_S(H)|$ so we can dominate $|J(H, H_1) - J(H, H_2)|$ by $M/P^2_S(H)$. Since $p < p_c$, $|M/P^2_S(H)|^q$ is integrable.

We shall give two examples illustrating typical ranges of p for which mean convergence results hold.

EXAMPLE 4.10. $G = SU(2)$, dim $\mathfrak{h} = 1$, $P(H) = h$ and computation gives $p_c = 3/2$. There is only one way to sum the series and for that method $r = 1$. Then $(2r + 1)/(r + 1) = 3/2$ and complete mean convergence information is obtained except at $p = 3/2$. This is also a special case of Pollard's theorem.

EXAMPLE 4.11. $G = SU(3)$, dim $\mathfrak{h} = 2$ and $\mathfrak{h}^+$ is a $60°$ cone. Take the the hyperplane σ to be a line perpendicular to the $H^*_{\omega_1}$ axis. Using $H^*_{\omega_1}, H^*_{\omega_2}$ for coordinates,

$$D(H) = c \sin x \sin y \sin(x + y).$$

Then $r = 2$ and $(2r + 1)/(r + 1) = 5/3$. The Euclidean multiplier m is the characteristic function of the triangular region shown in Figure 1.
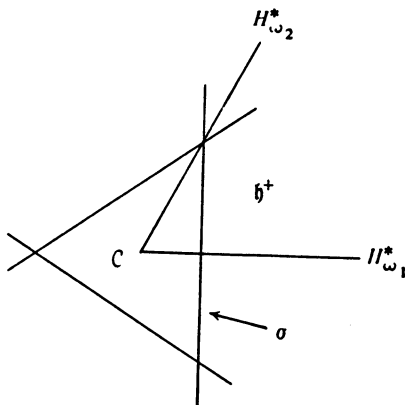

FIGURE 1

Suppose σ was chosen to form an equilateral triangle with the $H^*_{\omega_1}$ and $H^*_{\omega_2}$ axes. Using $H^*_{\alpha+\beta}$, $H^{*\perp}_{\alpha+\beta}$ for coordinates we get

$$D(H) = c \sin x \sin(x - \sqrt{3}\,y)\sin(x + \sqrt{3}\,y).$$

Here σ corresponds to the y variable so $r = 3$ and $(2r + 1)/(r + 1) = 7/4$. The Euclidean multiplier is the characteristic function of the hexagonal region shown in Figure 2.
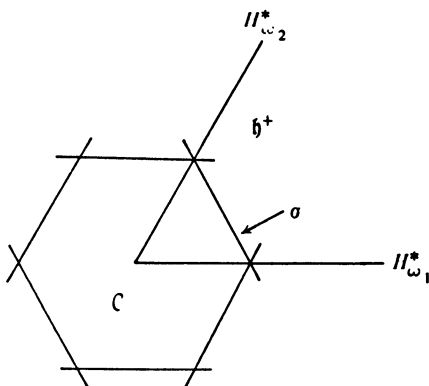

FIGURE 2

BIBLIOGRAPHY

1. N. Bourbaki, *Éléments de mathématique*. Fasc. XXXIV. *Groupes et algèbres de Lie.* Chaps. IV–VI, Actualités Sci. Indust., no. 1337, Hermann, Paris, 1968. MR 39 #1590.

2. Jean-Louis Clerc, *Les sommes partielles de la décomposition en harmoniques sphériques ne convergent pas dans L^p* ($p \neq 2$), C. R. Acad. Sci. Paris Sér. A–B 274 (1972), A59–A61. MR 45 #2405.

2'. ———, *Sommes de Riesz sur un groupe de Lie compact*, C. R. Acad. Sci. Paris Sér. A–B 275 (1972), A591–A593. MR 46 #5931.

3. R. Coifman and G. Weiss, *Analyse harmonique noncommutative sur certaines espaces homogènes*, Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin, 1971.

4. ———, *Central multiplier theorems for compact Lie groups*, Bull. Amer. Math. Soc. 80 (1974), 124–126.

5. W. C. Connett and A. L. Schwartz, *Weak type multipliers for Hankel transforms* (preprint).

6. C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. (2) 94 (1971), 330–336. MR 45 #5661.

7. Harish-Chandra, *Fourier transforms on a semisimple Lie algebra*. I, Amer. J. Math. 79 (1957), 193–257. MR 19, 293.

8. ———, *Spherical functions on a semisimple Lie group*. I, Amer. J. Math. 80 (1958), 241–310. MR 20 #925.

9. S. Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., Vol. 12, Academic Press, New York, 1962. MR 26 #2986.

10. C. S. Herz, *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 996–999. MR 16, 127.

11. G. Hochschild, *The structure of Lie groups,* Holden-Day, San Francisco, Calif., 1965. MR 34 #7696.

12. R. A. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251. MR 47 #701.

13. S. Igari, *On the multipliers of Hankel transforms*, Tôhoku Math. J. (2) 24 (1972), 201–206. MR 48 #2684.

14. K. deLeeuw, *On L_p multipliers*, Ann. of Math. (2) 81 (1965), 364–379. MR 30 #5127.

15. H. Pollard, *The mean convergence of orthogonal series*. III, Duke Math. J. 16 (1949), 189–191. MR 10, 450.

16. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser., no. 30, Princeton Univ. Press, Princeton, N.J., 1970. MR 44 #7280.

17. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Ser., no. 32, Princeton Univ. Press, Princeton, N.J., 1971. MR 46 #4102.

18. R. Strichartz, *Multiplier transformations on compact Lie groups and Lie algebras* (to appear).

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001