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ON THE TRIVIAL EXTENSION OF EQUIVALENCE RELATIONS ON ANALYTIC SPACES

BY

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ABSTRACT. In this paper, we shall consider the problem: let X be a (reduced) analytic space and A a nowhere dense analytic set in X. And let R be a proper equivalence relation on A such that the quotient space A/R is an analytic space, and $\widetilde{R}$ the trivial extension of R to X. Then, is $X/\widetilde{R}$ an analytic space? To this, we have three sufficient conditions. Moreover, using this result we shall extend Satz 1 of H. Kerner [8].

1. **Introduction.** Let $(X, {}_X\mathcal{O})$ be an analytic space and R an equivalence relation on X. Then the local ringed quotient space $(X/R, {}_X\mathcal{O}/R)$ is defined and the problem, whether $(X/R, {}_X\mathcal{O}/R)$ is an analytic space, is studied by H. Cartan, H. Holmann, B. Kaup and others.

In this paper, we shall consider the problem: let X be a (reduced) analytic space and A a nowhere dense analytic set in X. And let R be a proper equivalence relation on A such that the quotient space A/R is an analytic space, and $\widetilde{R}$ the trivial extension of R to X. Then, is $X/\widetilde{R}$ an analytic space? To this, we have

THEOREM. *$X/\widetilde{R}$ is an analytic space, if one of the following three statements is satisfied*:

(1) *R is finite.*

(2) *A is contractible in X and the canonical mapping j: $A/R \longrightarrow X/\widetilde{R}$ is quasi-finite.*

(3) *A is contractible and retractable in X.*

Next, using Theorem, (3), we shall extend Satz 1 of H. Kerner [8]: let X_k be a connected complex manifold, A_k a contractible and retractable analytic set in X_k and R_k a proper equivalence relation on A_k such that A_k/R_k is an analytic space and $\dim_a R_k(a) > 0$ for any $a \in A_k$ $(k = 1, 2)$. Then, we have the following diagrams of analytic spaces:

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$$\begin{array}{ccc} X_k & \xrightarrow{\widetilde{p}_k} & X_k/\widetilde{R}_k \\ {\scriptstyle i_k}\uparrow & & \uparrow{\scriptstyle j_k} \\ A_k & \xrightarrow{p_k} & A_k/R_k \end{array}$$

Here p_k: $A_k \longrightarrow A_k/R_k$, $\widetilde{p}_k$: $X_k \longrightarrow X_k/\widetilde{R}_k$ are natural projections, i_k: $A_k \longrightarrow X_k$ is the injection and j_k: $A_k/R_k \longrightarrow X_k/\widetilde{R}_k$ is the canonical mapping. Let r_k: $X_k \longrightarrow A_k$ be the holomorphic retraction. Then, we have

THEOREM. *Suppose that f.m.d.* $r_2 \geqslant \dim A_1 + 2$. *If* $X_1/\widetilde{R}_1$ *and* $X_2/\widetilde{R}_2$ *are analytically equivalent, then the above two diagrams are analytically equivalent.*

H. Kerner has treated the case that r_k: $X_k \longrightarrow A_k$ is a weakly negative vector bundle and $R_k(a) = A_k$ for any $a \in A_k$.

2. **Trivial extension of equivalence relations.** Let L be the category of local ringed spaces [6]: objects in L are local ringed spaces and morphisms in L are morphisms of local ringed spaces.

DEFINITION 1. A commutative diagram of morphisms in L:

$$\begin{array}{ccc} Z & \xrightarrow{b} & P \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle a} \\ X & \xrightarrow{r} & Y \end{array}$$

is called a pushout (and P is called the pushout for r and s), if for any object A and morphisms u: $Y \longrightarrow A$, v: $Z \longrightarrow A$ in L with $v \circ s = u \circ r$, there exists the unique morphism p: $P \longrightarrow A$ such that $p \circ b = v$ and $p \circ a = u$.

Let $(X, {}_X\mathcal{O})$ be a (reduced) analytic space and R an equivalence relation on X. Then there exists the local ringed quotient space $(X/R, {}_X\mathcal{O}/R)$ and the natural projection p: $X \longrightarrow X/R$ is a morphism of local ringed spaces, where X/R is the quotient topological space of X by R and ${}_X\mathcal{O}/R$, the structure sheaf on X/R, is defined as follows: for any open set $U \subset X/R$, $({}_X\mathcal{O}/R)(U) := \{f\colon U \longrightarrow \mathbf{C},\ f \circ p \in \Gamma(p^{-1}(U), {}_X\mathcal{O})\}$.

DEFINITION 2. An equivalence relation R on X is called proper if for any compact set $K \subset X$, the R-saturated set $R(K)$ (i.e. the union of all equivalence classes meeting K) is also compact.

This condition is equivalent that X/R is locally compact and the natural projection p: $X \longrightarrow X/R$ is proper.

DEFINITION 3. Let A be a subset of X and R an equivalence relation on A. The trivial extension $\widetilde{R}$ of R to X, an equivalence relation on X, is defined by

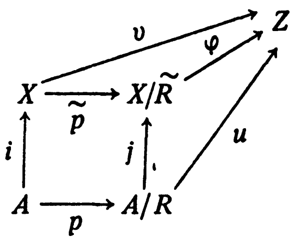
$$\widetilde{R}(x) := \begin{cases} R(x), & \text{for } x \in A, \\ \{x\}, & \text{for } x \notin A, \end{cases}$$

where $R(x), x \in A$, denotes the equivalence class by R containing x.

Let $(A, {}_A\mathcal{O})$ be a nowhere dense analytic set of $(X, {}_X\mathcal{O})$ and R an equivalence relation on A. Then we have the local ringed quotient spaces $(A/R, {}_A\mathcal{O}/R), (X/\widetilde{R}, {}_X\mathcal{O}/\widetilde{R})$. Let $p: A \longrightarrow A/R$, $\widetilde{p}: X \longrightarrow X/\widetilde{R}$ be natural projections and $i: A \longrightarrow X$ the injection. Then there exists the canonical mapping $j: A/R \longrightarrow X/\widetilde{R}$ $(\widetilde{p} \circ i = j \circ p)$ and j is a morphism in L.

Lemma 1. *$X/\widetilde{R}$ is the pushout for i and p in L.*

Proof. For any object Z and morphisms $u: A/R \longrightarrow Z$, $v: X \longrightarrow Z$ in L with $v \circ i = u \circ p$, we define the mapping as follows: for any $\widetilde{x} \in X/\widetilde{R}$, we put $\varphi(\widetilde{x}) := v(x)$ $(x \in \widetilde{p}^{-1}(\widetilde{x}))$. Then this is well defined. In fact $\widetilde{p}(x) = \widetilde{p}(x')$ $(x, x' \in X)$ implies $v(x) = v(x')$. Now φ is continuous with $v = \varphi \circ \widetilde{p}$, and $u = \varphi \circ j$ since $u \circ p = \varphi \circ j \circ p$ and p is surjective.



For any $f \in {}_Z\mathcal{O}_{\varphi(\widetilde{x})}(\widetilde{x} \in X/\widetilde{R})$, there exists $\widetilde{f} \in ({}_X\mathcal{O}/\widetilde{R})_{\widetilde{x}}$ with $v_x^*(f) = \widetilde{f} \circ \widetilde{p}$. And we put $\varphi_{\widetilde{x}}^*(f) := \widetilde{f}$. Then φ^* holds commutativity and is unique. Hence $X/\widetilde{R}$ is the pushout in L for i and p. Q.E.D.

Definition 4. An analytic set $A \subset X$ is called contractible in X if A is nowhere discrete, compact and if there exist an analytic space Y and a surjective proper holomorphic mapping $\psi: X \longrightarrow Y$ such that $\psi(A) =: y_A \in Y$ and the restriction $\psi \mid (X - A) \longrightarrow (Y - \{y_A\})$ is biholomorphic.

Definition 5. An analytic set $A \subset X$ is called retractable if there exists a holomorphic retraction of X to A (i.e. a surjective holomorphic mapping r: $X \longrightarrow A$ with $r|A = \mathrm{id}_A$).

Definition 6. A morphism f: $(X, {}_X\mathcal{O}) \longrightarrow (Y, {}_Y\mathcal{O})$ in L is called quasi-finite if for any $x \in X$, ${}_X\mathcal{O}_x/(f_x^*(M_{f(x)}))$ is a finite dimensional vector space over $\mathbf{C}$, where $M_{f(x)}$ is the maximal ideal of ${}_Y\mathcal{O}_{f(x)}$.

Let $(A, {}_A\mathcal{O})$ be an analytic set in $(X, {}_X\mathcal{O})$ and R a proper equivalence relation on A such that A/R is an analytic space. Using the results by B. Kaup [6] and the method of H. Kerner [8], we shall show the sufficient conditions under which $X/\widetilde{R}$ is an analytic space.

THEOREM 1. *$X/\widetilde{R}$ is an analytic space, if one of the following statements is satisfied*:

(1) *R is finite (i.e. every equivalence class of A by R is a finite set).*

(2) *A is contractible in X and the canonical mapping j: $A/R \longrightarrow X/\widetilde{R}$ is quasi-finite.*

(3) *A is contractible and retractable in X.*

PROOF. (1) From Lemma 1, $X/\widetilde{R}$ is the pushout for the injection $i: A \longrightarrow X$ and the natural projection $p: A \longrightarrow A/R$. Hence, by B. Kaup [6, Satz 1.8], $X/\widetilde{R}$ is an analytic space.

(2) If A is contractible in X, A is exceptional in X in the sense of B. Kaup [6]. Hence, by Lemma 1 and B. Kaup [6, Aussage 1.11], $X/\widetilde{R}$ is an analytic space.

(3) $\widetilde{R}$ is proper since, for any compact set $K \subset X$, $\widetilde{R}(K) = K \cup R(K)$ is also compact in X.

By the assumption, there exist an analytic space Y, a surjective proper holomorphic mapping $\psi: X \longrightarrow Y$ and a holomorphic retraction $r: X \longrightarrow A$. Then we have a surjective morphism $\widetilde{r}: X/\widetilde{R} \longrightarrow A/R$ with $\widetilde{r} \circ \widetilde{p} = p \circ r$. In fact, for any $\widetilde{x} \in X/\widetilde{R}$, we put

$$\widetilde{r}(\widetilde{x}) := p \circ r(x) \qquad (x \in \widetilde{p}^{-1}(\widetilde{x})).$$

Then $\widetilde{r}$: $X/\widetilde{R} \longrightarrow A/R$ is well defined.

$$\begin{array}{ccc} X & \xrightarrow{\ \widetilde{p}\ } & X/\widetilde{R} \\ i\uparrow\downarrow r & & \widetilde{r}\downarrow\uparrow j \\ A & \xrightarrow{\ p\ } & A/R \end{array}$$

Now, we claim that $(X/\widetilde{R}, {}_X\mathcal{O}/\widetilde{R})$ is locally morph-separable (i.e. for any $\widetilde{x} \in X/\widetilde{R}$, there exists an open neighborhood $U \subset X/\widetilde{R}$ such that $\Gamma(U, {}_X\mathcal{O}/\widetilde{R})$ separates points of U). Then $(X/\widetilde{R}, {}_X\mathcal{O}/\widetilde{R})$ is an analytic space by H. Cartan [1, Main Theorem].

Let $\widetilde{x}$ be a point of $X/\widetilde{R}$. We may assume that $\widetilde{x} \in j(A/R)$. Then there exists an open neighborhood V of $x := \widetilde{r}(\widetilde{x})$ such that $\Gamma(V, {}_A\mathcal{O}/R)$ separates points of V and also there exists an open neighborhood $O \subset Y$ of y_A such that $\Gamma(O, {}_Y\mathcal{O})$ separates points of O. Since $W := \psi^{-1}(O) \subset X$ is an open neighborhood of A, we have $\widetilde{p}^{-1}(\widetilde{p}(W)) = W$, hence $\widetilde{p}(W)$ is an open neighborhood of $\widetilde{x}$. Thus, so is $U := \widetilde{p}(W) \cap \widetilde{r}^{-1}(V) \subset X/\widetilde{R}$. We can show that U satisfies the above statement. Let $\widetilde{y}$, $\widetilde{z}$ be any distinct points in U. Then there exist two distinct points y, z in X such that $\widetilde{p}(y) = \widetilde{y}$, $\widetilde{p}(z) = \widetilde{z}$. If $\psi(y) \neq \psi(z)$, we have $f \in \Gamma(O, {}_Y\mathcal{O})$ with $f \circ \psi(y) \neq f \circ \psi(z)$. And $f \circ \psi \in \Gamma(W, {}_X\mathcal{O})$ is constant on A. Put

$$F(\widetilde{w}) := \begin{cases} f \circ \psi \circ (\widetilde{p}|W-A)^{-1}(\widetilde{w}), & \text{for } \widetilde{w} \in \widetilde{p}(W-A), \\ f(y_A), & \text{for } \widetilde{w} \in \widetilde{p}(A). \end{cases}$$

Then $F \in \Gamma(\widetilde{p}(W), {}_X0/\widetilde{R}) \subset \Gamma(U, {}_X0/\widetilde{R})$ and $f \circ \psi = F \circ \widetilde{p}$ in W. Therefore $F(\widetilde{y}) \neq F(\widetilde{z})$. If $\psi(y) = \psi(z)$, then $y, z \in A$ and $p(y) \neq p(z)$. Hence we have $g \in \Gamma(V, {}_A0/R)$ with $g \circ p(y) \neq g \circ p(z)$. Put in U, $G := g \circ \widetilde{r}$; then $G \in \Gamma(U, {}_X0/\widetilde{R})$ with $G(\widetilde{y}) \neq G(\widetilde{z})$, since $r: X \longrightarrow A$ is a holomorphic retraction. Thus $(X/\widetilde{R}, {}_X0/\widetilde{R})$ is locally morph-separable. Q.E.D.

REMARK 1. We can easily find the examples such that $X/\widetilde{R}$ is not an analytic space, in the case that R is not finite in (1), or A is not contractible in (2), (3) respectively.

COROLLARY 1. *Let $(X, {}_X0)$, $(A, {}_A0)$ and R be as in Theorem* 1, (1) *or* (3). *Then A/R is embedded in $X/\widetilde{R}$. In particular, in the case of* (3), *A/R is contractible and retractable in $X/\widetilde{R}$.*

PROOF. The canonical mapping $j: A/R \longrightarrow j(A/R)$ is a holomorphic homeomorphism since j is proper. We assert that for any $\widetilde{a} \in A/R$, $j^*_{\widetilde{a}}: ({}_X0/\widetilde{R})_{j(\widetilde{a})} \longrightarrow ({}_A0/R)_{\widetilde{a}}$ is surjective.

(1) For any $f \in ({}_A0/R)_{\widetilde{a}}$ $(\widetilde{a} \in A/R)$, we have $p^*_a(f) \in {}_A0_a$ $(a \in p^{-1}(\widetilde{a}))$. Then there exists $g \in {}_X0_a$ with $i^*_a(g) = p^*_a(f)$. Since p is finite proper, we have $G \in ({}_X0/\widetilde{R})_{j(\widetilde{a})}$ with $\widetilde{p}^*_a(G) = g$. Then it follows that $j^*_{\widetilde{a}}(G) = f$.

(3) Since $\widetilde{r} \circ j = \mathrm{id}_{A/R}$, surjectiveness of $j^*_{\widetilde{a}}$ is evident and in particular $\widetilde{r}$ is a holomorphic retraction. Therefore A/R is retractable and contractible in $X/\widetilde{R}$. Q.E.D.

3. **Applications.** We now consider the following problem: Let $(X, {}_X0)$ and $(M, {}_M0)$ be analytic spaces, A a nowhere dense analytic set in X and h: $A \longrightarrow M$ a surjective proper holomorphic mapping. Then, does an analytic space Y exist with the following property (P)?

(P) There exist a surjective proper holomorphic mapping $\widetilde{h}: X \longrightarrow Y$ and an injection $j: M \longrightarrow Y$ such that the restriction $\widetilde{h}|A = j \circ h$ and $\widetilde{h}|(X-A) \longrightarrow (Y-\widetilde{A})$ $(\widetilde{A} := \widetilde{h}(A))$ is biholomorphic.

DEFINITION 7. We say that a reduced analytic space X is maximal if, for any open set $U \subset X$ and a nowhere dense analytic set $S \subset U$, every continuous function on U which is holomorphic on $U - S$ is actually holomorphic on U.

REMARK 2. If an analytic space $(X, {}_X0)$ is maximal, ${}_X0$ is the maximal reduced complex structure on X.

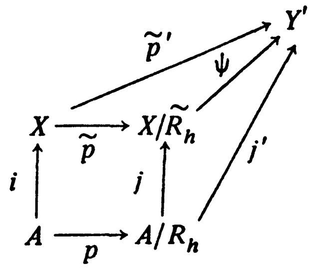
Let X, A and R be as in Theorem 1 (1) or (2) or (3). If X is maximal, so is $X/\widetilde{R}$.

Let R_h be the equivalence relation on A defined by $h: A \longrightarrow M$ (i.e. for any $u, v \in A$, $u\, R_h\, v$ means $h(u) = h(v)$). Then R_h is proper and, if M is maximal we can show that A/R_h, M are isomorphic. Thus from Theorem 1 and Corollary 1, we have

THEOREM 2. *If* (1) *or* (3) *in Theorem* 1 *is satisfied for* X, A, R_h *and* M *is maximal, there exists an analytic space* Y *with the property* (P).

COROLLARY 2. *Let* X, A, M *and* R_h *be as in Theorem* 2. *Suppose that* X *is maximal. Then any maximal analytic space* Y' *with the property* (P) *is biholomorphically equivalent to* $X/\widetilde{R}_h$.

PROOF. Let $\widetilde{p}': X \longrightarrow Y'$ be a surjective proper holomorphic mapping and $j': A/R_h \longrightarrow Y'$ an injection such that the restriction $\widetilde{p}'|A = j' \circ p$ and $\widetilde{p}'|(X - A) \longrightarrow (Y' - \widetilde{p}'(A))$ is biholomorphic. Then, from Lemma 1, we have the unique holomorphic mapping $\psi: X/\widetilde{R}_h \longrightarrow Y'$ with $\widetilde{p}' = \psi \circ \widetilde{p}$, $j' = \psi \circ j$.



Since the restriction $\psi|(X/\widetilde{R}_h - \widetilde{p}(A)) \longrightarrow (Y' - \widetilde{p}'(A))$ and $\psi|j(A/R_h) \longrightarrow j'(A/R_h)$ are biholomorphic, ψ is bijective. Moreover, ψ^{-1} is continuous since $\widetilde{p}'$ is proper. Hence ψ is a holomorphic homeomorphism. By assumption, Y' is maximal, thus ψ is biholomorphic. Q.E.D.

Now, using Theorem 1, (3), we shall extend Satz 1 of H. Kerner [8]. Let X_k be a connected complex manifold and A_k a contractible and retractable analytic set in X_k. Let R_k be an equivalence relation on A_k such that A_k/R_k is an analytic space and $\dim_a R_k(a) > 0$ for any $a \in A_k$ $(k = 1, 2)$. If R_k is proper, $X_k/\widetilde{R}_k$ is an analytic space and the natural projection $\widetilde{p}_k: X_k \longrightarrow X_k/\widetilde{R}_k$ is proper holomorphic. Let $r_k: X_k \longrightarrow A_k$ be the holomorphic retraction. Then we use the following result.

LEMMA 2 (H. HOLMANN [5]). *Let* X *be a complex manifold and* A *an analytic set in* X. *Suppose that* $r: X \longrightarrow A$ *is a holomorphic retraction. Then* A *is a closed complex submanifold of* X *and, for any* $a \in A$, *there exists an open neighborhood* $U \subset X$ *such that the restriction* $r|U$ *is a holomorphic projection* (*i.e. there exist two complex manifolds* M_1, M_2 *and a biholomorphic mapping* $T: U \longrightarrow M_1 \times M_2$ *such that* $\mathrm{pr} = T \circ r \circ T^{-1}$, *where* $\mathrm{pr}: M_1 \times M_2 \longrightarrow M_1 \times$

M_2, $\mathrm{pr}(x_1, x_2) = (x_1, x_2^0)$ *for any* $(x_1, x_2) \in M_1 \times M_2$, x_2^0 *is a fixed point).*

If φ is a holomorphic mapping of an analytic space X into an analytic space Y, we put f.m.d. $\varphi := \min_{x \in X} \dim_x \varphi^{-1}(\varphi(x))$. Then using Lemma 2 and the assumption $\dim_a R_k(a) > 0$, we can prove the next lemma in almost like manner as in [8].

LEMMA 3. *Suppose that f.m.d.* $r_k \geqslant 2$. *Then* $\widetilde{A}_k := \widetilde{p}_k(A_k)$ *is the set of all singular points of* $X_k/\widetilde{R}_k$.

THEOREM 3. *Suppose that f.m.d.* $r_2 \geqslant \dim A_1 + 2$. *If* $X_1/\widetilde{R}_1$ *and* $X_2/\widetilde{R}_2$ *are analytically equivalent, the following diagrams* ($k = 1, 2$) *are analytically equivalent.*

PROOF. We first show that

$$(*) \qquad \text{f.m.d. } r_1 \geqslant \dim A_2 + 2$$

in some open neighborhood of A_1.

$$\begin{array}{ccc} X_k & \xrightarrow{\ \widetilde{p}_k\ } & X_k/\widetilde{R}_k \\ {\scriptstyle i_k}\uparrow & & \uparrow{\scriptstyle j_k} \\ A_k & \xrightarrow{\ p_k\ } & A_k/R_k \end{array}$$

By assumption, any point of A_k ($k = 1, 2$) has an open neighborhood with the property stated in Lemma 2. Let O_k be the union of all such open neighborhoods. Then

$$\dim O_2 - \dim A_2 \geqslant \text{f.m.d. } r_2 \geqslant \dim A_1 + 2.$$

Since $\dim O_1 = \dim O_2$, it follows that

$$\text{f.m.d. } (r_1 | O_1) = \dim O_1 - \dim A_1 \geqslant \dim A_2 + 2.$$

Hence, by Lemma 3, $\widetilde{A}_k := \widetilde{p}_k(A_k)$ ($k = 1, 2$) is the set of all singular points of $X_k/\widetilde{R}_k$. Let $\psi\colon X_1/\widetilde{R}_1 \longrightarrow X_2/\widetilde{R}_2$ be the biholomorphic mapping. Then $\psi(\widetilde{A}_1) = \widetilde{A}_2$ and there exists an open neighborhood $U_k \subset X_k/\widetilde{R}_k$ of $\widetilde{A}_k$ with $U_k^\wedge := \widetilde{p}_k^{-1}(U_k) \subset O_k$.

We now assert that there exists a holomorphic mapping $\psi^\wedge\colon U_1^\wedge \longrightarrow U_2^\wedge$ such that $\psi \circ \widetilde{p}_1 = \widetilde{p}_2 \circ \psi^\wedge$. We put

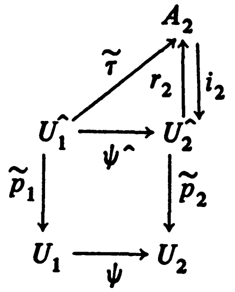
$$\psi^\vee := \psi | (U_1 - \widetilde{A}_1) \longrightarrow (U_2 - \widetilde{A}_2),$$

$$\widetilde{p}_k^\vee := \widetilde{p}_k | (U_k^\wedge - A_k) \longrightarrow (U_k - \widetilde{A}_k) \qquad (k = 1, 2).$$

These mappings are biholomorphic. And we put, on $U_1^\wedge - A_1$, $\tau := r_2 \circ (\tilde{p}_2^\vee)^{-1} \circ \psi^\vee \circ \tilde{p}_1^\vee$. Then τ: $(U_1^\wedge - A_1) \longrightarrow A_2$ is also holomorphic. Since f.m.d. $\tau \geqslant \dim A_1 + 2$ on $U_1^\wedge - A_1$, we have the holomorphic mapping $\tilde{\tau}$: $U_1^\wedge \longrightarrow A_2$ such that $\tilde{\tau} | (U_1^\wedge - A_1) = \tau$ [9, Satz 2]. Define the mapping $\psi^\wedge$: $U_1^\wedge \longrightarrow U_2^\wedge$ as follows:

$$\psi^\wedge(x) := \begin{cases} (\tilde{p}_2^\vee)^{-1} \circ \psi^\vee \circ \tilde{p}_1^\vee(x), & \text{for } x \in U_1^\wedge - A_1, \\ i_2 \circ \tilde{\tau}(x), & \text{for } x \in A_1, \end{cases}$$

where i_2: $A_2 \longrightarrow U_2^\wedge$ is the injection. Remark that $\tilde{\tau} = r_2 \circ \psi^\wedge$ on $U_1^\wedge$.



Then we can show that $\psi^\wedge$: $U_1^\wedge \longrightarrow U_2^\wedge$ is continuous. To show this, it suffices to say that $\psi^\wedge$ is continuous at any $a \in A_1$, and hence, for any sequence $\{a_n\} \subset U_1^\wedge - A_1$ which converges to a, $\{\psi^\wedge(a_n)\}$ converges and $\lim_{n\to\infty} \psi^\wedge(a_n) = \psi^\wedge(a)$.

$\{\psi^\wedge(a_n)\} = \{\tilde{p}_2^{-1}(\psi \circ \tilde{p}_1(a_n))\} \subset U_2^\wedge - A_2$ has cluster points in $U_2^\wedge$ since $\tilde{p}_2$ is proper, and they must be contained in A_2. Further, the cluster points are unique and coincide with $\psi^\wedge(a)$. In fact, if α is a cluster point of $\{\psi^\wedge(a_n)\}$, we have a subsequence $\{a'_n\}$ of $\{a_n\}$ with $\lim_{n\to\infty} \tilde{p}_2^{-1} \circ \psi \circ \tilde{p}_1(a'_n) = \alpha$. Then

$$\begin{aligned} \alpha = r_2(\alpha) &= r_2\left(\lim_{n\to\infty} \tilde{p}_2^{-1} \circ \psi \circ \tilde{p}_1(a'_n)\right) \\ &= \lim_{n\to\infty} r_2 \circ \tilde{p}_2^{-1} \circ \psi \circ \tilde{p}_1(a'_n) = \lim_{n\to\infty} \tau(a'_n) \\ &= \lim_{n\to\infty} \tilde{\tau}(a'_n) = \tilde{\tau}(a) = \psi^\wedge(a). \end{aligned}$$

Hence $\lim_{n\to\infty} \psi^\wedge(a_n) = \psi^\wedge(a)$. Therefore $\psi^\wedge$ is continuous. Since $U_k^\wedge$ is a complex manifold ($k = 1, 2$) and $\psi^\wedge | (U_1^\wedge - A_1)$ is holomorphic on $U_1^\wedge - A_1$, $\psi^\wedge$ is holomorphic on $U_1^\wedge$. Further, $\psi \circ \tilde{p}_1 = \tilde{p}_2 \circ \psi^\wedge$ on $U_1^\wedge$.

To complete the proof of the theorem, it suffices to show that $\psi^\wedge$ is bijective and its inverse is holomorphic. By (*), we also have the holomorphic mapping $(\psi^{-1})^\wedge$: $U_2^\wedge \longrightarrow U_1^\wedge$ such that $\psi^{-1} \circ \tilde{p}_2 = \tilde{p}_1 \circ (\psi^{-1})^\wedge$ on $U_2^\wedge$. Then it follows that

$$(\psi^{-1})^{\wedge} \circ \psi^{\wedge} = \mathrm{id} \quad \text{on } U_1^{\wedge},$$

$$\psi^{\wedge} \circ (\psi^{-1})^{\wedge} = \mathrm{id} \quad \text{on } U_2^{\wedge}.$$

Hence $\psi^{\wedge}: U_1^{\wedge} \longrightarrow U_2^{\wedge}$ is biholomorphic and, in particular, $\psi^{\wedge}(A_1) = A_2$. Therefore A_k, X_k and A_k/R_k $(k = 1, 2)$ are analytically equivalent respectively, and the two diagrams are analytically equivalent. Q.E.D.

REMARK 3. H. Kerner [8] has treated the case that $r_k: X_k \longrightarrow A_k$ $(k = 1, 2)$ is a weakly negative vector bundle and $R_k(a) = A_k$ for any $a \in A_k$.

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