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# ON THE INTEGRABILITY OF JACOBI FIELDS ON MINIMAL SUBMANIFOLDS

BY

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ABSTRACT. Let $M$ be a minimal submanifold of a Riemannian manifold. It is proved that every Jacobi field on $M$ is locally the deformation vector field along $M$ of some one-parameter families of minimal submanifolds. This fact follows from a theorem on nonlinear elliptic systems which is also proved in this paper. The related global problems are also discussed briefly.

1. **Introduction.** In response to a question posed in [7], the following theorem was proved in [4]: *Under the assumption of real analyticity, locally every Jacobi field on a minimal submanifold can be realized as the deformation vector field of some one-parameter families of minimal submanifolds.* However, the technique used in [4] cannot be used to handle the $C^\infty$ case. In this note we will prove the above theorem under the assumption that all the data are $C^\infty$. It will follow from a theorem on nonlinear elliptic systems which will be proved here. The related global problems on Jacobi fields will also be discussed briefly. The author would like to thank S. S. Chern for his encouragement as well as R. D. Moyer and M. S. Baouendi for some helpful conversations.

2. **Preliminaries.** In this section we will recall some pertinent facts about strongly elliptic operators.

Let $V$ be a connected bounded open set of $\mathbf{R}^n$ whose boundary $\partial V$ is a sufficiently smooth, say, $C^\infty$ submanifold of $\mathbf{R}^n$. For any nonnegative integer $r$, and real number $0 < \alpha < 1$, let $C^{r+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ denote the space of all $\mathbf{C}^N$ valued functions on $\overline{V}$ whose components possess continuous derivatives up to order $r$ on $\overline{V}$ and their derivatives of order $r$ satisfy a uniform Hölder condition of order $\alpha$. $\beta = (\beta^1, \ldots, \beta^n)$ denotes a vector with integral components $\beta^i \geqslant 0$, $|\beta| = \Sigma_i \beta^i$ and $D^\beta = \partial^{|\beta|}/(\partial x^1)^{\beta_1} \cdots (\partial x^n)^{\beta_n}$. For $u \in C^{r+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ with components $u^j$ we define the norm

$$|u|_{r+\alpha} = \sum_{j=1}^{N} \{[u^j]_r + H_{r+\alpha}(u^j)\}$$

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where

$$[u^j]_r = \sum_{|\beta|\leq r} \sup_{x\in V} |D^\beta u^j(x)|$$

and

$$H_{r+\alpha}(u^j) = \text{l.u.b.}\ \frac{|D^\beta u^j(P) - D^\beta u^j(Q)|}{|P-Q|^\alpha}$$

where the l.u.b. is over $P \neq Q$ in $\overline{V}$ and $|\beta| = r$. We write $|\ |_{r+\alpha}$ as $|\ |_{r+\alpha}^{\overline{V}}$ when it is necessary to emphasize the domain over which it is defined.

Let $L$ be a uniformly strongly elliptic linear differential operator of order $2m$ on $C^{2m+\alpha}(W) \otimes \mathbf{C}^N$ whose coefficients are $C^\infty$ on an open set $W$ of $\mathbf{R}^n$ such that $\overline{V} \subset W$. We will formulate the Dirichlet problem on the domain $V$ corresponding to the operator $L$ as follows: For given functions $f, g \in C^\alpha(W) \otimes \mathbf{C}^N$ find a function $u \in C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ such that

$$\begin{aligned} &Lu = f \quad \text{in } V, \qquad u|_{\partial V} = g|_{\partial V}, \\ &\partial^r u/\partial \mathbf{n}^r|_{\partial V} = \partial^r g/\partial \mathbf{n}^r|_{\partial V}, \qquad r = 1, \ldots, m-1; \end{aligned} \tag{2.1}$$

$\partial/\partial\mathbf{n}$ denotes differentiation in the normal direction. The following facts are well known and can be easily obtained from the basic *a priori* estimates in [1] and the standard techniques in the theory of elliptic operators.

PROPOSITION 2.1. *If $L$ has constant coefficients and consists only of terms of order $2m$, then the Dirichlet problem* (2.1) *admits a unique solution.*

PROPOSITION 2.2. *If the domain $V$ is of sufficiently small diameter, the Dirichlet problem for the operator $L$ on $V$ admits a unique solution.*

PROPOSITION 2.3. *Given $V$ for which one has existence and uniqueness of Dirichlet problem. There is a constant $C > 0$ such that for $u \in C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ and $\partial^h u/\partial\mathbf{n}^h = 0$, $h = 0, 1, \ldots, m-1$ we have*

$$|u|_{2m+\alpha}^{\overline{V}} \leq C|Lu|_\alpha^{\overline{V}}. \tag{2.2}$$

**3. Deformations of solutions of nonlinear strongly elliptic systems.** Consider the solution $\zeta = (\zeta^k)$ ($k = 1, \ldots, N$, $\zeta^k \in C^\infty(U) \otimes \mathbf{C}$, $U \subset \mathbf{R}^n$ is open) of the following system of order $2m$,

$$F_j(x, D^\beta u) = 0, \qquad |\beta| \leq 2m, \quad j = 1, \ldots, N, \tag{3.1}$$

in which the $F_j$ are $C^\infty$ for all values of their arguments near the values of those arguments along the solution $\zeta$. We will also assume that the system (3.1) is strongly elliptic along the solution $\zeta$ in the sense that the linear equations of variation

$$\sum_{|\beta|\leq 2m} \sum_{k=1}^{N} \frac{\partial F_j}{\partial D^\beta u^k}(x, D^\beta \zeta) D^\beta w^k = 0, \quad j = 1, \ldots, N, \tag{3.2}$$

is a strongly elliptic system. A function $v = (v^k)$ is called a *variational vector field* along the solution $\zeta$ if it is a solution of the linear equations (3.2). If $\zeta_t = (\zeta_t^k)$ is a one-parameter family of solutions of (3.1) such that $\zeta_0 = \zeta$, then obviously the *deformation vector field* $v = \partial\zeta_t/\partial t|_{t=0}$ is a variational vector field along the solution $\zeta$. That the converse is true, at least locally, is the content of the following theorem.

THEOREM 3.1. *Let $\zeta = (\zeta^k)$, $1 \leq k \leq N$, be a $C^\infty$ solution to a $C^\infty$ system of the form* (3.1) *such that it is strongly elliptic along $\zeta$. Then locally every $C^\infty$ variational vector field $v$ along $\zeta$ is the deformation vector field along $\zeta$ of some one-parameter families of $C^\infty$ solutions of* (3.1).

There is no loss of generality to assume that $\zeta$ is defined on an open set in $\mathbf{R}^n$ which contains the unit ball centered at $0 \in \mathbf{R}^n$. We will prove that in a neighborhood of 0, $v$ is the deformation vector field of some one-parameter families of solutions of (3.1). We will begin with the following lemmas.

Let $L$ be the linear differential operator with constant coefficients defined by

$$L \equiv \sum_{|\beta|=2m} \frac{\partial F}{\partial D^\beta u}(0, D^\beta \zeta(0)) D_y^\beta \tag{3.3}$$

where $\partial F/\partial D^\beta u$ is the $N \times N$ matrix $(\partial^{F_i}/\partial D^\beta u^j)$, $F_i$ as in (3.1) and $D_y^\beta = \partial^{|\beta|}/(\partial y^1)^{\beta_1} \cdots (\partial y^n)^{\beta_n}$. It follows that $L$ is a strongly elliptic operator. Let $V = \{y \in \mathbf{R}^n \mid \|y\| < 1\}$. For any $\phi \in C^{2m+\alpha}(W) \otimes \mathbf{C}^N$ where $W$ is a connected open set in $\mathbf{R}^n$ which contains $\overline{V}$, let $\Gamma_\phi$ be the operator such that for any $f \in C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$, $\Gamma_\phi(f)$ is the unique solution of the equation $Lu = f$ with Dirichlet data $\partial^h\phi/\partial n^h$, $h = 0, 1, \ldots, m-1$, on $\partial V$. Such an operator exists by Proposition 2.1. The following lemma follows readily from some simple computations and the basic estimate (2.2).

LEMMA 3.2. (a) *The mapping*

$$\Gamma_\phi : C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N \longrightarrow C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N \tag{3.4}$$

*of the Banach space $C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ with norm $|\ |_{2m+\alpha}^V$ into itself has continuous Fréchet derivative $\Gamma'_\phi$. In fact for $f, w \in C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$, $\Gamma'_\phi(f)w$ is the unique solution of the equation $Lu = w$ with homogeneous Dirichlet data on $\partial V$, hence $\Gamma'_\phi(f)$ is independent of $\phi$ and $f$.*

(b) *If $\phi_s$ depends differentiably on a parameter $s$, then so does $\Gamma_{\phi_s}(f)$. In*

*fact* $\partial/\partial s(\Gamma_{\phi_s}(f))$ *is a solution of the equation* $Lu = 0$ *with Dirichlet data* $\partial/\partial s(\partial^h \phi_s/\partial \mathrm{n}^h)$ *on* $\partial V$ *hence is independent of* $f$.

Consider the following operator introduced by L. Nirenberg in [5, p. 16] (from now on, we write equations (3.1) in vector form as $F = 0$):

$$\theta_{\epsilon,\phi}(w) = \Gamma_\phi[Lw - \epsilon^{2m}F(\epsilon y, \epsilon^{-|\beta|}D_y^\beta w)], \tag{3.5}$$

where $0 < \epsilon < 1$. It is clear that for a fixed $\epsilon$, $\theta_{\epsilon,\phi}$ is continuously differentiable in the sense of Fréchet as mapping of the Banach space $C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ into itself. It is also important to note that if $\theta_{\epsilon,\phi}$ has $\eta$ as a fixed point if and only if $\eta$ is a solution of the equation,

$$F(\epsilon y, \epsilon^{-|\beta|}D_y^\beta w) = 0, \tag{3.6}$$

with Dirichlet boundary data $\partial^h\phi/\partial \mathrm{n}^h$ on $\partial V$. By a homothety this will give a solution of (3.1) defined for $|x| \leqslant \epsilon$. Let $\phi_\epsilon \in C^{2m+\alpha}(W) \otimes \mathbf{C}^N$ be defined by

$$\phi_\epsilon(y) = \zeta(\epsilon y). \tag{3.7}$$

Let $A\colon \mathcal{B} \longrightarrow \mathcal{B}$ be a continuous linear map of a Banach space $\mathcal{B}$ with norm $|\ |$. We define the norm $\|A\|$ of $A$ to be the greatest lower bound of all numbers $K$ such that $|Ax| \leqslant K|x|$ for all $x \in \mathcal{B}$.

LEMMA 3.3. *There exist real numbers* $b$, $\delta > 0$ *such that for all* $u \in C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ *with* $|u - \phi_b|_{2m+\alpha} < \delta$ *we have*

$$\|\theta'_{b,\phi_b}(u)\| \leqslant \tfrac{1}{2}.$$

PROOF. We have by a straightforward computation

$$\theta'_{\epsilon,\phi_\epsilon}(\phi_\epsilon) = \Gamma'_{\phi_\epsilon} \circ P^\epsilon_{\phi_\epsilon} \tag{3.8}$$

where the operator $P^\epsilon_\phi$, for $0 < \epsilon < 1$ and $\phi \in C^{2m}(W) \otimes \mathbf{C}^N$, is defined by

$$P^\epsilon_\phi = L - \sum_{|\beta| \leqslant 2m} \epsilon^{2m-|\beta|} \frac{\partial F}{\partial D^\alpha u}(\epsilon y, \epsilon^{-|\beta|}D_y^\beta \phi) D_y^\beta. \tag{3.9}$$

Again here $\partial F/\partial D^\beta u$ is the $N \times N$ matrix $(\partial F_j/\partial D^\beta u^k)$. It follows from the definition of the operator $L$ that the coefficients of the operator $P^\epsilon_{\phi_\epsilon}$ are $O(\epsilon)$ as $\epsilon \longrightarrow 0$. Using (2.2) and Lemma 3.2 we have

$$|\theta'_{\epsilon,\phi_\epsilon}(\phi_\epsilon)w|_{2m+\alpha} \leqslant C|P^\epsilon_{\phi_\epsilon}(w)|_\alpha \leqslant \epsilon\tilde{c}|w|_{2m+\alpha} \tag{3.10}$$

for some constant $\tilde{c}$. Therefore by choosing $b$ sufficiently small we have $\|\theta'_{b,\phi_b}(\phi_b)\| < 1/3$. Since $\theta'_{b,\phi_b}(u)$ depends continuously on $u$, we can find $\delta$ such that for $|u - \phi_b| < \delta$, then $\|\theta'_{b,\phi_b}(u)\| < \tfrac{1}{2}$. Q.E.D.

Lemmas 3.4 and 3.5 will be needed in the construction of the one-parameter family of solutions of (3.1) which depends differentiably on the parameter. Lemma 3.4 is in [6, p. 14].

LEMMA 3.4 (CONTRACTING MAPPING PRINCIPLE). *Let $X$ be a complete metric space and $\psi\colon \mathcal{Q} \longrightarrow X$. $\mathcal{Q}$ open in $X$ and assume $d(\psi(x), \psi(y)) \leqslant \alpha d(x, y)$ with $0 \leqslant \alpha < 1$ where $d(x, y)$ is the distance between $x$ and $y$. Moreover suppose there exists $Z_0 \in \mathcal{Q}$ such that $d(Z_0, X\backslash\mathcal{Q}) > \delta$ and $d(Z_0, \psi(Z_0)) < \delta(1-\alpha)$ for some $\delta > 0$. If we define a sequence $\{Z_n\}_{n=0}^{\infty}$ inductively by $Z_{n+1} = \psi(Z_n)$, then each $Z_n$ lies in $\mathcal{Q}$, $\lim_{n\to\infty} Z_n = Z_\infty$ is a fixed point of $\psi$ and $d(Z_0, Z_\infty) < \delta$.*

REMARK. In our application of this lemma, $X$ is a Banach space with norm $|\ |$ and $\psi\colon \mathcal{Q} \longrightarrow X$ is a mapping possessing continuous Fréchet derivative $\psi'(u)$ for all $u \in \mathcal{Q}$ such that $\|\psi'(u)\| \leqslant \frac{1}{2}$. It then follows from the mean value theorem that $|\psi(x) - \psi(y)| \leqslant \frac{1}{2}|x - y|$.

LEMMA 3.5. *Let $\{a_n\}_{n=0}^{\infty}$ be some sequence in a Banach space with norm $|\ |$ such that for some positive constants $c_1$, $c_2$ and $\alpha$ the following inequalities are satisfied,*

$$
\begin{aligned}
&|a_{n+1}| \leqslant c_1 + e^{-\alpha}|a_n|,\\
&|a_{n+1} - a_n| \leqslant c_2 e^{-n\alpha}|a_n| + e^{-\alpha}|a_n - a_{n-1}|.
\end{aligned}
\tag{3.11}
$$

*Then $\{a_n\}$ is a Cauchy sequence.*

PROOF. A simple induction argument gives

$$
|a_n| \leqslant c_1 \sum_{k=0}^{n-2} e^{-k\alpha} + e^{-(n-1)\alpha}|a_0|. \tag{3.12}
$$

The sequence $\{a_n\}$ is therefore bounded by $c_1/(1 - e^{-\alpha}) + |a_0|$. So we can assume

$$
|a_{n+1} - a_n| < Ke^{-n\alpha} + e^{-\alpha}|a_n - a_{n-1}| \tag{3.13}
$$

for some positive constant $K$. Another simple induction gives

$$
|a_{n+1} - a_n| \leqslant nKe^{-n\alpha} + e^{-n\alpha}|a_1 - a_0|.
$$

Therefore $\Sigma_{n=0}^{\infty} |a_{n+1} - a_n|$ is a convergent series. $\{a_n\}$ is thus a Cauchy sequence. Q.E.D.

PROOF OF THEOREM 3.1. Define a function $\phi_{\epsilon,t}$ in an open neighborhood of $\overline{V}$ by

$$
\phi_{\epsilon,t}(y) = \zeta(\epsilon y) + tv(\epsilon y), \qquad t \in (-1, 1). \tag{3.14}
$$

Let $\psi_{\epsilon,t}$: $C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N \longrightarrow C^{2m+\alpha}(\overline{V}) \otimes \mathbf{C}^N$ be defined by

$$\psi_{\epsilon,t}(w) = \Gamma_{\phi_{\epsilon,t}}[Lw - \epsilon^{2m}F(\epsilon y, \epsilon^{-|\beta|}D_y^\beta w)]. \tag{3.15}$$

By Lemma 3.2(a) we have $\psi'_{b,t}(u) = \theta'_{b,\phi_b}(u)$, hence by Lemma 3.3 we have $\|\psi'_{b,t}(u)\| \leqslant ½$ for $|u - \phi_b|_{2m+\alpha} < \delta$ (and for all $t$). Using Lemma 3.2(b), we have that $\psi_{b,t}(\phi_b)$ depends continuously on $t$. Since $\psi_{b,0}(\phi_b) = \phi_b$, there exists a number $\rho$ such that for $|t| \leqslant \rho$, we have

$$|\psi_{b,t}(\phi_b) - \phi_b|_{2m+\alpha} < ½\delta. \tag{3.16}$$

For every $|t| \leqslant \rho$, define a sequence $\{w_{n,t}\}$ by setting $w_{0,t} = \phi_{b,t}$ and $w_{n+1,t} = \psi_{b,t}(w_{n,t})$. By Lemma 3.5 and the remark after it we have $|w_{n,t} - \phi_b|_{2m+\alpha} < \delta$ and the sequence $\{w_{n,t}\}$ converges in the norm $|\ |_{2m+\alpha}$. In fact, for $|t| \leqslant \rho$, $\{w_{n,t}\}$ converges uniformly with respect to $t$. If we put $w_t = \lim_{n\to\infty} w_{n,t}$ for $|t| \leqslant \rho$, we have a one-parameter family of solutions of the equations (3.6) with $w_0 = \phi_b$. We will next prove that $w_t$ depends differentiably on $t$. Using Lemma 3.2, we have that for $n$ finite, $|t| \leqslant \rho$, $w_{n,t}$ depends differentiably on $t$. It follows that ($P^\epsilon_\phi$ as defined in (3.9) and (3.8))

$$\begin{aligned} \partial/\partial t(w_{n+1,t}) &= \partial/\partial s(\psi_{b,s}(w_{n,t}))|_{s=t} + \Gamma'_{\phi_{b,t}} \circ P^b_{w_{n,t}}[\partial/\partial t(w_{n,t})] \\ &= \partial/\partial s(\psi_{b,s}(w_{n,t}))|_{s=t} + \psi'_{b,t}(w_{n,t})[\partial/\partial t(w_{n,t})]. \end{aligned} \tag{3.17}$$

From Lemma 3.2(b) and (3.15) it follows that the first term in the left-hand side of (3.17) is independent of $w_{n,t}$. Therefore there exists a constant $c_1$ such that

$$|\partial/\partial t(w_{n+1,t})|_{2m+\alpha} < c_1 + ½|\partial/\partial t(w_{n,t})|_{2m+\alpha} \tag{3.18}$$

for $|t| \leqslant \rho$. We have also

$$\begin{aligned} &\partial/\partial t(w_{n+1,t}) - \partial/\partial t(w_{n,t}) \\ &= \Gamma'_{\phi_{b,t}}\{P^b_{w_{n,t}}[\partial/\partial t(w_{n,t})] - P^b_{w_{n-1,t}}[\partial/\partial t(w_{n-1,t})]\} \\ &= \Gamma'_{\phi_{b,t}}\{P^b_{w_{n,t}}[\partial/\partial t(w_{n,t}) - \partial/\partial t(w_{n-1,t})] \\ &\qquad + (P^b_{w_{n,t}} - P^b_{w_{n-1,t}})[\partial/\partial t(w_{n-1,t})]\} \\ &= \psi'_{b,t}(w_{n,t})[\partial/\partial t(w_{n,t}) - \partial/\partial t(w_{n-1,t})] \\ &\quad + \Gamma'_{\phi_{b,t}} \circ (P^b_{w_{n,t}} - P^b_{w_{n-1,t}})[\partial/\partial t(w_{n-1,t})]. \end{aligned} \tag{3.20}$$

Since $|w_{n,t} - \phi_b|_{2m} \leqslant \delta$ for all $n$ and $|t| \leqslant \rho$, the moduli of the coefficients of the linear operator $P^b_{w_{n,t}} - P^b_{w_{n-1,t}}$ are bounded by

$$c_3|w_{n,t} - w_{n-1,t}|_{2m+\alpha} \leqslant (\tfrac{1}{2})^n c_3|w_{1,t} - w_{0,t}|_{2m+\alpha} \tag{3.21}$$

for some constant $c_3$. Using (2.2) and Lemma 3.2(a) we have

$$\begin{aligned} &|\Gamma'_{\phi_{b,t}} \circ (P^b_{w_{n,t}} - P^b_{w_{n-1,t}})[\partial/\partial t(w_{n-1,t})]|_{2m+\alpha} \\ &\qquad \leqslant c|(P^b_{w_{n,t}} - P^b_{w_{n-1,t}})[\partial/\partial t(w_{n-1,t})]|_\alpha \\ &\qquad \leqslant (\tfrac{1}{2})^n c\,\tilde{c}_3|w_{1,t} - w_{0,t}|_{2m+\alpha}|\partial/\partial t(w_{n-1,t})|_{2m+\alpha} \\ &\qquad \leqslant (\tfrac{1}{2})^n c_2|\partial/\partial t(w_{n,t})|_{2m+\alpha} \end{aligned} \tag{3.22}$$

for some constants $c_2$ and $\tilde{c}_3$.

Since $\|\psi'_{b,t}(w_{n,t})\| \leqslant \frac{1}{2}$, we have from (3.20) and (3.22) that

$$\begin{aligned} &|\partial/\partial t(w_{n+1,t}) - \partial/\partial t(w_{n,t})|_{2m+\alpha} \\ &\quad < (\tfrac{1}{2})^n c_2|\partial/\partial t(w_{n,t})|_{2m+\alpha} + \tfrac{1}{2}|\partial/\partial t(w_{n,t}) - \partial/\partial t(w_{n-1,t})|_{2m+\alpha}. \end{aligned} \tag{3.23}$$

It follows from Lemma 3.5 that for $|t| \leqslant \rho$, the sequence $\{\partial/\partial t(w_{n,t})\}$ converges. Since the constants in (3.18) and (3.23) are independent of $t$, $\{\partial/\partial t(w_{n,t})\}$ converges uniformly with respect to $t$. Hence by a standard theorem in analysis, we can conclude that $w_t$ depends differentiably on $t$ and, for $|t| \leqslant \rho$, $\partial w_t/\partial t = \lim_{n\to\infty} \partial/\partial t(w_{n,t})$.

Let $V_b = \{x \in \mathbf{R}^n \mid |x| < b\}$. For $|t| \leqslant \rho$, define $\zeta_t \in C^{2m}(\overline{V}_b) \otimes \mathbf{C}^N$ by $\zeta_t(x) = w_t(x/b)$. Obviously, $\zeta_t$ is a one-parameter family of solutions of (3.1) such that $\zeta_0(x) = \zeta(x)$ for $x \in \overline{V}_b$. It remains to show that $\partial\zeta_t(x)/\partial t|_{t=0} = v(x)$ for $x \in \overline{V}_b$. We can assume by making $b$ smaller if necessary that the Dirichlet problem for (3.2) on $V_b$ has a unique solution (cf. Proposition 2.2). Now it follows from the way $\zeta_t$ was constructed and Lemma 3.2 (6) that $\partial\zeta_t/\partial t|_{t=0}$ has the same Dirichlet data as $v$ on $\partial V_b$. $\partial\zeta_t/\partial t|_{t=0}$ also satisfies (3.2). Therefore $\partial\zeta_t/\partial t|_{t=0} = v$ on $V_b$. Finally it follows from Theorem 12.1 in [1] that, for each fixed $t$, $\zeta_t$ is in fact a $C^\infty$ solution of (3.1). Q.E.D.

REMARKS. 1. Theorem 3.1 holds true under weaker differentiability assumptions on the equations (3.1) as well as on the data $\zeta$ and $v$. However, this is sufficient for our purpose in the present paper where we assume, as is usually done in differential geometry, all the data to be $C^\infty$.

2. If the system (3.1) as well as the data in Theorem 3.1 are real, then obviously the one-parameter family of solutions of (3.1) constructed above is also real.

4. **Local integrability of Jacobi fields on minimal submanifolds.** Let $f: M \longrightarrow N$ be a $C^\infty$ minimal submanifold of a $C^\infty$ Riemannian manifold $N$. If the dimension of $M$ is greater than one, in terms of local coordinates, $f(M)$ can be considered as solutions to a system of nonlinear partial differential equations $F = 0$ (see, for example, [2, p. 178]). It is not difficult to check that $F = 0$ is strongly elliptic along $f(M)$. The following fact is well known ([7, Theorem 3.31]): *Let $f_t: M \longrightarrow N$, $t \in (-a, a)$ and $a > 0$, be a one-parameter family of minimal submanifolds which depends differentiably on $t$. Then the normal components of its deformation vector field along $f_0(M)$ define a Jacobi field on $f_0(M)$.* A proof of this fact under different notations was given in [4]. (The equations of variation were defined in a somewhat different context in [4], but when applied to minimal submainfolds, it is equivalent to our present definition.) In fact, using a computation similar to the one indicated in [4, p. 357] (see also errata to [4] at the end of this paper), it is not difficult to show that the equations of variation of the system $F = 0$ (or the equivalent exterior differential system) along $f(M)$ are precisely the Jacobi equations of the minimal submanifold $f(M)$, that is, the tangential components of the variational vector field do not enter into the equations of variation. After this observation, the following theorem is a corollary of Theorem 3.1 and the remarks after it.

THEOREM 4.1. *Let $M$ be a $C^\infty$ minimal submanifold of a $C^\infty$ Riemannian manifold, then every Jacobi field on $M$ is locally the deformation vector field along $M$ of some one-parameter families of $C^\infty$ minimal submanifolds.*

Of course we can only apply Theorem 3.1 when dimension of $M$ is greater than one. However, when dimension of $M$ is one, Theorem 4.1 is classical. From now on we will say that a Jacobi field on a minimal submanifold $M$ of a Riemannian manifold is *integrable* if it is the deformation vector field along $M$ of a one-parameter family of minimal submanifolds. Thus Theorem 4.1 says that every $C^\infty$ Jacobi field is locally integrable.

5. **Remarks on global integrability of Jacobi fields.** The global problem on the integrability of Jacobi fields is quite different. In fact, it is not true that all $C^\infty$ Jacobi fields are globally integrable; not even in the one-dimensional case, in spite of the fact that Jacobi fields on open geodesic segments are always integrable. We will next describe some examples due to F. J. Almgren.

Let $M$ be the surface in $E^3$ defined by the equation $x^2 + y^2 + z^4 = 1$ and

$C$ be the geodesic on $M$ defined by the equations $x^2 + y^2 = 1, z = 0$. It is easy to check that $M$ has zero Gaussian curvature along $C$ and positive Gaussian curvature elsewhere. Using these facts, it is straightforward to see that the vector field $\partial/\partial z$, when restricted to $C$, defines a Jacobi field which will be denoted by $v$. We claim that *$v$ is not integrable.*

PROOF. Suppose $v$ is integrable. Let $C_t$, $t \in (-a, a)$ and $a > 0$, be a $C^\infty$ one-parameter family of closed geodesic such that $C_0 = C$ and its deformation vector field on $C$ is $v$. Since $v \neq 0$ on $C$, there exists a $0 < \epsilon < a$ such that $C_\epsilon$ does not intersect $C$. Then using an argument like that in the proof of Theorem 1 of [3], the compactness of $C_\epsilon$ and $C$, the fact that $M$ has Gaussian curvature everywhere positive except on $C$, and the second variation formula of Synge, we have a contradiction. Q.E.D.

Appropriate Cartesian products of the above example with copies of $S^1$ yield corresponding counterexamples of any dimension and codimension.

On the other hand, one can prove that every Jacobi field on a totally geodesic submanifold $M$ is equal to the projection into the normal space of $M$ of some Killing vector fields of $S^n$ (Lemma 5.17 and Lemma 5.19 of [7]). Using this, it is not difficult to see that all the Jacobi fields on $M$ are induced by some one-parameter groups of isometries of $S^n$.

There still remains the interesting problem of investigating the sufficient conditions for the global integrability of Jacobi fields on compact minimal submanifolds. We hope to return to this problem in the future.

6. **Errata for** [4]. We like to take this opportunity to correct some errors in [4].

Page 338, line 22: "$T_m(M)$" should read "$T_m(N)$".

Page 356, line 11 should read:

$$\tau \lrcorner d\Theta_\alpha = \sum_{\beta,i} \sum_{j \neq i} \tau_\alpha \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge \omega_{j\beta} \wedge \omega_{j+1}.$$

Page 356, line 15 should read: "+ terms at least linear in $\omega_\gamma$ or $\Theta_\alpha$".

Page 357, line 15: "$\bar{R}_{\alpha\beta} = e^*(\sigma_{\alpha\beta})$" should read "$e^*(\bar{R}_{\alpha\beta})$ and $e^*(\sigma_{\alpha\beta})$ also as $\bar{R}_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ respectively".

## BIBLIOGRAPHY

1. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for the solutions of elliptic differential equations satisfying general boundary conditions.* II, Comm. Pure Appl. Math. 17 (1964), 35–92. MR 28 #5252.

2. L. P. Eisenhart, *Riemannian geometry,* 2nd ed., Princeton Univ. Press, Princeton, N. J., 1949. MR 11, 687.

3. T. Frankel, *Manifolds with positive curvature,* Pacific J. Math. 11 (1961), 165–174. MR 23 #A600.

4. D. S. P. Leung, *Deformations of integrals of exterior differential systems*, Trans. Amer. Math. Soc. 170 (1972), 333–358. MR 47 #2634.

5. L. Nirenberg, *Lectures on linear partial differential equations*, Conf. Board Math. Sci. Regional Conf. Ser. in Math., no. 17, Amer. Math. Soc., Providence, R. I., 1972.

6. J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

7. J. Simons, *Minimal subvariety in Riemannian manifolds*, Ann. of Math. (2) 88 (1968), 62–105. MR 38 #1617.

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