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NORMING $C(\Omega)$ AND RELATED ALGEBRAS

BY

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ABSTRACT. The first result of the paper is that the question of defining a submultiplicative seminorm on the commutative unital C^* algebra $C(\Omega)$ is equivalent to that of putting a nontrivial submultiplicative seminorm on the algebra of infinitesimals in some nonstandard model of C. The extent to which the existence of such a norm on one $C(\Omega)$ implies the existence for others is investigated. Using the continuum hypothesis it is shown that the algebras of infinitesimals are isomorphic and that if such an algebra has a submultiplicative norm (or, equivalently, seminorm) then, for any totally ordered field $\mathfrak{k}$ containing R, the R-algebra of infinitesimals in $\mathfrak{k}$ has a norm. A result of Allan is extended to show that in the particular case when $\mathfrak{k}$ is a certain field of Laurent series in several (possibly infinitely many) unknowns then the infinitesimals have a submultiplicative seminorm.

1. **Introduction.** In this paper we prove a number of results related to the question of whether $C(\Omega)$, the algebra of continuous complex valued functions on the compact space Ω, has a submultiplicative norm not equivalent to the usual sup norm. Bade and Curtis [2] showed that this is equivalent to the problem of putting a nontrivial seminorm on the algebra $C(\Omega)/\mathfrak{I}(F)$ where F is a finite subset of Ω and $\mathfrak{I}(F)$ is the set of all functions in $C(\Omega)$ which are zero in some neighbourhood of F. We extend this by replacing $\mathfrak{I}(F)$ by $\mathfrak{I}(E)$ where E is a finite subset of the complement of $\Omega\backslash F$ in its Stone-Čech compactification and $\mathfrak{I}(E)$ is the set of $f \in C(\Omega)$ which, when restricted to $\Omega\backslash F$ and then extended to $\beta(\Omega\backslash F)$, are zero in a neighbourhood of E. Since $C(\Omega)/\mathfrak{I}(E)$ is a direct sum of the $C(\Omega)/\mathfrak{I}(\omega)$, $\omega \in E$, the question is one of norming $C(\Omega)/\mathfrak{I}(\omega)$. When Ω is the Stone-Čech compactification of the integers the algebras $C(\Omega)/\mathfrak{I}(\omega)$ are the complexification of the algebra of finite elements of a nonstandard model of the real numbers.

In §3 we see the extent to which the existence of a nontrivial seminorm on one $C(\Omega)/\mathfrak{I}(\omega)$ implies the same for all others. We see that if $C(\Omega)$ has an incomplete submultiplicative seminorm for some compact Ω then it has such a seminorm for all compact metric Ω. The results in this section were inspired by some unpublished work of A. M. Sinclair. The central result in §4 extends a result of Erdös, Gillman and Henriksen [4, Theorem 13.13] which states that,

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under the continuum hypothesis, all real-closed η_1-fields of cardinality $2^{\aleph_0}$ are isomorphic. The extension is that if the fields both contain **R** then the isomorphism can be chosen to be the identity on **R**. As a corollary to this the results in §2 can be improved; e.g. if any $C(\Omega)$ has an incomplete norm so does every other infinite dimensional $C(\Omega)$. Another corollary is that the algebra of infinitesimals in a nonstandard model of **R** has a norm if and only if the algebra of infinitesimals in any totally ordered field containing **R** has a norm. In §5 we extend the result of G. R. Allan [1, Theorem 2] to show that certain algebras of power series with several variables (possibly infinitely many) can be normed. The general method is that of [1]. These algebras are algebras of infinitesimals in a totally ordered field and so, by the results of §4, norming these algebras is necessary if $C(\Omega)$ is to have an incomplete norm. In §6 we embed the power series algebra in the algebra of finite elements in an ultrapower of **R** without using the continuum hypothesis.

2. **Extension of the results of Bade and Curtis.** Let Ω be a compact topological space and ν an algebra homomorphism $C(\Omega) \longrightarrow \mathfrak{B}$ where $\mathfrak{B}$ is a Banach algebra. We shall extend Theorem 4.3 of [2] by making a more detailed analysis of the open sets G such that the restriction of ν to

$$\{f; f \in C(\Omega), f(\omega) = 0 \ \forall \omega \in \Omega\backslash G\} = \mathfrak{M}(\Omega\backslash G)$$

is continuous. We shall denote the set of all these sets by $\mathcal{G}$ (this is not the same object as $\mathfrak{G}$ in [2]).

LEMMA 2.1. *If $G_1, G_2, \ldots$ are disjoint open sets in Ω then $G_i \in \mathcal{G}$ for all but a finite number of values of i.*

This is merely a corollary of [2, Corollary 5.3].

LEMMA 2.2. *If G_1, G_2 are cozero sets in $\mathcal{G}$ then $G_1 \cup G_2 \in \mathcal{G}$.*

A cozero set is a set of the form $f^{-1}(\mathbf{C}\backslash\{0\})$ where $f \in C(\Omega)$.

PROOF. If $g, h \in C(\Omega)$, $G_1 = g^{-1}(\mathbf{C}\backslash\{0\})$, $G_2 = h^{-1}(\mathbf{C}\backslash\{0\})$ and $f \in \mathfrak{M}(\Omega\backslash(G_1 \cup G_2))$ then $f = |g|(|g| + |h|)^{-1}f + |h|(|g| + |h|)^{-1}f$ on $G_1 \cup G_2$. Put $g_1 = |g|(|g| + |h|^{-1})f$, $g_2 = |h|(|g| + |h|)^{-1}f$ on $G_1 \cup G_2$ and extend to Ω by putting $g_1 = g_2 = 0$ on $\Omega\backslash(G_1 \cup G_2)$. Then $g_1 \in C(\Omega)$ and so $g_1 \in \mathfrak{M}(\Omega\backslash G_i)$; the map $f \longrightarrow g_1$ is continuous so the map $f \longrightarrow (g_1, g_2) \longrightarrow (\nu(g_1), \nu(g_2)) \longrightarrow \nu(f) = \nu(g_1) + \nu(g_2)$ is continuous $\mathfrak{M}(\Omega\backslash(G_1 \cup G_2)) \longrightarrow \mathfrak{B}$.

LEMMA 2.3. $\sup_{G \in \mathcal{G}} \|\nu|\mathfrak{M}(\Omega\backslash G)\| < \infty$.

PROOF. Let $p(G) = \|\nu|\mathfrak{M}(\Omega\backslash G)\|$ for $G \in \mathcal{G}$. The proof of Lemma 2.2 shows that for cozero sets G_1, G_2, $p(G_1) + p(G_2) \geqslant p(G_1 \cup G_2)$. Since $\mathfrak{F}(\Omega\backslash G) = \{f; f \in C(\Omega), \operatorname{supp} f \subseteq G\}$ is dense in $\mathfrak{M}(\Omega\backslash G)$ we have, for $G \in \mathcal{G}$, $p(G) =$

sup $p(G^*)$ where G^* ranges over the cozero sets with closure in G. If the lemma is false we choose a sequence $\{G_i\}$ of cozero sets from $\mathcal{G}$ inductively by taking $G_0 = \varnothing$, G'_{i+1} a cozero set with $p(G'_{i+1}) \geqslant i + p(G_i)$, $G_{i+1} = G_i \cup G'_{i+1}$, and hence $\{G_i\}$ is increasing, $p(G_{i+1}) \geqslant i + p(G_i)$. Now choose $G_0^* = \varnothing$ and for each $i > 0$, G_i^* a cozero set in $\mathcal{G}$ with $\bar{G}_i^* \subseteq G_i$, $\bar{G}_{i-1}^* \subseteq G_i^*$, $p(G_i^*) \geqslant p(G_i) - 1$. We have

$$p(G_i^* \backslash \bar{G}_{i-1}^*) \geqslant p(G_i^*) - p(G_{i-1}) \geqslant p(G_i) - 1 - p(G_{i-1}) \geqslant i - 2,$$

which is impossible by [2], Corollary 5.3 as the sets $G_i^* \backslash \bar{G}_{i-1}^*$ are disjoint.

LEMMA 2.4. *Let Ω_0 be an open subset of Ω. Then $\beta\Omega_0$ contains a finite subset E such that if G is open in Ω_0 then either $\bar{G} \cap E$ is nonvoid or $G \in \mathcal{G}$.*

Here $\bar{G}$ denotes the closure of G in $\beta\Omega_0$.

PROOF. Let $E = \{\omega; \omega \in \beta\Omega_0, N \cap \Omega_0 \notin \mathcal{G}$ for all open neighbourhoods N of $\omega\}$. E is finite because if it were not we could find a sequence $\{\omega_i\}$ in $\beta\Omega_0$ and an open neighbourhood N_i of ω_i for each i with $N_i \cap N_j = \varnothing$ for $i \neq j$. We would then have $\{N_i \cap \Omega_0\}$ as a disjoint sequence of open sets in Ω, none of which is in $\mathcal{G}$, contradicting Lemma 2.1.

If $\bar{G} \cap E = \varnothing$ choose an open neighbourhood N_ω for each $\omega \in \bar{G}$ with $N_\omega \cap \Omega_0 \in \mathcal{G}$. Let $f_\omega \in C(\beta\Omega_0)$ have $f_\omega(\omega) = 1$, $f_\omega = 0$ outside N_ω and put $N'_\omega = f_\omega^{-1}(\mathbf{C}\backslash\{0\})$, so that $\omega \in N'_\omega \subseteq N_\omega$ and, in particular, $N'_\omega \cap \Omega_0 \in \mathcal{G}$. By compactness there are $\omega_1, \ldots, \omega_n \in \bar{G}$ with $\bar{G} \subseteq N'_{\omega_1} \cup \cdots \cup N'_{\omega_n}$. Put $G_i = \Omega_0 \cap N'_{\omega_i}$. Let $G' \subseteq G$ be a cozero set in Ω. Because G' is a cozero set in Ω with $G' \subseteq \Omega_0$ and each G_i is a cozero set in Ω_0, $G_i \cap G'$ is a cozero set in Ω (it is $(fg)^{-1}(\mathbf{C}\backslash\{0\})$ where $f = 0$ on $\mathbf{C}\backslash\Omega_0$, $f = f_{\omega_i}$ on Ω_0, $g \in C(\Omega)$ with $G' = g^{-1}(\mathbf{C}\backslash\{0\})$; fg is in fact continuous on Ω). Thus $G' = (G_1 \cap G') \cup \cdots \cup (G_n \cap G') \in \mathcal{G}$ by Lemma 2.2. Hence, using Lemma 2.3, we see $G \in \mathcal{G}$.

One might seek to determine the collections $\mathcal{G}'$ of open sets in Ω which satisfy the conclusion of Lemma 2.1 and other, more elementary, properties of $\mathcal{G}$, for example any open subset of a set in $\mathcal{G}$ is also in $\mathcal{G}$ and any finite open set in Ω is in $\mathcal{G}$. If $\Omega_0 = \mathbf{N}$, and Ω is the one point compactification of $\mathbf{N}$ then the alternatives in Lemma 2.4 are mutually exclusive and only possible collections $\mathcal{G}'$ are the set of all open subsets of Ω and

$$\{S; S \subseteq \mathbf{N}, S \notin \mathcal{U}_i \quad i = 1, \ldots, n\},$$

where $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are free ultrafilters on $\mathbf{N}$. For more general spaces the situation is not so simple and the conclusion of Lemma 2.4 with $\Omega_0 = \bigcup \mathcal{G} = \Omega\backslash F$, where F is the singularity set of Bade and Curtis, gives a sufficient condition for a set to be in $\mathcal{G}$ which is by no means necessary.

We now introduce some more notation. As F does not contain any isolated points of Ω, Ω is a compactification of $\Omega\backslash F$. Thus there is a continuous map $\alpha\colon \beta(\Omega\backslash F) \longrightarrow \Omega$ which is the identity on $\Omega\backslash F$. Applying Lemma 2.4 with $\Omega_0 = \Omega\backslash F$ we have $\alpha E = F$. Let $U \in E$ and put

$$\mathfrak{M}(F) = \{f; f \in C(\Omega), f(\omega) = 0 \text{ for all } \omega \in F\},$$

$$\mathfrak{I}(U) = \{f; f \in \mathfrak{M}(F), f \circ \alpha = 0 \text{ on some neighbourhood of } U \text{ in } \beta(\Omega\backslash F)\},$$

$$I(U) = \mathfrak{M}(F)/\mathfrak{I}(U) = \mathfrak{M}(\alpha U)/\mathfrak{I}(U).$$

$I(U)$ depends on U but is unaltered if F is replaced by any other finite set containing αU.

Lemmas 2.3 and 2.4 and the same methods as are used in [2, Theorem 4.3] enable us to prove

THEOREM 2.5. *Let ν be an algebra homomorphism of $C(\Omega)$ into a dense subsubalgebra of a Banach algebra $\mathfrak{B}$. Then there is a closed subset Ω' of Ω, a finite set $F \subseteq \Omega'$, a finite set $E \subseteq \beta(\Omega\backslash F)\backslash(\Omega\backslash F)$ and an algebraic and topological isomorphism φ of $\mathfrak{C}$ onto $\mathfrak{B}$ such that $\nu = \varphi\psi$, where*

(i) *For $U \in E$, $\mathfrak{R}_U$ is the completion of $I(U)$ in a nontrivial submultiplicative seminorm, i.e., the completion of the quotient of $I(U)$ by the kernel of such a seminorm.*

(ii) $\mathfrak{C} = C(\Omega') \oplus \mathfrak{R}_{U_1} \oplus \cdots \oplus \mathfrak{R}_{U_n}$

where $U_1, \ldots, U_n$ are the points of E, the Banach space structure is the direct sum and multiplication is the commutative operation determined by

$$(f, 0, \ldots, 0)(g, 0, \ldots, 0) = (fg, 0, \ldots, 0),$$

$$(0, r_1, \ldots, r_n)(0, s_1, \ldots, s_n) = (0, r_1 s_1, \ldots, r_n s_n),$$

$$(f, 0, \ldots, 0)(0, r_1, \ldots, r_n) = (0, f(\alpha(U_1))r_1, \ldots, f(\alpha(U_n))r_n),$$

$$f, g \in C(\Omega'), r_i, s_i \in \mathfrak{R}_{U_i}.$$

(iii) $\psi(f) = (f|\Omega', r_1, \ldots, r_n)$

r_i being the element of $\mathfrak{R}_{U_i}$ corresponding to any element of $\mathfrak{M}(F)$ which is equal to $f - f(\alpha(U_i))$ in some neighborhood of $\alpha(U_i)$.

The important feature of this result is that it demonstrates the equivalence between the problem of putting an incomplete algebra norm on $C(\Omega)$ and the problem of putting a nontrivial algebra seminorm on the algebras $I(U)$.

3. Norming $C(\Omega)$ and norming algebras of infinitesimals. Let V be a free ultrafilter on N, s the algebra of all sequences of complex numbers with pointwise operations, $\mathfrak{I}(V) = \{a; a \in s, a^{-1}(0) \in V\}$ a maximal ideal in s, ${}^*C(V) =$

$s/\mathfrak{F}(V)$, $\mathfrak{i}(V)$ the image in ${}^*C(V)$ of those elements a of s with $\lim_V a = 0$ and $\mathfrak{i}_0(V)$ the image of c_0 in ${}^*C(V)$. We shall frequently write merely $\mathfrak{i}$, $\mathfrak{i}_0$, ${}^*C$ for $\mathfrak{i}(V), \mathfrak{i}_0(V)$, ${}^*C(V)$.

LEMMA 3.1. *Let Ω be an infinite metrisable compact space, ω_0 a nonisolated point of Ω and $U \in \beta(\Omega\setminus\{\omega_0\})\setminus(\Omega\setminus\{\omega_0\})$. Then there is an algebra homomorphism φ: $c_0 \longrightarrow \mathfrak{I}(U)$ with $\mathfrak{I}(U)$. Im $\varphi = \mathfrak{I}(U)$.*

PROOF. Let d be a metric in Ω giving the topology and let

$$G_n = \{\omega;\, \omega \in \Omega, (2n)^{-1} > d(\omega, \omega_0) > (2n+3)^{-1}\},$$

$$G_0 = \{\omega;\, \omega \in \Omega, d(\omega, \omega_0) > 3^{-1}\}$$

and put $G^o = \bigcup_{n=0}^{\infty} G_{2n+1}$, $G^e = \bigcup_{n=0}^{\infty} G_{2n}$. U can be considered as an ultrafilter of relatively closed sets in $\Omega\setminus\{\omega_0\}$. We have $G^o = G \cap (\Omega\setminus\{\omega_0\})$ for some neighbourhood G of U in $\beta(\Omega\setminus\{\omega_0\})$ if and only if G^o contains a set from U. Thus either $G^o = G \cap (\Omega\setminus\{\omega_0\})$ or $G^e = G \cap (\Omega\setminus\{\omega_0\})$ for some open set G in $\beta(\Omega\setminus\{\omega_0\})$ containing U. For definiteness we assume $G^e = G \cap (\Omega\setminus\{\omega_0\})$.

Let $\rho_n \in C(\Omega)$ with $\rho_n = 1$ on G_n, $\rho_n = 0$ on G_m for $m \neq n-1, n, n+1$. Let q be the quotient map $\mathfrak{M}(\omega_0) \longrightarrow \mathfrak{I}(U)$ and ψ the map $c_0 \longrightarrow C(\Omega)$ defined by $\psi(a) = \Sigma\, a_n \rho_{2n}$. Then ψ is linear and, on G^e, $\psi(ab) = \psi(a)\psi(b)$, $a, b \in c_0$. Thus $\varphi = q\psi$ is the required algebra homomorphism. If $f \in \mathfrak{M}(\omega_0)$ put $b_n = \sup_{\omega \in G_{2n}} |f(\omega)|$ ($b_n = 0$ if $G_{2n} = \varnothing$). Then $b \in c_0$ and there exists $c, d \in c_0$, $c_n \geqslant 0$, $d_n > 0$ with $b_n = c_n d_n$. Let $g_0 = d_n^{-1} f$ on $\overline{G}_{2n}$, $g_0(\omega_0) = 0$ so that $g_0 \in C((\bigcup G_{2n})^-)$ and has a continuous extension $g \in C(\Omega)$. We have $g \in \mathfrak{M}(\omega_0)$ and on G_{2n}, $\psi(d)g = d_n d_n^{-1} f = f$ so $\varphi(d)q(g) = q(f)$.

The following result is related to [8, Corollary 4.4].

THEOREM 3.2. (i) *If Ω is a compact topological space such that $C(\Omega)$ has an incomplete submultiplicative seminorm then $\mathfrak{i}_0(V)$ has a nontrivial submultiplicative seminorm for some ultrafilter V on* N.

(ii) *If $\mathfrak{i}_0(V)$ has a nontrivial submultiplicative seminorm for some ultrafilter V on* N *then $C(\Omega)$ has an incomplete submultiplicative norm for every infinite compact metric space Ω, and more generally, for every compact topological space with a nontrivial convergent sequence.*

PROOF. (i) If $C(\Omega)$ has an incomplete seminorm then there is a discontinuous homomorphism ν: $C(\Omega) \longrightarrow \mathfrak{B}$ [2, Theorem 4.4]. Hence there is a separable closed * subalgebra $\mathfrak{A}$ of $C(\Omega)$ containing 1 with $\nu|\mathfrak{A}$ discontinuous. We have $\mathfrak{A} \approx C(\Omega^*)$ where Ω^* is a compact metric space. Now apply Theorem 2.5 giving $\omega_0 \in \Omega^*$ and $U \in \beta(\Omega^*\setminus\{\omega_0\})$ such that $\mathfrak{I}(U)$ has a nontrivial seminorm p. Let

φ be the map in Lemma 3.1. Then $p \circ \varphi$ is an algebra seminorm on c_0 giving rise to a homomorphism ν': $c_0 \longrightarrow \mathfrak{B}'$ for some Banach algebra $\mathfrak{B}'$. The kernel of p is a proper ideal in $\mathcal{I}(\mathcal{U})$ whereas Im φ lies in no proper ideal. Thus $\nu' \neq 0$. However if $a \in c_0$ has finite support then $\psi(a)$ is 0 in a neighbourhood of ω_0 so $\nu'(a) = 0$. This shows that ν' is not continuous and so, applying Theorem 2.5 we see that there is an ultrafilter $\mathcal{V}$ on N such that $i_0(\mathcal{V})$ carries a nontrivial multiplicative seminorm.

(ii) Let $\{\omega_n\}$ be a sequence of distinct points of Ω with limit ω where, for all n, $\omega \neq \omega_n$. The map $\theta : f \longrightarrow \{ f(\omega_n)\}$ is a continuous algebra homomorphism of $C(\Omega)$ onto c. If $\mathfrak{i}_0(\mathcal{V})$ has a nontrivial submultiplicative seminorm then the construction in Theorem 2.4 gives a discontinuous homomorphism ν: $c \longrightarrow \mathfrak{B}$ for some Banach algebra $\mathfrak{B}$. $\nu\theta$ is then a discontinuous homomorphism of $C(\Omega)$. into $\mathfrak{B}$.

THEOREM 3.3. *If there is a nontrivial submultiplicative seminorm p on $\mathfrak{i}(\mathcal{V})$ for some $\mathcal{V}$ and Ω is an infinite compact topological space then there is a discontinuous homomorphism of $C(\Omega)$ into a Banach algebra.*

PROOF. First of all we show that there is an ultrafilter $\mathcal{V}'$ on N and a seminorm on $\mathfrak{i}(\mathcal{V}')$ which does not vanish on $\mathfrak{i}_0(\mathcal{V}')$. Let $a \in l^\infty(\mathrm{N})$ with $\|a\| \leqslant 1$, $\lim_{\mathcal{V}} a_n = 0$ and $pqa \neq 0$ where q is the quotient map $s \longrightarrow {}^*\mathrm{C}(\mathcal{V})$. Put $V_j = \{n; n \in \mathrm{N}, |a'_n| \leqslant j^{-1}\}$ and let α: N $\longrightarrow$ N be defined by $\alpha(n) = j$ if $n \in V_j \setminus V_{j+1}$. Define $(\alpha^* b)_n = b_{\alpha(n)}$ so that α^* is a unital homomorphism $l^\infty(\mathrm{N}) \longrightarrow l^\infty(\mathrm{N})$. $\alpha\mathcal{V} = \{\alpha V; V \in \mathcal{V}\}$ is the base for an ultrafilter $\mathcal{V}'$ on N which is free because $V_j \in \mathcal{V}$ and $\alpha(V_j) \subseteq \{ j, j+1, j+2, \ldots \}$. $pq\alpha^*$ is an algebra seminorm on the subalgebra of $l^\infty(\mathrm{N})$ of elements b with $\lim_{\mathcal{V}} b = 0$ which is zero for any b with $b^{-1}(0) \in \mathcal{V}'$. Thus it induces a seminorm p' on $\mathfrak{i}(\mathcal{V}')$. As in the proof of Lemma 3.1 we show $a \in q\alpha^* c_0$ $\mathfrak{i}(\mathcal{V})$ so p is not trivial on $q\alpha^* c_0$. Thus p' is not trivial on $\mathfrak{i}_0(\mathcal{V}')$.

Now choose a sequence $\{G_i\}$ of disjoint open nonvoid sets from Ω and for each i, $\omega_i \in G_i$. Let $\varphi(f)_n = f(\omega_n)$. Then φ is a continuous algebra homomorphism $C(\Omega) \longrightarrow l^\infty(N)$ with closed range containing c. Writing $\mathfrak{M} = \{ f; f \in C(\Omega), \lim_{\mathcal{V}'} \varphi(f) = 0\}$, $pq\alpha^*\varphi$ is a seminorm on $\mathfrak{M}$ which is not continuous with respect to the sup norm as $pq\alpha^*$ is not continuous on c_0 and φ is an open mapping with range containing c_0. As $C(\Omega)$ is the algebra obtained by adjoining a unit to $\mathfrak{M}$, $pq\alpha^*\varphi$ extends to a submultiplicative seminorm on $C(\Omega)$ which is discontinuous.

4. Results using the continuum hypothesis. We now turn to a study of the algebras $\mathcal{I}(\mathcal{U})$, $\mathfrak{i}(\mathcal{V})$. Since each is the complexification of a real subalgebra $\mathcal{J}(\mathcal{U})$ or $\mathfrak{j}(\mathcal{V})$, the subalgebra of elements corresponding to real valued functions or sequences, and a real algebra carries a nontrivial seminorm if and only if its

complexification does, we study $\mathcal{J}$ and $\mathfrak{j}$ instead. If $\mathfrak{k}$ is a totally ordered field containing $\mathbf{R}$ as a subfield then

$$\mathfrak{k}^0 = \{k; k \in \mathfrak{k}, -q < k < q \text{ for all } q \in \mathbf{Q}^+\}$$

is the algebra of infinitesimals in $\mathfrak{k}$. We use $\mathbf{Q}^+$ rather than $\mathbf{R}^+$ in the definition because $\mathbf{Q}$ can be embedded in $\mathfrak{k}$ in only one way whereas the embedding of $\mathbf{R}$ is not unique. $\mathfrak{k}^0$ is a real algebra if we take αk, $\alpha \in \mathbf{R}$, $k \in \mathfrak{k}^0$ to be a product in $\mathfrak{k}$. Similar remarks apply to

$$\mathfrak{k}^\# = \{k; k \in \mathfrak{k}, -q < k < q \text{ for some } q \in \mathbf{Q}^+\}$$

the algebra of finite elements of $\mathfrak{k}$. $\mathfrak{k}^\#$ is the algebra obtained by adjoining a unit to $\mathfrak{k}^0$ so every element a of $\mathfrak{k}^\#$ can be expressed uniquely as $f(a) + b$ where $f(a) \in \mathbf{R}$, $b \in \mathfrak{k}^0$. f is an order preserving homomorphism of $\mathfrak{k}^\#$ onto $\mathbf{R}$. When V is an ultrafilter on $\mathbf{N}$, $\mathfrak{j}(V) = {}^*\mathbf{R}(V)^0$. If U is a free ultrafilter of zero sets in $\Omega \setminus \{\omega_0\}$, and

$$\mathfrak{H}(U) = \{f; f \in C_{\mathbf{R}}(\Omega \setminus \{\omega_0\}), f^{-1}(0) \in U\}$$

then ${}^*\mathbf{R}(U) = C_{\mathbf{R}}(\Omega \setminus \{\omega_0\})/\mathfrak{H}(U)$ is a nonstandard model of $\mathbf{R}$ and there is a homomorphism $\mathcal{J}(U) \longrightarrow {}^*\mathbf{R}(U)_0$ because every real valued function in $\mathfrak{K}(U)$ is in $\mathfrak{H}(U)$.

We now show that the various algebras $\mathfrak{i}(V)$ are isomorphic. It follows from [4, Theorem 13.13] that they are isomorphic as rings but to show that they are isomorphic as algebras we need to show that there is an isomorphism between the fields ${}^*\mathbf{R}(V)$ which leaves $\mathbf{R}$ invariant.

LEMMA 4.1. *Let $\mathfrak{k}$, $\mathfrak{l}$ be totally ordered fields containing* $\mathbf{R}$ *and let u be an order preserving isomorphism $\mathfrak{k} \longrightarrow \mathfrak{l}$. Then $gu = f$ where f is the map $\mathfrak{k}^\# \longrightarrow \mathbf{R}$ defined above and g is the corresponding map $\mathfrak{l}^\# \longrightarrow \mathbf{R}$.*

PROOF. For $a \in \mathfrak{k}^\#$, $L_a = \{q; q \in \mathbf{Q}, q < a\}$ and $R_a = \{q; q \in \mathbf{Q}, q > a\}$ define a Dedekind cut which is the cut associated with $f(a)$. As u is the identity on $\mathbf{Q}$, $uL_a = L_a$, $uR_a = R_a$ but because u is order preserving $uL_a = L_{ua}$, $uR_a = R_{ua}$ so L_a, R_a is the same cut as L_{ua}, R_{ua}, which defines gua. Thus $fa = gua$.

If $\mathfrak{k}$ is a field containing $\mathbf{R}$ and $\mathfrak{k}'$ is a subfield then $\mathfrak{k}'$ is *full* if $\mathfrak{k}' \cap \mathfrak{k}^\#$ is closed under f. If $A, B, \ldots$ are subsets of a field then $K[A, B, \ldots]$ is the subfield generated by $A, B, \ldots$. If $\mathfrak{k}'$ is a subfield of $\mathfrak{k}$ then $\overline{\text{alg}}\ \mathfrak{k}'$ is the set of all elements of $\mathfrak{k}$ which are algebraic over $\mathfrak{k}'$.

LEMMA 4.2. *Let $\mathfrak{k}$ be a totally ordered field containing* $\mathbf{R}$. *Let $\mathfrak{k}'$ be a full subfield of $\mathfrak{k}$, $w \in \mathbf{R}$. Then $\mathfrak{k}'' = K[\mathfrak{k}', w]$ is full.*

PROOF. We suppose $w \notin \mathfrak{k}'$ as otherwise there is nothing to prove. Let $z \in \mathfrak{k}'' \cap \mathfrak{k}^\#$. z can be expressed as a rational function of w with coefficients in

$\mathfrak{k}'$ and, if w is algebraic over $\mathfrak{k}'$, we can assume that the degrees of the denominator and numerator are less than the degree of w. Dividing numerator and denominator by the coefficient of the denominator with largest absolute value we have

$$z = \frac{P(w)}{Q(w)} = \frac{a_0 + a_1 w + \cdots + a_k w^k}{b_0 + b_1 w + \cdots + b_l w^l}$$

where $-1 \leqslant b_i \leqslant +1$ and one b_i is 1. We have $fQ(w) = f(b_0) + f(b_1)w + \cdots + f(b_l)w^l \neq 0$ because the coefficients are in $\mathfrak{k}'$, are not all zero and if w is algebraic over $\mathfrak{k}'$ it is of degree $> l$. Thus $Q(w)^{-1} \in \mathfrak{k}^{\#}$. Similarly we can write $P(w) = c(a_0' + a_1' w + \cdots + a_k' w^k) = cR(w)$ where $c \in \mathfrak{k}'$, $a_i' \in \mathfrak{k}' \cap \mathfrak{k}^{\#}$ and one a_i' is 1, unless $P(w) = 0$ in which case clearly $f(z) \in \mathfrak{k}''$. We have $R(w)$ and $R(w)^{-1} \in \mathfrak{k}^{\#}$. Thus $c = zQ(w)^{-1} \in \mathfrak{k}^{\#}$ and hence $P(w) = cR(w) \in \mathfrak{k}^{\#}$. This shows

$$f(z) = \frac{f(a_0) + \cdots + f(a_k)w^k}{f(b_0) + \cdots + f(b_l)w^l}$$

where the coefficients are in $\mathfrak{k}'$ so $f(z) \in \mathfrak{k}''$.

LEMMA 4.3. *Let $\mathfrak{k}$ be a totally ordered field containing* $\mathbf{R}$, *and $\mathfrak{k}'$ a subfield of $\mathfrak{k}$. Then there is a smallest full extension $\mathfrak{k}''$ of $\mathfrak{k}'$,*

$$\mathfrak{k}'' = K(\mathfrak{k}', \mathfrak{k}'' \cap \mathbf{R})$$

and if $\mathfrak{k}'$ is denumerable so is $\mathfrak{k}''$.

PROOF. For any subfield $\mathfrak{l}$ of $\mathfrak{k}$ put $E(\mathfrak{l}) = K(\mathfrak{l}, f(\mathfrak{l} \cap \mathfrak{k}^{\#}))$. Take $\mathfrak{k}'' = \bigcup_{i=1}^{\infty} E^i(\mathfrak{k}')$.

LEMMA 4.4. *Let $\mathfrak{k}$ be a totally ordered field containing* $\mathbf{R}$ *and $\mathfrak{k}'$ a full subfield of $\mathfrak{k}$. Then $\overline{\text{alg}}\ \mathfrak{k}'$ is full.*

PROOF. If $z \in \mathfrak{k}^{\#} \cap \overline{\text{alg}}\ \mathfrak{k}'$ then we have a nontrivial relation $a_0 + a_1 z + \cdots + a_n z^n = 0$, $a_i \in \mathfrak{k}'$. Dividing by the coefficient of largest absolute value we can assume $a_i \in \mathfrak{k}^{\#} \cap \mathfrak{k}'$ and one of the a_i is 1. Thus

$$f(a_0) + f(a_1)f(z) + \cdots + f(a_n)\,(f(z))^n = 0$$

where $f(a_i) \in \mathfrak{k}'$ and some $f(a_i)$ is 1 so $f(z) \in \overline{\text{alg}}\ \mathfrak{k}'$.

LEMMA 4.5. *Let $\mathfrak{k}$, $\mathfrak{l}$ be ordered fields containing* $\mathbf{R}$, *$\mathfrak{k}'$ a full subfield of $\mathfrak{k}$ and u an order preserving isomorphism of $\mathfrak{k}'$ into $\mathfrak{l}$ with u = identity on $\mathfrak{k}' \cap \mathbf{R}$. Then $u(\mathfrak{k}' \cap \mathbf{R}) = u(\mathfrak{k}') \cap \mathbf{R}$.*

PROOF. Suppose $a \in \mathfrak{k}'$ with $u(a) \in \mathbf{R}$. If $p, q \in \mathbf{Q}$ with $p < u(a) < q$ then $p < a < q$ so $a \in \mathfrak{k}^{\#}$ and $p < f(a) < q$. Thus $f(a) = u(a)$. Since $f(a) \in \mathfrak{k}'$

$\cap \mathbf{R}$, $u(f(a)) = f(a)$ we have $u(a) = u(f(a))$ so $a = f(a) \in \mathfrak{k}' \cap \mathbf{R}$.

LEMMA 4.6. *Let $\mathfrak{k}$ be a real-closed η_1-field containing* $\mathbf{R}$. *Then $\mathfrak{k}$ has a dense transcendence base over* $\mathbf{R}$.

This is an extension of [4, Lemma 13.11]. Our terminology is that of [4], in particular §§13.5–13.8.

PROOF. First of all we show that if $\aleph$ is the cardinal of $\mathfrak{k}$ then the transcendence degree of $\mathfrak{k}$ over $\mathbf{R}$ is $\aleph$. If $\aleph > 2^{\aleph_0}$ then any transcendental extension of $\mathbf{R}$ of degree less than $\aleph$ has cardinality less than $\aleph$ and any algebraic extension of an infinite field has the same cardinality as the field [4, p. 172].

Suppose $\aleph \leqslant 2^{\aleph_0}$. Since $\mathbf{R} \subseteq \mathfrak{k}$ we have $\aleph = 2^{\aleph_0}$. Because $\mathfrak{k}$ is an η_1-field, $\mathfrak{k}^0$ has strictly positive elements. Let y be one and put $x = y^{-1}$. The map $p \longrightarrow x^p$ where, if $p = rs^{-1}$, $r, s \in \mathbf{Z}$, $s > 0$, then x^p is the unique positive solution z of $z^s = x^r$, is an order preserving map of $\mathbf{Q}$ into $\mathfrak{k}$ with $x^p \in \mathfrak{k} \setminus \mathfrak{k}^{\#}$ if $p > 0$. Let $A \subseteq \mathbf{R}^+$ be a basis for $\mathbf{R}$ considered as a vector space over $\mathbf{Q}$ and for $a \in A$ choose $x(a) \in \mathfrak{k}$ with $x^p < x(a) < x^q$ whenever $p, q \in \mathbf{Q}$ with $p < a < q$. We shall show that the elements $\{x(a); a \in A\}$ are algebraically independent over $\mathbf{R}$. Suppose $a_1, \ldots, a_k \in A$, $m_1, \ldots, m_k$ are nonnegative integers and $p \in \mathbf{Q}$ with $p < m_1 a_1 + \cdots + m_k a_k$. Then there exists $p_1, \ldots, p_k$ in $\mathbf{Q}$ with $p_1 < a_i$ and $p < m_1 p_1 + \cdots + m_k p_k$ so $x^p < (x^{p_1})^{m_1} \cdots (x^{p_k})^{m_k} < x(a_1)^{m_1} \cdots x(a_k)^{m_k}$. Similarly if $m_1 a_1 + \cdots + m_k a_k < q \in \mathbf{Q}$ then $x(a_1)^{m_1} \cdots x(a_k)^{m_k} < x^q$. Thus putting $v(M) = m_1 a_1 + \cdots + m_k a_k$ if $M = x(a_1)^{m_1} \cdots x(a_k)^{m_k}$ we have $x^p < M < x^q$ whenever $p, q \in \mathbf{Q}$ with $p < v(M) < q$. In particular, as the a_i are rationally independent, v is well defined. Suppose there is a nonzero polynomial P with real coefficients and $P(x(a_1), \ldots, x(a_k)) = 0$ for distinct $a_1, \ldots, a_k \in A$. Omitting all terms with zero coefficient and dividing by the coefficient of the monomial M for which v takes its highest value (as the a_i are rationally independent M is uniquely determined) we can assume that M has coefficient 1. Let $p, q \in \mathbf{Q}$ with $v(M) > p > q >$ value of v at all other monomials in P, and let $-\beta$ be the sum of the negative coefficients in P. We have $x^p < M = M - P(x(a_1), \ldots, x(a_k)) < \beta x^q < x^{p-q} x^q = x^p$, a contradiction.

We complete the proof by taking the smallest ordinal ω_* of cardinality $\aleph$, indexing the intervals (a, b), $a, b \in \mathfrak{k}$, $a < b$ by the ordinals $< \omega_*$ and choosing for each $\omega < \omega_*$ an element x_ω of I_ω which is transcendental over $K[\mathbf{R}, x_\alpha; \alpha < \omega]$ by transfinite induction. This choice is possible because $K[\mathbf{R}, x_\alpha; \alpha < \omega]$ and hence its algebraic closure in $\mathfrak{k}$, is of transcendence degree $< \aleph$ over $\mathbf{R}$ whereas $\mathfrak{k}$ is of transcendence degree $\aleph$, so $\mathfrak{k} \neq \overline{\text{alg}}\, K[\mathbf{R}, x_\alpha; \alpha < \omega] = \mathfrak{k}'$. Since $\mathfrak{k}' \neq \mathfrak{k}$ we cannot have $I_\omega \subseteq \mathfrak{k}'$. $\{x_\omega; \omega < \omega_*\}$ is a dense set of elements of $\mathfrak{k}$ which are algebraically independent over $\mathbf{R}$ and so can be completed to form a dense transcendence base.

The following result was first obtained by a different method by N. J. Block (thesis, University of Rochester, 1965, but otherwise unpublished; see also [5, p. 145]). The author wishes to thank the referee for drawing his attention to this reference.

THEOREM 4.7. *Let $\mathfrak{k}$, $\mathfrak{l}$ be real-closed η_1-fields containing* $\mathbf{R}$ *and of cardinality $\aleph_1$. Then there is an isomorphism u of $\mathfrak{k}$ onto $\mathfrak{l}$ with $u|\mathbf{R}$ = identity.*

PROOF (cf. [4, Theorem 13.13]). Let ω_1 be the first uncountable ordinal, S, T dense transcendence bases of $\mathfrak{k}$, $\mathfrak{l}$ over $\mathbf{R}$. Index S, T by the countable ordinals so $S = \{s_\alpha; \alpha < \omega_1\}$, $T = \{t_\alpha; \alpha < \omega_1\}$. We shall define for each $\alpha < \omega_1$, by transfinite induction, countable full algebraically closed subfields $\mathfrak{k}_\alpha$, $\mathfrak{l}_\alpha$ of $\mathfrak{k}$, $\mathfrak{l}$ and an isomorphism u_α of $\mathfrak{k}_\alpha$ onto $\mathfrak{l}_\alpha$ with u_α = identity on $\mathfrak{k}_\alpha \cap \mathbf{R}$ and, for $\beta < \alpha < \omega_1$, $s_\beta \in \mathfrak{k}_\alpha$, $t_\beta \in \mathfrak{l}_\alpha$, $\mathfrak{k}_\beta \subseteq \mathfrak{k}_\alpha$, $\mathfrak{l}_\beta \subseteq \mathfrak{l}_\alpha$, $u_\alpha|\mathfrak{k}_\beta = u_\beta$. We start the induction by taking $\mathfrak{k}_0 = \mathfrak{l}_0$ = algebraic numbers, u_0 = identity. If $\alpha < \omega_1$ is a limit ordinal and $\mathfrak{k}_\beta$, $\mathfrak{l}_\beta$, u_β have been defined for $\beta < \alpha$ we put $\mathfrak{k}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{k}_\beta$, $\mathfrak{l}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{l}_\beta$ and take u_α to be the unique map $\mathfrak{k}_\alpha \longrightarrow \mathfrak{l}_\alpha$ with $u_\alpha|\mathfrak{k}_\beta = u_\beta$ for $\beta < \alpha$.

Suppose $\alpha = \beta + 1$ and $\mathfrak{k}_\gamma$, $\mathfrak{l}_\gamma$, u_γ have been defined for $\gamma \leqslant \beta$. First of all we shall extend u_β to $\mathfrak{k}'$, the smallest full algebraically closed subfield of $\mathfrak{k}$ containing $\mathfrak{k}_\beta$ and s_β. If $s_\beta \in \mathfrak{k}_\beta$ then $\mathfrak{k}' = \mathfrak{k}_\beta$ so we put $u' = u_\beta$. Suppose $s_\beta \notin \mathfrak{k}_\beta$ and let $r_1, r_2, \ldots,$ be a transcendence base of $\mathfrak{k}' \cap \mathbf{R}$ over $\mathfrak{k}_\beta \cap \mathbf{R}$ (the base is countable because $\mathfrak{k}'$ is countable, the base might be finite or even void). We shall put $\mathfrak{k}^n = \overline{\text{alg}}\, K[\mathfrak{k}_\beta, r_1, \ldots, r_n]$, $\mathfrak{l}^n = \overline{\text{alg}}\, K[\mathfrak{l}_\beta, r_1, \ldots, r_n]$ and define bijective isomorphisms u^n; $\mathfrak{k}^n \longrightarrow \mathfrak{l}^n$ with $u^0 = u_\beta$, $u^{n+1}|\mathfrak{k}^n = u^n$, u^n = identity on $\mathfrak{k}'' \cap \mathbf{R}$, $u^n(r_n) = r_n$ by ordinary induction. Suppose u^n has been defined. $\mathfrak{k}^n$ and $\mathfrak{l}^n$ are full (Lemmas 4.2 and 4.4) so, by Lemma 4.5, $\mathfrak{k}^n \cap \mathbf{R} = u^n(\mathfrak{k}^n \cap \mathbf{R}) = \mathfrak{l}^n \cap \mathbf{R}$. As r_{n+1} is transcendental over $\mathfrak{k}^n$ it is transcendental over $\mathfrak{k}^n \cap \mathbf{R} = \mathfrak{l}^n \cap \mathbf{R}$ and the proof of Lemma 4.4 with $z = r_{n+1} = f(z)$ shows that r_{n+1} is transcendental over $\mathfrak{l}^n$. Thus u^n has a unique extension to an isomorphism v of $K[\mathfrak{k}^n, r_{n+1}]$ onto $K[\mathfrak{l}^n, r_{n+1}]$ with $v(r_{n+1}) = r_{n+1}$. If $x \in \mathfrak{k}^{n\#}$, $x > r_{n+1}$ then $f(x) \geqslant r_{n+1}$. As $\mathfrak{k}^n$ is full, $f(x) \in \mathfrak{k}^n$ whereas $r_{n+1} \notin \mathfrak{k}^n$ so $f(x) > r_{n+1}$. Thus there is $q \in \mathbf{Q}$ with $f(x) > q > r_{n+1}$ and so $x > q > r_{n+1}$. This conclusion holds trivially if $x \in \mathfrak{k}^n \setminus \mathfrak{k}^{n\#}$ and $x > r_{n+1}$ so if $x \in \mathfrak{k}^n$, $x > r_{n+1}$ then $u^n(x) > u^n(q) = q > r_{n+1}$. Similarly if $x < r_{n+1}$ then $u^n(x) < r_{n+1}$ showing that v is order preserving on $\mathfrak{k}^n \cup \{r_{n+1}\}$ and hence on $K[\mathfrak{k}^n, r_{n+1}]$ [4, Lemma 13.12]. By the Artin-Schreier theorem [7, p. 285] v has an extension u^{n+1} which maps $\mathfrak{k}^{n+1}$, the real-closure of $K[\mathfrak{k}^n, r_{n+1}]$ onto $\mathfrak{l}^{n+1}$, the real-closure of $K[\mathfrak{l}^n, r_{n+1}]$. The proof of Lemma 4.2 shows that if $a \in K[\mathfrak{k}^n, r_{n+1}] \cap \mathbf{R}$ then $a \in K[\mathfrak{k}^n \cap \mathbf{R}, r_{n+1}] = \mathfrak{r}$ so $v(a) = a$. If $a \in \mathfrak{k}^{n+1} \cap \mathbf{R}$ then the proof of Lemma 4.3 shows that $a \in \overline{\text{alg}}\, \mathfrak{r}$. Both u^{n+1} and

the identity are order preserving isomorphisms alg $\mathfrak{c} \longrightarrow \mathfrak{l}$ which agree on $\mathfrak{r}$ so by [7, Corollary, p. 172] they agree on alg $\mathfrak{r}$. Thus u^{n+1} = identity on $\mathfrak{k}^{n+1} \cap \mathbf{R}$.

The u^n give an extension $\tilde{u}$ of u_β to $\tilde{\mathfrak{k}} = \bigcup \mathfrak{k}^n = \overline{\text{alg}}\, K[\mathfrak{k}_\beta, r_1, \ldots]$. If $s_\beta \in \tilde{\mathfrak{k}}$ then $\mathfrak{k}' = \tilde{\mathfrak{k}}$ and $u' = \tilde{u}$. If $s_\beta \notin \tilde{\mathfrak{k}}$ then it is transcendental over $\tilde{\mathfrak{k}}$. Now consider the sets

$$A = \{a; a \in \tilde{\mathfrak{k}}, a < s_\beta\} \qquad B = \{a; a \in \tilde{\mathfrak{k}}, a > s_\beta\}.$$

Because $\tilde{\mathfrak{k}}$ is algebraically closed in $\mathfrak{k}$ it is real-closed, hence $\tilde{\mathfrak{l}} = \tilde{u}(\tilde{\mathfrak{k}})$ is real-closed and so algebraically closed in $\mathfrak{l}$. Put

$$A' = \tilde{u}(A) \cup \{t; t \in T \cap \tilde{\mathfrak{l}}, t \leqslant \tilde{u}(a) \text{ for some } a \in A\},$$

$$B' = \tilde{u}(B) \cup \{t; t \in T \cap \tilde{\mathfrak{l}}, t > \tilde{u}(a) \text{ for all } a \in A\}.$$

Then A', B' are countable sets. Let $t \in T$ with $A' < t < B'$. Since $t \notin \tilde{\mathfrak{l}}$, t is transcendental over $\tilde{\mathfrak{l}}$ and $\tilde{u}$ extends to an isomorphism u' of $K[\tilde{\mathfrak{k}}, s_\beta]$ onto $K[\tilde{\mathfrak{l}}, t]$ with $u'(s_\beta) = t$. u' is clearly order preserving on $\tilde{\mathfrak{k}} \cup \{s_\beta\}$ so by [4, Lemma 13.12] on $K[\tilde{\mathfrak{k}}, s_\beta]$. Again using the Artin-Schreier theorem we extend u' to the algebraic closure of $K[\tilde{\mathfrak{k}}, s_\beta]$ in $\mathfrak{k}$, which is $\mathfrak{k}'$. If $a \in \mathfrak{k}' \cap \mathbf{R}$ then a is algebraic over $K[\mathfrak{k}_\beta \cap \mathbf{R}, r_1, r_2, \ldots]$ so $a \in \tilde{\mathfrak{k}}$ and $u'(a) = u(a) = a$. Put $\mathfrak{l}' = u'(\mathfrak{k}')$. By the argument used above to show $\mathfrak{k}^n \cap \mathbf{R} = \mathfrak{l}^n \cap \mathbf{R}$ we see $\mathfrak{k}' \cap \mathbf{R} = \mathfrak{l}' \cap \mathbf{R}$. By Lemma 4.1 $gu' = f$ so $\mathfrak{l}' \cap \mathbf{R} = \mathfrak{k}' \cap \mathbf{R} = f(\mathfrak{k}'^{\#}) = g(\mathfrak{l}'^{\#})$ and $\mathfrak{l}'$ is full. $\mathfrak{l}'$ is algebraically closed for the same reason that $\tilde{\mathfrak{l}}$ was.

We now repeat the whole procedure with $\mathfrak{k}_\beta$, $\mathfrak{l}_\beta$, s_β, u_β replaced by $\mathfrak{l}'$, $\mathfrak{k}'$, t_β, $(u')^{-1}$ and take u_α to be the inverse of the resulting map. This completes the inductive step.

We define $u: \bigcup_{\alpha<\omega_1} \mathfrak{k}_\alpha \longrightarrow \bigcup_{\alpha<\omega_1} \mathfrak{l}_\alpha$ by $u|\mathfrak{k}_\alpha = u_\alpha$. $\bigcup_{\alpha<\omega_1} \mathfrak{k}_\alpha$ is full and contains S. Since S is order dense, for each $a \in \mathbf{R}$ there is $s \in S \cap \mathfrak{k}^{\#}$ with $f(s) = a$. Thus $\bigcup_{\alpha<\omega_1} \mathfrak{k}_\alpha$ contains $\mathbf{R}$. Since it is also algebraically closed in $\mathfrak{k}$ it is $\mathfrak{k}$. Similarly $\mathfrak{l} = \bigcup_{\alpha<\omega_1} \mathfrak{l}_\alpha$.

COROLLARY 4.8. *Let $\mathfrak{k}$ be an ordered field containing* $\mathbf{R}$ *and of cardinality* $\aleph_1$. *Let $\mathfrak{l}$ be a real-closed* η_1-field *containing* $\mathbf{R}$. *Then there is an order preserving isomorphism u of $\mathfrak{k}$ into $\mathfrak{l}$ with $u|\mathbf{R}$ = identity.*

PROOF. Remarks similar to those at the end of §§13.9 and 13.5 of [4] apply. By the Artin-Schreier theorem we can assume that $\mathfrak{k}$ is real-closed. In the proof we take S to be any transcendence base for $\mathfrak{k}$ over $\mathbf{R}$ and when $\alpha = \beta + 1$ we take $\mathfrak{k}^\alpha = \mathfrak{k}'$ and the induction proceeds as before except that if S is countable, $\bigcup_{\alpha<\omega_1} \mathfrak{k}_\alpha$ is replaced by $\bigcup_{n=1}^{\infty} \mathfrak{k}_n$. For the final extension to $\mathfrak{k}$ we use the same procedure as was used to extend from $\mathfrak{k}_\beta$ to $\mathfrak{k}'$, applied transfinitely if necessary.

COROLLARY 4.9. *If* $\mathfrak{j}(V_0)$ *has an algebra norm for one free ultrafilter* V_0 *on* N *then, under the continuum hypothesis,* $\mathfrak{j}(V^*)$ *has an algebra norm for every free ultrafilter* V *on* N.

PROOF. $\mathfrak{j}(V) = {}^*\mathbf{R}(V)^0$ and ${}^*\mathbf{R}(V)$ is a real-closed η_1-field of cardinality $2^{\aleph_0}$ containing **R**.

COROLLARY 4.10. *If* V *is a free ultrafilter on* N *and* $\mathfrak{j}(V)$ *has a nontrivial algebra seminorm then, under the continuum hypothesis, it has an algebra norm* (*cf.* [9, §4]).

PROOF. Let $\mathfrak{k} = K[{}^*\mathbf{R}(V), X]$, the algebra of rational expressions in the indeterminate X with coefficients from ${}^*\mathbf{R}(V)$. We order the polynomial ring lexicographically, $P \geqslant 0$ if the term of lowest degree with nonvanishing coefficient has a positive coefficient, and take the induced order on the quotient field $\mathfrak{k}$. $\mathfrak{k}$ is a totally ordered field containing ${}^*\mathbf{R}(V)$ as the polynomials of zero degree, and hence containing **R**. The cardinality of $\mathfrak{k}$ is $2^{\aleph_0}$. Put $\mathfrak{l} = {}^*\mathbf{R}(V)$ and let p be a nontrivial seminorm on $\mathfrak{l}^0$. Suppose $t_0 \in \mathfrak{l}^0$ with $p(t_0) \neq 0$ and $t_0 > 0$. In Corollary 4.8 we can take $s_0 = X$. $K[\mathfrak{k}_0, s_0]$ is the field of rational functions with coefficients from the field of algebraic numbers. Any nonzero element z can be written in the form

$$\frac{X^m(a_0 + a_1 X + \cdots + a_k X^k)}{(b_0 + b_1 X + \cdots + b_l X^l)}$$

with $a_i, b_i \in \mathfrak{k}_0$, $a_0 \neq 0$, $b_0 \neq 0$, $m \in \mathbf{Z}$, $z \in \mathfrak{k}^{\#}$ if and only if $m \geqslant 0$ and if $m = 0$, $f(z) = a_0/b_0 \in \mathfrak{k}_0$, if $m > 0$, $f(z) = 0 \in \mathfrak{k}_0$. Thus $K[\mathfrak{k}_0, s_0]$ is full and hence, by Lemma 4.3, $\mathfrak{k}_1 = \overline{\text{alg}}\, K[\mathfrak{k}_0, s_0]$. If $z \in \mathfrak{k}_1 \cap \mathbf{R}$ then the proof of 4.3 shows that $z = f(z)$ is algebraic over $f(K[\mathfrak{k}_0, s_0]^{\#})$ which we have just shown to be $\mathfrak{k}_0$. Thus $\mathfrak{k}_1 \cap \mathbf{R} = \mathfrak{k}_0 \cap \mathbf{R}$ and $\mathfrak{k}_0 = \tilde{\mathfrak{k}}$. The sets A, B are the set of nonpositive algebraic numbers and the set of positive algebraic numbers respectively, also $A = A'$, $B = B'$. Thus we may begin our induction by taking $u_1(s_0) = t_0$, that is, there is an order preserving isomorphism u; $\mathfrak{k} \longrightarrow {}^*\mathbf{R}(V)$ with u = identity on **R**, $u(X) = t_0$.

We show that $p \circ u|\mathfrak{j}$ is really a norm. Suppose $a \in \mathfrak{j}$, $a > 0$. As $X < qa$ for all $q \in \mathbf{Q}^+$, we have $a^{-1}X \in \mathfrak{j}$. Thus $0 < p(t_0) \leqslant p(u(a))p(u(a^{-1}X))$ showing $p(u(a)) \neq 0$. A similar argument applies if $a < 0$.

COROLLARY 4.11. *Let* V *be a free ultrafilter on* N. *Under the continuum hypothesis, if* $\mathfrak{j}_0(V)$ *has a nontrivial algebra seminorm* p *then so does* $\mathfrak{j}(V)$.

PROOF. Let $\mathfrak{k} = K[{}^*\mathbf{R}(V), X, Y]$, $\mathfrak{l} = {}^*\mathbf{R}(V)$ where X, Y are indeterminates. First of all we put an order on the ring of polynomials in Y with coefficients from ${}^*\mathbf{R}$. We say $P > 0$ if $P = c(a_0 + a_1 Y + \cdots + a_k Y^k)$ where $c \in$

${}^*\mathbf{R}$, $c > 0$, $a_i \in {}^*\mathbf{R}^{\#}$ for all i, $a_i \notin {}^*\mathbf{R}^0$ for some i and the first coefficient not in ${}^*\mathbf{R}^0$ is positive. This gives an order on the quotient field $K[{}^*\mathbf{R}, Y]$ and we order $\mathfrak{k} = K[K[{}^*\mathbf{R}, Y], X]$ lexicographically as in 4.10.

Let $t \in \mathfrak{j}_0$ with $t > 0$, $p(t) > 0$. There is $w \in \mathfrak{j}_0$ with $w > t^{1/n}$ for all $n \in \mathbf{N}$ because for each sequence $\{a_n\}$ of elements from c_0 there is $b \in c_0$ with $b_j > a_{nj}$ for all but a finite number of values of j. Define $u_0 | \mathbf{A} =$ identity, $u_0(X) = t$, $u_0(Y) = w$ we have an isomorphism of $K[\mathbf{A}, X, Y]$ into $\mathfrak{l} = {}^*\mathbf{R}$ where $\mathbf{A}$ is the set of algebraic numbers. As u_0 is order preserving it extends to $\mathfrak{k}_0 = \overline{\mathrm{alg}}\, K[\mathbf{A}, X, Y]$. $\mathfrak{k}_0$ is full. Applying the induction in 4.7 and 4.8 starting with this definition of $\mathfrak{k}_0$, u_0 we extend u_0 to an order preserving homomorphism u; $\mathfrak{k} \longrightarrow \mathfrak{l}$. If $a \in \mathfrak{j}$ we have $-Y < a < Y$ in $\mathfrak{k}$ so $-w < u(a) < w$ which implies $u_{\mathfrak{j}} \subseteq \mathfrak{j}_0$. Thus $pu|\mathfrak{j}$ is a seminorm on $\mathfrak{j}$ and an argument similar to that at the end of 4.10 shows that it is a norm.

5. **Norming algebras of power series.** Although they involve the continuum hypothesis, the results of the previous section suggest the problem of norming $\mathfrak{k}^{\#}$ for any ordered field $\mathfrak{k}$ containing $\mathbf{R}$. One such algebra, the algebra of formal power series is one variable, has been normed [1]. In this section we extend these results. The field $\mathfrak{L}(X, \mathfrak{k})$ of formal Laurent series in X with coefficients from a field $\mathfrak{k}$ is the set of all expressions $\Sigma_{n=j}^{\infty} a_n X^n$ where $j \in \mathbf{Z}$, $a_n \in \mathfrak{k}$. If $i < j$ and $a_n = 0$, $i \leqslant n < j$ then $\Sigma_{n=i}^{\infty} a_n X^n$ and $\Sigma_{n=j}^{\infty} a_n X^n$ are identified. We define

$$\sum_{n=j}^{\infty} a_n X^n + \sum_{n=j}^{\infty} b_n X^n = \sum_{n=j}^{\infty} (a_n + b_n) X^n,$$

$$\sum_{m=j}^{\infty} a_m X^m \sum_{n=k}^{\infty} b_n X^n = \sum_{p=j+k}^{\infty} \left(\sum_{m+n=p} a_m b_n \right) X^p.$$

With these operations $\mathfrak{L}(X)$ is a field containing a copy of $\mathfrak{k}$, the series with $a_n = 0$ for $n \neq 0$, which we identify with $\mathfrak{k}$. If $\mathfrak{k}$ is an ordered field $\mathfrak{L}(X)$ is also an ordered field under the lexicographic ordering. If $\mathfrak{k}$ is ordered and contains $\mathbf{R}$ we put $\mathfrak{F}(X) = \{\Sigma_{i=0}^{\infty} a_i X^i; a_i \in \mathfrak{k}, a_0 \in \mathfrak{k}^{\#}\}$, $\mathfrak{F}(X)$ is an algebra over $\mathbf{R}$ and $\mathfrak{F}(X) = \mathfrak{L}(X)^{\#}$.

We now define ordered fields $\mathfrak{L}_\alpha$ for each ordinal α, corresponding algebras $\mathfrak{F}_\alpha$ and projections $p_{\alpha\beta}$; $\mathfrak{F}_\alpha \longrightarrow \mathfrak{F}_\beta$ for $\alpha \geqslant \beta$. Put $\mathfrak{F}_0 = \mathbf{R} = \mathfrak{L}_0$, $\mathfrak{L}_{\alpha+1} = \mathfrak{L}(X_\alpha, \mathfrak{L}_\alpha)$, $\mathfrak{F}_{\alpha+1} = \mathfrak{F}(X_\alpha, \mathfrak{L}_\alpha)$ where for each ordinal α, X_α is an indeterminate. We define $p_{\alpha+1,\alpha+1} =$ identity on $\mathfrak{F}_{\alpha+1}$, $p_{\alpha+1,\alpha}(\Sigma_{i=0}^{\infty} a_i X_\alpha^i) = a_0$, $p_{\alpha+1,\beta} = p_{\alpha\beta} p_{\alpha+1,\alpha}$ for $\alpha > \beta$. For a limit ordinal α, $\mathfrak{F}_\alpha$ is the projective limit of the system $\{\mathfrak{F}_\beta; p_{\beta\gamma}, \alpha > \beta > \gamma\}$, an integral domain because each $\mathfrak{F}_\beta$ is, by transfinite induction, and $\mathfrak{L}_\alpha$ is the quotient field of $\mathfrak{F}_\alpha$. If $\alpha > \beta$, $p_{\alpha\beta}$ is the projection of $\mathfrak{F}_\alpha$ onto the $\mathfrak{F}_\beta$ coordinate. $p_{\alpha\alpha}$ is the identity on $\mathfrak{F}_\alpha$.

For $\alpha > \beta$ we define injections $i_{\beta\alpha}$ of $\mathfrak{F}_\beta$ into $\mathfrak{F}_\alpha$ by transfinite induction; $i_{\alpha,\alpha+1}$ is the injection $\mathfrak{L}_\alpha \subsetneq \mathfrak{L}(X_\alpha, \mathfrak{L}_\alpha)$ restricted to $\mathfrak{F}_\alpha$, $i_{\beta,\alpha+1} = i_{\alpha,\alpha+1} i_{\beta,\alpha}$ and for α a limit ordinal, $\beta < \alpha$, $A \in \mathfrak{F}_\beta$, $i_{\beta\alpha}(A)$ is the element B of $\mathfrak{F}_\alpha$ with $p_{\alpha\gamma}(B) = i_{\beta\gamma}(A)$ if $\alpha > \gamma > \beta$, $p_{\alpha\gamma}(B) = p_{\beta\gamma}(A)$ if $\beta \geqslant \gamma$. We have $p_{\alpha\beta} i_{\beta\alpha} =$ identity on $\mathfrak{F}_\beta$ for $\beta < \alpha$. Since $\mathfrak{L}_\alpha$ is the quotient field of $\mathfrak{F}_\alpha$ for all α, $i_{\beta\alpha}$ extends to an isomorphsim $\mathfrak{L}_\beta \longrightarrow \mathfrak{L}_\alpha$. We shall identify $\mathfrak{L}_\beta$ with its image under this map so that the $\mathfrak{L}_\alpha$ are an expanding system of fields. We put $\mathfrak{L} = \bigcup_{\alpha<\omega_1} \mathfrak{L}_\alpha$, $\mathfrak{F} = \bigcup_{\alpha<\omega_1} \mathfrak{F}_\alpha$ where ω_1 is the first uncountable ordinal.

Denote the complexification of $\mathfrak{L}_\alpha, \mathfrak{F}_\alpha, \mathfrak{L}, \mathfrak{F}$ by $\mathfrak{K}_\alpha, \mathfrak{C}_\alpha, \mathfrak{K}, \mathfrak{C}$. These can be defined in the same way as the $\mathfrak{F}$'s and $\mathfrak{L}$'s but starting from $\mathbf{C}$ in place of $\mathbf{R}$. They are not totally ordered. We are going to map these algebras into a complex Banach algebra $\mathfrak{A}$ and it seems more natural to work with the complex algebras $\mathfrak{C}_\alpha$ throughout. We shall use $p_{\alpha\beta}$ to denote the map $\mathfrak{C}_\alpha \longrightarrow \mathfrak{C}_\beta$ obtained by complexifying the original projections.

The term *monomial* is defined by transfinite induction. 1 is a monomial. An element of $\mathfrak{K}_{\alpha+1}$ is a monomial if it is of the form aX_α^n where a is a monomial in $\mathfrak{K}_\alpha$ and $n \in Z$. An element of $\mathfrak{K}_\alpha$ where α is a limit ordinal is a monomial if it is a monomial in $\mathfrak{K}_\beta$ for some $\beta < \alpha$. Every monomial can be written $X_{\alpha_1}^{n_1} X_{\alpha_2}^{n_2} \cdots X_{\alpha_k}^{n_k}$ where the n_i are integers.

LEMMA 5.1. *Let α be a countable ordinal and let $A \in \mathfrak{C}_\alpha$. Then A is regular in $\mathfrak{C}_\alpha$ if and only if $p_{\alpha 0}(A) \neq 0$.*

PROOF. As $p_{\alpha 0}$ is a unital homomorphism, if $p_{\alpha 0}(A) = 0$ then A is not regular. We prove the converse implication by induction on α. It is obvious for $\alpha = 0$. If it is true for some value of α and $A \in \mathfrak{C}_{\alpha+1}$ with $p_{\alpha+1,0}(A) \neq 0$ then $A = \Sigma_{i=0}^\infty a_i X_\alpha^i$ where $a_0 \in \mathfrak{C}_\alpha$ and $p_{\alpha,0}(a_0) = p_{\alpha,0} p_{\alpha+1,\alpha}(A) = p_{\alpha+1,0}(A) \neq 0$. Thus a_0 is regular in $\mathfrak{C}_\alpha$ and so the inverse of A in $\mathfrak{K}_{\alpha+1}$ is an element of $\mathfrak{C}_{\alpha+1}$. If α is a limit ordinal and the result is true for all $\beta < \alpha$, let $A \in \mathfrak{C}_\alpha$. For $\beta < \alpha$, $p_{\beta 0} p_{\alpha\beta} A = p_{\alpha 0} A \neq 0$ so $(p_{\alpha\beta} A)^{-1}$ exists in $\mathfrak{C}_\beta$. The $(p_{\alpha\beta} A)^{-1}$ form the inverse of A in $\mathfrak{C}_\alpha$.

LEMMA 5.2. *Let α be a countable ordinal and let $A \in \mathfrak{C}_\alpha$, $A \neq 0$. Then there is a monomial M in $\mathfrak{C}_\alpha$ and a regular element B in $\mathfrak{C}_\alpha$ with $A = MB$.*

PROOF. The result is obvious for $\alpha = 0$. Suppose it holds for all $\beta < \alpha$ and let γ be the first value of β with $p_{\alpha\beta}(A) \neq 0$. γ is not a limit ordinal so $p_{\alpha\gamma}(A) = \Sigma_{i=0}^\infty a_i X_{\gamma-1}^i$. Let j be the first value of i with $a_j \neq 0$. j is not 0. Since $\mathfrak{K}_{\gamma-1}$ is the quotient field of $\mathfrak{C}_{\gamma-1}$ we have $a_j = M'B'$ where B' is regular in $\mathfrak{C}_{\gamma-1}$ and M' is a monomial in $\mathfrak{K}_{\gamma-1}$. Thus $p_{\alpha\gamma}(A) = MB''$ where $M =$

$M'X^j_{\gamma-1}$ is a monomial in $\mathfrak{E}_\gamma$ and $B'' = B' + \Sigma_{i=j+1}^{\infty} M'^{-1}a_i X^{i-j}_{\gamma-1}$ is regular in $\mathfrak{E}_\gamma$ because $p_{\gamma 0}B'' = p_{\gamma-1,0}B'$.

We now show that for $\gamma \leqslant \beta \leqslant \alpha$ there is a regular element B_β in $\mathfrak{E}_\beta$ with $p_{\alpha\beta}(A) = MB_\beta$. We have shown this for $\beta = \gamma$. If it holds for one value of β less than α then $p_{\alpha\beta+1}(A) = p_{\alpha\beta}(A) + \Sigma_{n=1}^{\infty} b_n X_\beta^n = MB_\beta + \Sigma_{n=1}^{\infty} b_n X_\beta^n = MB_{\beta+1}$ where $B_{\beta+1} = B_\beta + \Sigma_{n=1}^{\infty} M^{-1}b_n X_\beta^n$ is invertible in $\mathfrak{E}_{\beta+1}$. If δ is a limit ordinal with $\gamma < \delta \leqslant \alpha$ and the result holds for all β with $\gamma \leqslant \beta < \delta$ then $Mp_{\beta\beta'}(B_\beta) = p_{\beta\beta'}(MB_\beta) = MB_{\beta'}$ where $\gamma \leqslant \beta' \leqslant \beta < \delta$ so, since $\mathfrak{E}_{\beta'}$ is an integral domain, $p_{\beta\beta'}(B_\beta) = B_{\beta'}$ and the B_β define a regular element B_δ of $\mathfrak{E}_\delta$ with $MB_\delta = p_{\alpha\delta}(A)$.

Since $\mathfrak{K}_\alpha$ is the quotient field of $\mathfrak{E}_\alpha$ m if A is a nonzero element of $\mathfrak{K}_\alpha$ then $A = MB$ where M is a monomial in $\mathfrak{K}_\alpha$ and B is regular in $\mathfrak{E}_\alpha$.

$\mathfrak{A}$ is the algebra $L^1(0, 1)$ with convolution multiplication and $\mathfrak{A}^1$ the algebra obtained by adjoining a unit to $\mathfrak{A}$. Every element of $\mathfrak{A}$ is a topological nilpotent so $\mathfrak{A} =$ Radical $\mathfrak{A}^1$. An element a of $\mathfrak{A}$ is a zero divisor if and only if $\int_0^\epsilon |a(t)|dt = 0$ for some $\epsilon > 0$. It is easy to see from this that a is a zero divisor if and only if it is properly nilpotent. If a is not a zero divisor then $\overline{a\mathfrak{A}} = \mathfrak{A}$ and conversely. The nontrivial parts of these results follow from the Titchmarsh convolution theorem by methods similar to those of Donoghue [3]. We now select elements x_α of $\mathfrak{A}$ which will be the images of the elements X_α of $\mathfrak{E}$.

LEMMA 5.3. *There is a system* $\{x_\alpha; \alpha < \omega_1\}$ *of elements of* $\mathfrak{A}$ *indexed by the countable ordinals with* $\overline{x_\alpha \mathfrak{A}} = \mathfrak{A}$ *and* $x_\alpha \mathfrak{A}^1 \subseteq x_\beta^n \mathfrak{A}$ *for* $\alpha > \beta, n \in \mathbf{Z}^+$.

PROOF. We proceed by transfinite induction. Let x_0 be any nonzero-divisor in $\mathfrak{A}$. Suppose x_α has been defined.

As $[x_\alpha x_\alpha^n \mathfrak{A}]^- = [x_\alpha [x_\alpha^n \mathfrak{A}]^-]^-$ we see $(x_\alpha^n \mathfrak{A})^- = \mathfrak{A}$ for all $n \in \mathbf{Z}^+$. Put $I_{\alpha+1} = \bigcap_{n=1}^{\infty} x_\alpha^n \mathfrak{A}$. Application of [8, Lemma 3.1] with R_n as left multiplication by x_α shows $I_{\alpha+1}$ is dense in $\mathfrak{A}$. On $I_{\alpha+1}$ we define $\|x\|_n = \|y\|$ where y is the element of $\mathfrak{A}$ with $x_\alpha^n y = x$. y is unique because x_α^n is not a zero divisor. We put $\|x\|_0 = \|x\|$. This sequence of norms makes $I_{\alpha+1}$ a complete metric space. The set $E_n = \{x; x \in \mathfrak{A}, x = 0 \text{ on } [0, n^{-1}]\}$ is a closed linear subspace in $\mathfrak{A}$ which is not dense in $\mathfrak{A}$. Hence $I_{\alpha+1} \cap E_n$ is a closed proper subspace of $I_{\alpha+1}$ and, in particular, is nowhere dense in $I_{\alpha+1}$. Thus, by the category theorem, $\bigcup_{n=1}^{\infty} (I_{\alpha+1} \cap E_n) \neq I_{\alpha+1}$ so $I_{\alpha+1}$ contains an element $x_{\alpha+1}$ which is not a zero divisor in $\mathfrak{A}$.

If α is a limit ordinal we make a similar construction. Writing the ordinals $< \alpha$ in a sequence $\{\beta_n\}$ and putting $I_\alpha = \bigcap_{\beta<\alpha} x_\beta \mathfrak{A} \supseteq \bigcap_{n=1}^{\infty} (x_{\beta_1} \cdots x_{\beta_n})\mathfrak{A}$ which is dense in $\mathfrak{A}$ by [8, Lemma 3.1] because $x_{\beta_1} \cdots x_{\beta_n} \mathfrak{A}$ is, we define norms $\| \ \|_\beta$ on I_α by $\|x\|_\beta = \|y\|$ where $x = x_\beta y$. Again I_α is a complete metric space

and so contains an element x_α which is not a zero divisor.

COROLLARY 5.4. *If $\beta < \alpha$ then $y \mapsto x_\beta y$ maps I_α one to one onto itself.*

PROOF. Since the I_α are ideals in $\mathfrak{A}$, $y \mapsto x_\beta y$ maps I_α into I_α. Because x_β is not a zero divisor the map is one to one. Suppose α is not a limit ordinal and $x \in I_\alpha$. Then for each $n \in \mathbf{Z}^+$ there is $y_n \in \mathfrak{A}$ with $x = x_{\alpha-1}^n y_n$. As there is $z \in \mathfrak{A}^1$ with $x_\beta z = x_{\alpha-1}$ we have $x = x_\beta x_{\alpha-1}^{n-1} y_n z = x_\beta y_1 z$. Thus, because x_β is not a zero divisor, $y_1 z = x_{\alpha-1}^{n-1} y_n z$, $n = 2, 3, \ldots$, so $y_1 z \in I_\alpha$. Suppose α is a limit ordinal and $x \in I_\alpha$. Then $x = x_\beta w$ for some $w \in \mathfrak{A}$. Also for each γ with $\alpha > \gamma > \beta$ there are $y_\gamma, z_\gamma \in \mathfrak{A}$ with $x = x_\gamma^2 y_\gamma$, $x_\gamma = x_\beta z_\gamma$. Thus $x_\beta w = x_\beta x_\gamma z_\gamma y_\gamma$ so that $w = x_\gamma y_\gamma z_\gamma$ for all γ with $\alpha > \gamma > \beta$. This shows $w \in I_\alpha$.

LEMMA 5.5. *Let α be a countable ordinal and θ_α an algebra isomorphism $\mathfrak{E}_\alpha \to \mathfrak{A}^1$ with $\theta_\alpha(X_\beta) = x_\beta$ for $\beta < \alpha$. Let π denote the quotient map $\mathfrak{A}^1 \to \mathfrak{A}^1/I_{\alpha+1}$. Then $\pi\theta_\alpha$ has an extension ψ to an algebra homomorphism $\mathfrak{E}_{\alpha+1} \to \mathfrak{A}^1/I_{\alpha+1}$ with $\psi(X_\alpha) = \pi(x_\alpha)$.*

PROOF. Let $a \in \mathfrak{K}_\alpha$. By Lemma 5.2 we have $a = X_{\beta_1}^{-1} \cdots X_{\beta_n}^{-1} B$ where $\beta_i < \alpha$ and $B \in \mathfrak{E}_\alpha$. If β is the largest of the β_i then $x_\alpha \mathfrak{A}^1 \subseteq x_\beta^n \mathfrak{A} \subseteq x_{\beta_1} \cdots x_{\beta_n} \mathfrak{A}$ so there is an element $\overline{\theta}(aX_\alpha)$ of $\mathfrak{A}$ with $x_{\beta_1} \cdots x_{\beta_n} \overline{\theta}(aX_\alpha) = x_\alpha \theta_\alpha(B)$. $\overline{\theta}(xX_\alpha)$ is independent of the choice of $\beta_1, \ldots, \beta_n$.

We have

$$\overline{\theta}(aX_\alpha) + \overline{\theta}(bX_\alpha) = \overline{\theta}((a+b)X_\alpha),$$

$$\overline{\theta}(aX_\alpha)\overline{\theta}(bX_\alpha) = \overline{\theta}(abX_\alpha)x_\alpha,$$

$$\overline{\theta}(aX_\alpha)\theta_\alpha(c) = \overline{\theta}(acX_\alpha), \quad a, b \in \mathfrak{K}_\alpha, c \in \mathfrak{E}_\alpha$$

from which if we define

$$\overline{\theta}(a_0 + a_1 X_\alpha + \cdots + a_m X_\alpha^m) = \theta_\alpha(a_0) + \overline{\theta}(a_1 X_\alpha) + \cdots + \overline{\theta}(a_m X_\alpha) x_\alpha^{m-1}$$

where $a_0 \in \mathfrak{E}_\alpha, a_1, \ldots, a_m \in \mathfrak{L}_\alpha$ then we have a homomorphism $\overline{\theta}$ from the polynomials in $\mathfrak{E}_{\alpha+1}$ into $\mathfrak{A}^1$.

Let $A \in \mathfrak{E}_{\alpha+1}$, $A = \Sigma_{i=0}^\infty a_i X_\alpha^i$ where $a_0 \in \mathfrak{E}_\alpha$, $a_1, a_2, \ldots \in \mathfrak{K}_\alpha$, $\psi(a)$ is the element w of $\mathfrak{A}^1/I_{\alpha+1}$ with

$$w - \pi\overline{\theta}(a_0 + a_1 X_\alpha + \cdots + a_m X_\alpha^m) \in x_\alpha^m \mathfrak{A}/I_{\alpha+1}$$

for all m in $\mathbf{Z}^+$. There is at most one element w with this property. To see that there is exactly one choose elements y_n of $\mathfrak{A}$ inductively so that

$$y_0 = \theta(a_0), \qquad \| y_n + \overline{\theta}(a_{n+1} X_\alpha) - y_{n+1} x_\alpha \| < 2^{-n} \| x_\alpha \|^{-n},$$

the choice being possible because $x_\alpha \mathfrak{A}$ is dense in $\mathfrak{A}$. Put

$$z_m = \sum_{n \geq m} (y_n + \overline{\theta}(a_{n+1}X_\alpha) - y_{n+1}x_\alpha)x_\alpha^{n-m}.$$

We then have

$$z_0 = \overline{\theta}(a_0 + a_1X_\alpha + \cdots + a_mX_\alpha^m) - y_mx_\alpha^m + z_mx_\alpha^m.$$

Hence πz_0 is an element of $\mathfrak{A}^1/I_{\alpha+1}$ with $\pi z_0 - \pi\overline{\theta}(a_0 + a_1X_\alpha + \cdots + a_mX_\alpha^m) \in x_\alpha^m\mathfrak{A}/I_{\alpha+1}$ for all m. We put $\psi(A) = \pi z_0$. ψ is a map $\mathfrak{E}_{\alpha+1} \longrightarrow \mathfrak{A}^1/I_{\alpha+1}$ and is an algebra homomorphism because $\psi(A) + \psi(B)$ and $\psi(A)\psi(B)$ satisfy the defining condition for $\psi(A + B)$ and $\psi(AB)$.

COROLLARY 5.6. *Under the same hypotheses as 5.5, if $\widetilde{\mathfrak{E}}$ is a subalgebra of $\mathfrak{E}_{\alpha+1}$ containing $\mathfrak{E}_\alpha$ and X_α and $\widetilde{\theta}; \widetilde{\mathfrak{E}} \longrightarrow \mathfrak{A}^1$ is an extension of θ_α with $\widetilde{\theta}(X_\alpha) = x_\alpha$ and $\pi\widetilde{\theta} = \psi$ on $\widetilde{\mathfrak{E}}$ then, for all nonzero A in $\widetilde{\mathfrak{E}}$, $x \longmapsto \theta(A)x$ maps $I_{\alpha+1}$ one to one onto itself. $\widetilde{\theta}(A)$ is not a zero divisor in $\mathfrak{A}^1$.*

PROOF. By Lemma 5.2 we have $A = MB$ where M is a monomial and B is regular in $\mathfrak{E}_{\alpha+1}$. Let $b \in \mathfrak{A}^1$ with $\pi(b) = \psi(B)$. By Corollary 5.4 $x \longmapsto \overline{\theta}(M)x$ maps $I_{\alpha+1}$ one to one onto itself as it is the composition of maps $x \longmapsto x_\beta x$ and their inverses. We have $\pi(\widetilde{\theta}(A) - \overline{\theta}(M)b) = \psi(A) - \psi(M)\psi(B) = 0$ so $\widetilde{\theta}(A) - \overline{\theta}(M)b \in I_{\alpha+1}$. Let $j \in I_{\alpha+1}$ with $\overline{\theta}(M)j = \widetilde{\theta}(A) - \overline{\theta}(M)b$. Then $\pi(b + j) = \psi(B)$ and $\widetilde{\theta}(A) = \overline{\theta}(M)(b + j)$. Because B is regular, $\psi(B)$ is regular in $\mathfrak{A}^1/I_{\alpha+1}$ and, as $I_{\alpha+1} \subseteq \mathfrak{A} = \operatorname{Rad} \mathfrak{A}^1$, $b + j$ is regular in $\mathfrak{A}^1$. Thus $x \longmapsto (b + j)x$ maps $I_{\alpha+1}$ one to one onto itself and hence so does $\widetilde{x} \longmapsto \widetilde{\theta}(A)$. If $y \in \mathfrak{A}^1$ with $\widetilde{\theta}(A)y = 0$ then $\widetilde{\theta}(A)x_{\alpha+1}y = 0$ so $x_{\alpha+1}y = 0$ and hence $y = 0$, because $x_{\alpha+1}$ is not a zero divisor and $x_{\alpha+1} \in I_{\alpha+1}$.

LEMMA 5.7. *Let α be a countable ordinal and θ_α an algebra isomorphism $\mathfrak{E}_\alpha \longmapsto \mathfrak{A}^1$ with $\theta_\alpha(X_\beta) = x_\beta$, $\beta < \alpha$. Then θ_α has an extension $\theta_{\alpha+1}$; $\mathfrak{E}_{\alpha+1} \longrightarrow \mathfrak{A}^1$ with $\theta_{\alpha+1}(X_\beta) = x_\beta$, $\beta < \alpha + 1$. $\theta_{\alpha+1}$ is an algebra isomorphism.*

PROOF. Construct ψ and $\overline{\theta}$ as in Lemma 5.5. $\overline{\theta}$ is an extension of θ_α to the polynomials in $\mathfrak{E}_{\alpha+1}$ with $\overline{\theta}(X_\beta) = x_\beta$, $\beta < \alpha$. Let $\widetilde{\theta} : \widetilde{\mathfrak{E}} \longrightarrow \mathfrak{A}^1$ be a maximal extension of $\overline{\theta}$ with $\pi\widetilde{\theta} = \psi$ to a subalgebra $\widetilde{\mathfrak{E}}$ of $\mathfrak{E}_{\alpha+1}$. Let $P, Q \in \widetilde{\mathfrak{E}}$ with $PQ^{-1} \in \mathfrak{E}_{\alpha+1}$ and let $a \in \mathfrak{A}^1$ with $\pi(a) = \psi(PQ^{-1})$. Then $\widetilde{\theta}(P) - \widetilde{\theta}(Q)a \in \ker \pi = I_{\alpha+1}$ so, by Corollary 5.6, there is $j \in I_{\alpha+1}$ with $\widetilde{\theta}(Q)j = \widetilde{\theta}(P) - \widetilde{\theta}(Q)a$. Thus $b = a + j$ satisfies $\widetilde{\theta}(P) = \widetilde{\theta}(Q)b$ and, since $\widetilde{\theta}(Q)$ is not a zero divisor (Corollary 5.7), is the only element of $\mathfrak{A}^1$ which does. Defining $\widetilde{\theta}(PQ^{-1}) = b$ we extend $\widetilde{\theta}$ to $\mathfrak{E}_{\alpha+1} \cap \mathfrak{K}$ where $\mathfrak{K}$ is the subfield of $\mathfrak{K}_{\alpha+1}$ generated by $\widetilde{\mathfrak{E}}$. Since $\widetilde{\theta}$ is already maximal, $\widetilde{\mathfrak{E}} = \mathfrak{E}_{\alpha+1} \cap \mathfrak{K}$.

If $A \in \mathfrak{E}_{\alpha+1} \subseteq \mathfrak{K}_{\alpha+1}$ is transcendental over $\widetilde{\mathfrak{K}}$ then choose $a \in \mathfrak{A}^1$ with

$\pi a = \psi A$ and extend $\widetilde{\theta}$ to the subalgebra of $\mathfrak{E}_{\alpha+1}$ generated by $\widetilde{\mathfrak{E}}$ and A by defining $\widetilde{\theta}(A) = a$. As $\widetilde{\theta}$ is already maximal we see that every element of $\widetilde{\mathfrak{E}}$ is algebraic over $\widetilde{\mathfrak{R}}$.

Let $A \in \mathfrak{E}_{\alpha+1} \backslash \widetilde{\mathfrak{E}}$. Then A is algebraic over $\widetilde{\mathfrak{R}}$. Let P be the minimum polynomial of A over $\mathfrak{R}$. The degree of P is at least 2. By the remark at the end of Lemma 5.2, by multiplying P by a suitable monomial element of $\mathfrak{R}_{\alpha+1}$, and hence in $\widetilde{\mathfrak{R}}$, we can assume that the coefficients of P are in $\widetilde{\mathfrak{E}}$ and at least one is regular in $\widetilde{\mathfrak{E}}$. Let Q be a polynomial over $\widetilde{\mathfrak{R}}$ with $Q(A) = 0$. Then $Q = PR$ for some polynomial R with coefficients from $\widetilde{\mathfrak{R}}$. If the coefficients of Q are in $\widetilde{\mathfrak{E}}$ then, by considering the coefficient of A^{i+j} in Q where A^i is the lowest degree term in P with regular coefficient and A^j is the lowest degree term in R with coefficient r_j such that $r_k/r_j \in \widetilde{\mathfrak{E}}$ for all coefficients r_k of R (some r_j have this property, e.g. the coefficient of maximum absolute value), we see that the coefficients in R are all in $\widetilde{\mathfrak{E}}$. Thus if we can find $a' \in \mathfrak{U}^1$ with $\pi a' = \psi A$, $(\widetilde{\theta} P)(a') = 0$, where $\widetilde{\theta}(\Sigma_{i=0}^n A_i Y^i) = \Sigma_{i=0}^n (\widetilde{\theta}(A_i) Y^i$, then we can extend $\widetilde{\theta}$ to the algebra generated by $\widetilde{\mathfrak{E}}$ and A by defining $\widetilde{\theta}(A) = a'$. This contradicts the maximality of $\widetilde{\theta}$ and shows $\widetilde{\mathfrak{E}} = \mathfrak{E}_{\alpha+1}$. $\widetilde{\theta}$ is one to one by Corollary 5.6.

Let $a \in \mathfrak{U}^1$ with $\pi a = \psi A$. We have $\pi(\widetilde{\theta} P)(a) = \psi P(A) = 0$ so $(\widetilde{\theta} P)(a) \in I_{\alpha+1}$. As $P(A) = 0$ is the polynomial of minimal positive degree satisfied by A, $P'(A) \neq 0$. Thus $P'(A) = MB$ where M is a monomial in $\mathfrak{E}_{\alpha+1}$, and hence in $\widetilde{\mathfrak{E}}$, and B is regular in $\mathfrak{E}_{\alpha+1}$ (Lemma 5.2). There exists $b \in \mathfrak{U}^1$ with $(\widetilde{\theta} P')(a) = \widetilde{\theta}(M) b$, $\pi b = \psi B$ because if $b_0 \in \mathfrak{U}^1$ has $\pi b_0 = \psi B$ then $(\widetilde{\theta} P')(a) - \widetilde{\theta}(M) b_0 \in \ker \pi = I_{\alpha+1}$ so we put $b = b_0 + j$ where $j \in I_{\alpha+1}$ satisfies $\widetilde{\theta}(M) j = (\widetilde{\theta} P')(a) - \widetilde{\theta}(M) b_0$ (Corollary 5.6). b is invertible because $\pi b = \psi B$ is invertible in $\mathfrak{U}^1/I_{\alpha+1}$ and $I_{\alpha+1} \subseteq \mathfrak{U} = \operatorname{Rad} \mathfrak{U}^1$. Again using Corollary 5.6 let $k \in I_{\alpha+1}$ with $\widetilde{\theta}(M)^3 k = (\widetilde{\theta} P)(a)$. We have, by Taylor's theorem for polynomials, $(\widetilde{\theta} P)(a + \widetilde{\theta}(M)^2 j) = \widetilde{\theta}(M)^3 [k + bj + \Sigma_{n=2}^m c_n j^n]$ where $k \in I_{\alpha+1} \subseteq \mathfrak{U} = \operatorname{Rad} \mathfrak{U}^1$, b is invertible, $c_2, \ldots, c_m \in \mathfrak{U}^1$. By [6, Lemma 3.28] there is $j \in \mathfrak{U}$ with $k + bj + \Sigma_{n=2}^m c_n j^n = 0$. As $\mathfrak{U}$ is the set of singular elements of $\mathfrak{U}^1$, $b \notin \mathfrak{U}$ and $\Sigma_{n=2}^m c_n j^{n-1} \in \mathfrak{U}$ we see that $b + \Sigma_{n=2}^m c_n j^{n-1} = c$ is regular in $\mathfrak{U}^1$ so $j = -c^{-1} k \in I_{\alpha+1}$. By Lemma 5.4, $\widetilde{\theta}(M)^2 j \in I_{\alpha+1}$ so if $a' = a + \widetilde{\theta}(M)^2 j$ then $\pi a' = \psi A$ and $(\theta P)(a') = 0$.

Lemma 5.7 provides the inductive step to extend from $\mathfrak{E}_\alpha$ to $\mathfrak{E}_{\alpha+1}$. Since $\mathfrak{E}_0 = \mathbf{C}$, Lemma 5.7, together with ordinary induction, shows that $\mathfrak{E}_\alpha$ can be normed for finite ordinals α. To complete the transfinite induction we need analogues of Lemmas 5.5 and 5.7 which deal with limit ordinals.

LEMMA 5.8. *Let α be a countable ordinal and for all $\beta < \alpha$ let θ_β be an algebra isomorphism $\mathfrak{E}_\beta \longrightarrow \mathfrak{U}^1$ with $\theta_\beta(X_\gamma) = x_\gamma$, $\theta_\beta | \mathfrak{E}_\gamma = \theta_\gamma$ for $\gamma < \beta$. Then*

there is an isomorphism $\psi\colon \mathfrak{E}_\alpha \longrightarrow \mathfrak{A}^1/I_\alpha$ *with* $\psi|\mathfrak{E}_\beta = \pi\theta_\beta$ *for* $\beta < \alpha$.

PROOF. If α is a successor ordinal then this is Lemma 5.5. If α is a limit ordinal let $\beta_0 < \beta_1 < \beta_2 \cdots$ be a sequence of ordinals less than α and such that if $\beta < \alpha$ then $\beta < \beta_i$ for some i. Let $A \in \mathfrak{E}_\alpha$. We show that there is exactly one element w of $\mathfrak{A}^1/I_\alpha$ with

$$w - \pi\theta_{\beta_{n+1}}(p_{\alpha\beta_{n+1}}A) \in x_{\beta_n}\mathfrak{A}^1/I_\alpha,$$

for all n. To do this we define elements y_n of $\mathfrak{A}$ inductively so that

$$y_0 = \theta_{\beta_0}(p_{\alpha\beta_0}A),$$

$$\|y_n + v_n - y_{n+1}x_{\beta_n}\| \leqslant 2^{-n}\|x_{\beta_0}\|^{-1} \cdots \|x_{\beta_{n-1}}\|^{-1},$$

where $v_n \in \mathfrak{A}$ such that

$$\begin{aligned} x_{\beta_0} \cdots x_{\beta_{n-1}} v_n &= \theta_{\beta_{n+1}}(p_{\alpha\beta_{n+1}}A) - \theta_{\beta_n}(p_{\alpha\beta_n}A) \\ &= \theta_{\beta_{n+1}}(p_{\alpha\beta_{n+1}}A - p_{\alpha\beta_n}A). \end{aligned}$$

To see that v_n exists consider $A' = p_{\alpha\beta_{n+1}}A - p_{\alpha\beta_n}A \in \mathfrak{E}_{\beta_{n+1}}$. We have $A' = MB$ where B is regular in $\mathfrak{E}_{\beta_{n+1}}$ and M is a monomial in $\mathfrak{E}_{\beta_{n+1}}$ by Lemma 5.2. As $p_{\beta_{n+1}\beta_n}A' = 0$ and $p_{\beta_{n+1}\beta_n}B$ is regular in $\mathfrak{E}_{\beta_n}$ we see $p_{\beta_{n+1}\beta_n}M = 0$. Thus if $M = X_{\gamma_1} \cdots X_{\gamma_k}X_{\delta_1}^{-1} \cdots X_{\delta_l}^{-1}$, in reduced form (that is none of the δ_j is a γ_i) with $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_k$ and $\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_l$ then $\beta_{n+1} > \delta_1 \geqslant \beta_n$ and $\gamma_1 > \delta_1$. By Lemma 5.3.

$$x_{\gamma_1} \cdots x_{\gamma_k}\theta_{\beta_{n+1}}(B) \in x_{\gamma_1}\mathfrak{A}^1 \subseteq x_{\delta_1}^{l+n}\mathfrak{A} \subseteq x_{\beta_0} \cdots x_{\beta_{n-1}}x_{\delta_1} \cdots x_{\delta_l}\mathfrak{A}$$

and hence for some $v_n \in \mathfrak{A}$

$$\begin{aligned} x_{\delta_1} \cdots x_{\delta_l}\theta_{\beta_{n+1}}(A') &= x_{\gamma_1} \cdots x_{\gamma_k}\theta_{\beta_{n+1}}(B) \\ &= x_{\beta_0} \cdots x_{\beta_{n-1}}x_{\delta_1} \cdots x_{\delta_l}v_n, \end{aligned}$$

and so, since the x_δ are not zero divisors, v_n has the required property.

Put

$$z_0 = \sum_{n=0}^{\infty} (y_n + v_n - y_{n+1}x_{\beta_n})x_{\beta_0} \cdots x_{\beta_{n-1}},$$

$$z_m = \sum_{n=1}^{\infty} (y_{n+m} + v_{n+m} - y_{n+m+1}x_{\beta_{n+m}})x_{\beta_{m+1}} \cdots x_{\beta_{n+m-1}}.$$

Then $z_0 = \theta_{\beta_{m+1}}(p_{\alpha\beta_{m+1}} A) + x_{\beta_0} \cdots x_{\beta_m}(z_m - y_{m+1})$ so $w = \pi z_0$ is as required.

The remainder of the proof is the same as that of 5.5.

LEMMA 5.9. *Let α be a countable ordinal and for all $\beta < \alpha$ let θ_β be an algebra isomorphism $\mathfrak{E}_\beta \longrightarrow \mathfrak{A}^1$ with $\theta_\beta(X_\gamma) = x_\gamma$ and $\theta_\beta | \mathfrak{E}_\gamma = \theta_\gamma$ for $\gamma < \beta$. Then there is an isomorphism $\theta_\alpha : \mathfrak{E}_\alpha \longrightarrow \mathfrak{A}^1$ with $\theta_\alpha | \mathfrak{E}_\beta = \theta_\beta$ for all $\beta < \alpha$.*

PROOF. We take $\overline{\mathfrak{E}} = \bigcup_{\beta<\alpha} \mathfrak{E}_\beta$, $\overline{\theta} : \overline{\mathfrak{E}} \longrightarrow \mathfrak{A}^1$ the map defined by $\overline{\theta} | \mathfrak{E}_\beta = \theta_\beta$ and ψ is given by Lemma 5.8. The proof now is almost identical with that of Lemma 5.7.

THEOREM 5.10. *There is an algebra isomorphism $\theta ; \mathfrak{E} \longrightarrow \mathfrak{A}^1$.*

PROOF. Lemmas 5.7 and 5.9 show, by transfinite induction, the existence of a system $\{\theta_\alpha ; \alpha < \omega_1\}$ of algebraic isomorphisms $\theta_\alpha ; \mathfrak{E}_\alpha \longrightarrow \mathfrak{A}^1$ with $\theta_\alpha | \mathfrak{E}_\beta = \theta_\beta$ for $\beta < \alpha$. As $\mathfrak{E} = \bigcup_{\alpha<\omega_1} \mathfrak{E}_\alpha$ we can define θ by $\theta | \mathfrak{E}_\alpha = \theta_\alpha$ for all $\alpha < \omega$. θ is then the required map.

COROLLARY 5.11. *There is an algebra norm on $\mathfrak{E}$.*

6. Embedding algebras of formal power series in $\mathfrak{j}_0^1$. In this section we show how $\mathfrak{F}_1$ can be embedded faithfully in $\mathfrak{j}_0^1$, the real algebra obtained by adjoining a unit to $\mathfrak{j}_0$ $(= \mathfrak{j}_0(V)$ for some ultrafilter V on $\mathbf{N})$. The process is similar to Allan's method for embedding $\mathfrak{F}_1$ in $\mathfrak{A}^1$ [1] in that $\mathfrak{F}_1$ is first embedded in a quotient algebra and this embedding is later lifted to an embedding in $\mathfrak{j}_0^1$. This result follows from the isomorphisms constructed in Corollaries 4.8 and 4.11 if we assume the continuum hypothesis. The construction in this section does not depend on the continuum hypothesis and can be extended in a similar way to the extensions in §5 of the results of Allan's paper.

LEMMA 6.1. *Let $x \in \mathfrak{j}_0, x > 0, I = \bigcap_{n=1}^{\infty} x^n \mathfrak{j}_0^1$. Then there is an isomorphism $\psi : \mathfrak{F}_1 \longrightarrow \mathfrak{j}_0^1/I$ with $\psi(X) = \pi x$ where π is the quotient map $\mathfrak{j}_0^1 \longrightarrow \mathfrak{j}_0^1/I$.*

PROOF. Put $\mathfrak{h} = \{h, h \in \mathfrak{j}_0^1$, for all $n \in \mathbf{Z}^+$ there is a polynomial P_n with real coefficients such that $h - P_n(x) \in x^n \mathfrak{j}_0\}$. $\mathfrak{h}$ is a subalgebra of $\mathfrak{j}_0$ and $P_{n+1}(x) - P_n(x) \in x^n \mathfrak{j}_0$. Because $\mathfrak{j}_0 \subseteq {}^*\mathbf{R}^0$, if Q is a nonzero polynomial of degree $\leqslant n$ and $a \in \mathfrak{j}_0$ then $|Q(x)| \geqslant x^n a$. Thus P_n and P_{n+1} have the same terms of degree $\leqslant n$.

For $h \in \mathfrak{h}$ let $\xi(h) = \Sigma_{m=0}^{\infty} a_m X^m$ where $P_n = \Sigma_{m=0}^{n} a_m X_m$ for all n. ξ is an algebra homomorphism $\mathfrak{h} \longrightarrow \mathfrak{F}_1$. We now show ξ is surjective. Let $a = \Sigma a_i X^i \in \mathfrak{F}_1$. Consider the sets

$$A = \left\{ -x^{n+1} + \sum_{i=0}^{n} a_n x^n ; n \in \mathbf{Z}^+ \right\}, \quad B = \left\{ x^{n+1} + \sum_{i=0}^{n} a_n x^n ; n \in \mathbf{Z}^+ \right\}$$

in $\mathfrak{j}_0^1$. As $\mathfrak{j}_0^1$ is an η_1-set (the proof of this is the same as the proof of [4, Lemma 13.7], and $A < B$, with A, B countable, there is $v \in \mathfrak{j}_0^1$ with $A < v < B$. Put $y_m = x^{-m}(v - \Sigma_{i=0}^m a_i x^i) \in {}^*\mathbf{R}$. Then $-x < y_m < x$ so $y_m \in \mathfrak{j}_0$ and $v \in \mathfrak{h}$. Clearly $\xi(v) = a$ and $\xi(x) = X$.

The kernel of ξ is I and we take ψ to be the inverse of the bijection $\mathfrak{h}_0^1/I \longrightarrow \mathfrak{F}_1$ induced by ξ.

THEOREM 6.2. *Let $x \in \mathfrak{I}_0, x > 0$. There is an algebra isomorphism θ; $\mathfrak{F}_1 \longrightarrow \mathfrak{j}_0^1$ with $\theta(X) = x$.*

PROOF. Define ψ as in the lemma and let $\overline{\theta}$ be the homomorphism from the unital subalgebra of $\mathfrak{F}_1$ containing X into $\mathfrak{j}_0^1$ with $\overline{\theta}(X) = x$. Let $\widetilde{\theta}$ be a maximal extension of $\overline{\theta}$ satisfying $\pi\widetilde{\theta} = \psi$. Denote the domain of θ by $\mathfrak{E}$. The proof now follows that of 5.7 up to the point at which we seek a solution $j \in I$ to the equation $k + bj + \Sigma_{n=2}^m c_n j^n = 0$ where $k \in I$, b is regular in $\mathfrak{j}_0^1$ and $c_2, \ldots, c_m \in \mathfrak{j}_0^1$.

To show that such a solution exists let $b = \lambda 1 + b'$, $b' \in \mathfrak{I}_0$ and choose preimages $\{\kappa_p\}$, $\{\beta_p\}$, $\{\gamma_{np}\}$ of k, b', c_n respectively in $c_{\mathbf{R}}$, the space of convergent sequences of real numbers. As $k = xk'$ for some $k' \in \mathfrak{j}_0^1$ we can assume $\{\kappa_p\} \in c_0$, we can also choose $\{\beta_p\} \in c_0$. Put $R_p = \max(p^{-1}, 3|\kappa_p||\lambda|^{-1})$. Since $\lambda \neq 0$, R_p is well defined and $R_p \longrightarrow 0$ as $p \longrightarrow \infty$. For each n, $\{\gamma_{np}\}$ is a bounded sequence so there is $K \in \mathbf{R}^+$ with $\Sigma_{n=2}^m |\gamma_{np}| < K$ for all p. If we choose p so large that $R_p < 1$, $|\beta_p| < |\lambda|/3$, $R_p < K^{-1}|\lambda|/3$ we have, for any complex number z with $|z| = R_p$,

$$\left| \kappa_p + (\lambda + \beta_p)z + \sum_{n=2}^{m} \gamma_{np} z^n - \lambda z \right|$$
$$< \frac{1}{3}|\lambda| R_p + \frac{1}{3}|\lambda| R_p + \left(\sum_{n=2}^{m} |\gamma_{np}| \right) R_p^2 < |\lambda z|.$$

This shows that the variation of $\operatorname{Arg}(\kappa_p + (\lambda + \beta_p)z + \Sigma_{n=2}^m \gamma_{np} z^n)$ as z describes the circle Γ_p centre 0 and radius R_p once in the positive direction is the same as the variation of $\operatorname{Arg} \lambda z$, that is 2π. Hence for large values of p, $\kappa_p + (\lambda + \beta_p)z + \Sigma_{n=2}^\infty \gamma_{np} z^n = 0$ has exactly one solution z_p inside Γ_p. As Γ_p is symmetric about the real axis and the coefficients are real, z_p is real. If j is the element of $\mathfrak{j}_0$ corresponding to $\{z_p\} \in c_0$ then j satisfies the polynomial equation and $j \in I$ because $j = -(b + \Sigma_{n=2}^m c_n j^{n-1})^{-1} k$ with $k \in I$, the inverse existing because b is invertible and $\Sigma_{n=2}^m c_n j^{n-1} \in \mathfrak{I}_0$.

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