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MULTIPLIERS OF L_E^p. I

BY

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ABSTRACT. Let X be an abelian group, the character group of a compact group G. For a subset E of X let L_E^p be the subspace of E-spectral functions in $L^p(G)$. We show that if X is infinite and $1 \leq p < 2$, then E can be chosen so that not every multiplier of $\widehat{L_E^p}$ extends to a multiplier of $\widehat{L^p}(G)$.

1. Let G be a compact abelian group with character group X. For $1 \leq p \leq \infty$, let $L^p(G)$ be the usual Lebesgue space with respect to normalized Haar measure on G, and for $E \subseteq X$, let L_E^p be the translation invariant subspace of $L^p(G)$ consisting of those functions whose Fourier transforms vanish off of E. Let M_E^p denote the set of functions in $l^\infty(E)$ which are multipliers for the Fourier transform space of L_E^p. Thus $\phi \in M_E^p$ if and only if for every $f \in L_E^p$ there exists $g \in L_E^p$ with $\hat{g}(x) = \phi(x)\hat{f}(x)$ for each $x \in E$. For $1 \leq p < \infty$, M_E^p can be identified with the space of operators on L_E^p which commute with translations by elements of G. Let $M^p|_E$ denote the set of restrictions to E of functions in M^p $(= M_X^p)$. Then clearly $M^p|_E \subseteq M_E^p$, and it follows from the Riesz-Thorin theorem that $M^p|_E \subseteq M^q|_E$ if $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p \leq \infty$. We are interested in the following questions:

(i) Does $M^p|_E = M_E^p$?

(ii) Does an analogue of the Riesz-Thorin theorem hold for the spaces L_E^p? I.e., for $1 \leq p_1 \leq p_2 \leq \infty$, are the interpolation spaces obtained by applying Calderón's complex method of interpolation to $L_E^{p_1}$ and $L_E^{p_2}$ actually the intermediate L_E^p spaces?

(iii) For $1 \leq p < q < 2$ or $2 < q < p \leq \infty$, is $M_E^p \subseteq M_E^q$?

Question (i) is posed for the circle group T in [2, pp. 280–281] and has an affirmative answer for any G if $p = 2$ (trivially) or if $p = \infty$ (for a very easy proof, see [10]). Question (ii) is inspired by [1, p. 344, Remarque], while (iii) seems natural in view of (i), (ii), and the case $E = X$.

In this paper we treat the case $1 \leq p < 2$ of (i). Our main result is the following theorem.

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THEOREM 1.1. *If G is an infinite compact abelian group and if $1 \leqslant p < 2$, then there exists $E \subseteq X$ such that $M^p|_E$ is a proper subset of M_E^p.*

Along the way to this theorem, we show that the answer to (ii) is sometimes negative when $p_1 = 1, p_2 = 2$.

In [10], a sequel to this paper, we prove an analogue of Theorem 1.1 for the case when $p > 2$ is an even integer. There we also show that the answer to (iii) can be negative. The results of the present paper and of [10] were announced in [9].

2. In this section we present a heuristic, if imprecise, sketch of the proof for Theorem 1.1. For unexplained definitions and notations the reader may consult [7].

We begin by observing that in view of [4, Theorems 2.1 and 2.3], it is sufficient to prove Theorem 1.1 for the cases $X = Z$, $X = Z(q^\infty)$ for some prime q, $X = \mathrm{P}^*{}_{n=1}^{\infty} Z(q_n)$ where $\{q_n\}_{n=1}^{\infty}$ is an increasing sequence of primes, and $X = \mathrm{P}^*{}_{n=1}^{\infty} Z(q)$ for some prime q. It turns out that the second and third cases can be deduced from the first and that this case reduces, in a certain sense, to $X = \mathrm{P}^*{}_{n=1}^{\infty} Z$. Thus it will really only be necessary to consider infinite product groups. This will be an important point.

Next we state

PROPOSITION 2.1. *Let $E \subseteq X$ be a $\Lambda(2)$ set and fix $1 \leqslant p < 2$. If $p > 1$, suppose that E is not a $\Lambda(q)$ set, where $p^{-1} + q^{-1} = 1$. If $p = 1$, suppose that E is not a Sidon set. Then $M_E^p = l^\infty(E) \neq M^p|_E$.*

PROOF. The statement $M_E^p = l^\infty(E)$ follows from $L_E^p = L_E^2$. If $p > 1$, $M^p|_E \neq l^\infty(E)$ follows from [6, Theorem 6], while if $p = 1$, this follows from Wendel's theorem and a well-known characterization of Sidon sets.

Proposition 2.1 and the existence in any infinite abelian group X of sets which are $\Lambda(2)$ but not Sidon (see, e.g., [4]) combine to prove Theorem 1.1 for $p = 1$. Thus we shall only consider $p > 1$. On the other hand, as it is not known whether there exists a $\Lambda(2)$ set which is not $\Lambda(4)$, Proposition 2.1 does not directly yield the proof of Theorem 1.1 for any $p \geqslant 4/3$. Nevertheless, the idea underlying the proof of Proposition 2.1—which we present as Lemma 2.2 below—is the key to the proof of our theorem. Before we can state Lemma 2.2, we need some notation. For $E \subseteq X$, $1 < p < 2, p^{-1} + q^{-1} = 1$, let $\lambda_p(E)$ denote the (possibly infinite) infimum of the set

$$\{K > 0\colon \|g\|_{L^q} \leqslant K\|g\|_{L^2} \text{ for all } g \in L_E^2\}.$$

Similarly, let $\sigma_p(E)$ denote

$$\inf\{K > 0\colon \text{for every } \phi \in l^\infty(E), \text{ there exists } \Phi \in M^p \text{ with } \Phi|_E = \phi,\ \|\Phi\|_{M^p} \leq K\|\phi\|_{l^\infty}\}.$$

LEMMA 2.2 [12, THEOREM 5.3]. *To each p $(1 < p < 2)$ there corresponds a positive number R_p such that $R_p\lambda_p(E) \leq \sigma_p(E) \leq \lambda_p(E)$ for any E contained in any X.*

Thus Lemma 2.2 is a quantitative version of [6, Theorem 6]. Now let $\mu_p(E)$ denote

$$\inf\{K > 0\colon \|g\|_{L^2} \leq K\|g\|_{L^p} \text{ for } g \in L_E^p\}.$$

If the ratio $\sigma_p(E)/\mu_p(E)$ is large, then it is possible to find a multiplier ϕ of norm one in M_E^p such that any $\Phi \in M^p$ satisfying $\Phi|_E = \phi$ has large norm in M^p. That is, if $\sigma_p(E)/\mu_p(E)$ is large, then there are multipliers in M_E^p which are hard to interpolate by functions in M^p. Our plan is first to produce sets $\{E_n\}_{n=1}^\infty$ with $\sigma_p(E_n)/\mu_p(E_n) \to \infty$ and then to piece these sets together and produce E for which $M^p|_E \neq M_E^p$. By Lemma 2.2, to produce sets $\{E_n\}_{n=1}^\infty$ with

$$\sigma_p(E_n)/\mu_p(E_n) \to \infty,$$

it is enough to produce sets $\{E_n\}_{n=1}^\infty$ with $\lambda_p(E_n)/\mu_p(E_n) \to \infty$. To this end, we note that the number $\lambda_p(E)/\mu_p(E)$ is the quotient of the norm of χ_E (characteristic function of E) as a multiplier of $\widehat{L^p}$ into $\widehat{L^2}$ by its norm as a multiplier of $\widehat{L_E^p}$ into $\widehat{L^2}$. In view of Lemma 2.3 below, in order to produce sets $\{E_n\}_{n=1}^\infty$ (in a suitable product group) with $\lambda_p(E_n)/\mu_p(E_n) \to \infty$ as desired, it is only necessary to find a set E_0 having $\lambda_p(E_0)/\mu_p(E_0)$ strictly greater than 1. This will be done by direct computation.

LEMMA 2.3 [1, CHAPITRE III, THÉORÈME 2, LEMME 1]. *For $i = 1, 2$, let G_i be a compact abelian group with character group X_i, let $E_i \subseteq X_i$, and let ϕ_i be a multiplier of $\widehat{L_{E_i}^{p_1}}$ into $\widehat{L_{E_i}^{p_2}}$, $1 \leq p_1 \leq p_2 < \infty$. Then $\phi_1 \circ \phi_2$, considered as a function on $E_1 \times E_2 \subseteq X_1 \times X_2$, is a multiplier of $\widehat{L_{E_1 \times E_2}^{p_1}}$ into $\widehat{L_{E_1 \times E_2}^{p_2}}$. Further $\|\phi_1 \circ \phi_2\| = \|\phi_1\| \circ \|\phi_2\|$.*

Actually Lemma 2.3 is slightly more general than the result given in [1], but the proofs are word for word the same.

In §§3, 4, and 5 below we give the details for the groups $X = \mathrm{P}^*{}_{n=1}^\infty Z(q)$, $X = Z$, $X = Z(q^\infty)$ and $X = \mathrm{P}^*{}_{n=1}^\infty Z(q_n)$.

3. We start with some lemmas about norms on R^N. For $1 \leq p < 2$, $N = 2, 3, \ldots$, let $\|\circ\|_{l^p}$ denote the l^p norm on R^N. Let $P = P(N) \subseteq R^N$ be the set of all $x = (x_1, \ldots, x_N)$ such that $x_1 + \cdots + x_N = 0$, and let

$$M = M(N, p) = \sup_{0 \neq x \in P} \frac{\|x\|_{l^2}}{\|x\|_{l^p}}.$$

Let $\pi = \pi(N)$ be the orthogonal projection of the Hilbert space R^N onto its subspace P, and let

$$\|\pi\|_{l^p - l^2} = \sup_{0 \neq x \in R^N} \frac{\|\pi x\|_{l^2}}{\|x\|_{l^p}}.$$

LEMMA 3.1. *For* $N = 2, 3, \ldots, M(N, 1) = 2^{-1/2}$.

PROOF. Fix $x = (x_1, \ldots, x_N) \in P(N)$. After reordering, there exists some N' with $1 \leq N' < N$ such that $|x_1| + \cdots + |x_{N'}| = |x_{N'+1}| + \cdots + |x_N|$, for $x_1 + \cdots + x_N = 0$. We must show that

$$\sum_1^N |x_i|^2 \leq \frac{1}{2}\left(\sum_{i=1}^N |x_i|\right)^2.$$

But $\Sigma_1^N |x_i| = 2\, \Sigma_1^{N'} |x_i|$, so

$$\left(\sum_1^N |x_i|\right)^2 = 4\left(\sum_1^{N'} |x_i|\right)^2 = 2\left[\left(\sum_1^{N'} |x_i|\right)^2 + \left(\sum_{N'+1}^{N} |x_i|\right)^2\right].$$

Thus

$$\sum_1^N |x_i|^2 = \sum_1^{N'} |x_i|^2 + \sum_{N'+1}^{N} |x_i|^2 \leq \left(\sum_1^{N'} |x_i|\right)^2 + \left(\sum_{N'+1}^{N} |x_i|\right)^2 = \frac{1}{2}\left(\sum_1^N |x_i|\right)^2.$$

LEMMA 3.2. *For* $N = 2, 3, \ldots$ *and* $1 \leq p < 2$, *either* $M(N, p) = 2^{1/2 - 1/p}$ *or* $M(N, p) < \|\pi(N)\|_{l^p - l^2}$.

PROOF. Because of Lemma 3.1 we can assume $p > 1$. Fix $x \in P$ with $\|x\|_{l^2}/\|x\|_{l^p} = M$. If the function of a real variable $f(t) = \|x + t(1, 1, \ldots, 1)\|_{l^p}^p$ does not have a minimum at $t = 0$, then $\|\pi\|_{l^p - l^2} > M$ because $\pi(1, 1, \ldots, 1) = 0$. Thus we may assume that:

(a) x minimizes $\|x\|_{l^p}^p$ subject to $x_1^2 + \cdots + x_N^2 = 1, x_1 + \cdots + x_N = 0$;

(b) $f(t)$ has a minimum at $t = 0$.

We will show that (a) and (b) imply $\|x\|_{l^2}/\|x\|_{l^p} \leq 2^{1/2 - 1/p}$.

From (a) and an application of Lagrange's method, it follows that there are real numbers λ_1, λ_2 with

$$(\operatorname{sgn} x_j)|x_j|^{p-1} = \lambda_1 x_j + \lambda_2 \quad \text{for } j = 1, \ldots, N. \tag{1}$$

From (b) it follows that

$$\sum_{j=1}^N (\operatorname{sgn} x_j)|x_j|^{p-1} = 0. \tag{2}$$

Adding (1) for $j = 1, \ldots, N$, and using (2) and $x_1 + \cdots + x_N = 0$, we get that $\lambda_2 = 0$. It then follows from (1) that each nonzero x_j has the same modulus. Since $x_1 + \cdots + x_N = 0$, there must be at least two nonzero x_j. Thus $\|x\|_{l^2}/\|x\|_{l^p} \leq 2^{1/2-1/p}$ follows.

LEMMA 3.3. *For* $1 \leq p < 2$, *there exists* $N_0 = N_0(p)$ *such that for* $N \geq N_0$, $M(N, p) < \|\pi(N)\|_{l^p - l^2}$.

PROOF. By Lemma 3.2 it suffices to show that there exists N_0 such that for $N \geq N_0$, $\|\pi(N)\|_{l^p - l^2} > 2^{1/2-1/p}$. But

$$\pi(N)(1, 0, 0, \ldots, 0) = \left(\frac{N-1}{N}, \frac{-1}{N}, \frac{-1}{N}, \ldots, \frac{-1}{N}\right).$$

Thus

$$\|\pi(N)(1, 0, 0, \ldots, 0)\|_{l^2} = ((N-1)/N)^{1/2},$$

and so $\|\pi(N)\|_{l^p - l^2} \geq ((N-1)/N)^{1/2}$. Hence we need only choose N_0 with $((N_0 - 1)/N_0)^{1/2} > 2^{1/2-1/p}$.

LEMMA 3.4. *There exists* $p_0 \in (1, 2)$ *such that* $M(N, p_0) > 2^{1/2-1/p_0}$ *for* $N \geq 7$.

PROOF. To have $M(N, p_0) \leq 2^{1/2-1/p_0}$ is equivalent to having $2 \leq 2^{p_0/2}\|x\|_{l^{p_0}}^{p_0}$ for $x \in P(N)$, $\|x\|_{l^2} = 1$. For such an x, define $F(x, p) = 2^{p/2} \sum_{i=1}^N |x_i|^p$ for $1 \leq p \leq 2$. Since $F(x, 2) = 2$, it is enough to show that there exists $x = (x_1, \ldots, x_7) \in P(7)$ with $\|x\|_{l^2} = 1$ and $dF/dp|_{p=2} > 0$. Let

$$x_1 = ((N-1)/N)^{1/2}, \quad x_2 = x_3 = \cdots = x_N = -(1/N(N-1))^{1/2}.$$

Then $x = (x_1, \ldots, x_N) \in P(N)$ and $\|x\|_{l^2} = 1$. Also

$$\begin{aligned}
\left.\frac{dF}{dp}\right|_{p=2} &= \sum_{i=1}^N (|2^{1/2}x_i|^2 \cdot \log|2^{1/2}x_i|) \\
&= \frac{N-1}{N} \log\left[\frac{2(N-1)}{N}\right] + \frac{1}{N} \log\left[\frac{2}{N(N-1)}\right] \\
&= \frac{1}{N} \log\left[\frac{2^N(N-1)^{N-2}}{N^N}\right] \\
&= \log 2 + \log\left(\frac{N-1}{N}\right) - \frac{2}{N}\log(N-1) \underset{N}{\to} \log 2.
\end{aligned}$$

Since $2^7 6^5/7^7 = 995{,}328/823{,}543 > 1$, $N = 7$ yields $dF/dp|_{p=2} > 0$.

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For an abelian group X, a set $E \subseteq X$, a function $\phi \in l^\infty(E)$, and numbers p_1, p_2 with $1 \leq p_1, p_2 \leq \infty$, we define $\|\phi\|_{E,p_1,p_2}$ to be the norm of ϕ as a multiplier of $\widehat{L_E^{p_1}}$ into $\widehat{L_E^{p_2}}$. The norm of ϕ (extended by $\phi = 0$ on $X \backslash E$) as a multiplier of $\widehat{L^{p_1}}$ into $\widehat{L^{p_2}}$ will be denoted $\|\phi\|_{p_1,p_2}$.

Fix a prime number $q \geq 2$, and let $X_m = \mathrm{P}_{n=1}^m Z(q)$ for $m = 1, 2, \ldots$.

LEMMA 3.5. *Fix p with $1 < p < 2$. There exist sequences $\{E_j\}_{j=1}^\infty$ and $\{n_j\}_{j=1}^\infty$ with $E_j \subseteq X_{n_j}$ and with $\lambda_p(E_j)/\mu_p(E_j) \to \infty$. (The numbers λ_p and μ_p are defined in* §1.)

PROOF. Let $N_0 = N_0(p)$ be as in Lemma 3.3 and fix an integer m with $q^m \geq N_0$. Let $E_1 \subseteq X_m$ be the set of all nonzero elements in the group X_m. Let G be the character group of X_m and let T be the translation invariant operator on $L^p(G)$ defined by the multiplier function χ_{E_1} on X_m. Then Lemma 3.3 implies that

$$(1) \qquad \sup_{0 \neq f \in L^p(G)} \frac{\|Tf\|_{L^2}}{\|f\|_{L^p}} > \sup_{0 \neq f \in L^p_{E_1}(G)} \frac{\|Tf\|_{L^2}}{\|f\|_{L^p}} = \sup_{0 \neq f \in L^p_{E_1}(G)} \frac{\|f\|_{L^2}}{\|f\|_{L^p}},$$

where the suprema are computed over the sets of real-valued functions in L^p and L^p_E. (This is so because T can be identified with $\pi(q^m)$ and the subspace of real-valued functions in L^p_E can be identified with $P(q^m)$. Lemmas 3.1–3.3 were worked out for counting measure, but changing the normalization of Haar measure on G only multiplies both sides of (1) by the same positive constant.)

By [1, Chapitre III, Lemme 2], (1) continues to hold when the suprema are taken over complex-valued functions, and so (1) implies that $\|\chi_{E_1}\|_{p,2} > \|\chi_{E_1}\|_{E_1,p,2}$. Thus Lemma 2.3 shows that for the sequence $\{E_j\}_{j=1}^\infty$ given by $E_j = \mathrm{P}_{n=1}^j E_1 \subseteq X_{m\cdot j}$, we have $\|\chi_{E_j}\|_{p,2}/\|\chi_{E_j}\|_{E_j,p,2} \to \infty$. Since $\lambda_p(E_j) = \|\chi_{E_j}\|_{p,2}$ and $\mu_p(E_j) = \|\chi_{E_j}\|_{E_j,p,2}$, this completes the proof of the lemma.

We identify the group $Z(q)$ with the set $\{0, 1, \ldots, q-1\}$. For $j = 1, 2, \ldots$, we define the subgroup H_j of $\mathrm{P}^*{}_{n=1}^\infty Z(q)$ by specifying that $x = (x_1, x_2, \ldots) \in H_j$ if and only if $x_{j+1} = x_{j+2} = \cdots = 0$. Thus $H_1 \subseteq H_2 \subseteq \cdots$.

LEMMA 3.6. *Let G be the character group of $\mathrm{P}^*{}_{n=1}^\infty Z(q)$ and fix p with $1 < p < 2$. There exists K_p such that for any $f \in L^p(G)$,*

$$\left(\sum_{j=2}^\infty \left\| \sum_{x \in H_j \backslash H_{j-1}} \hat{f}(x) x \right\|_{L^p}^2\right)^{1/2} \leq K_p \|f\|_{L^p}.$$

PROOF. This follows from [11, Théorème 3] and an application of Minkowski's inequality.

We can now prove a strengthened version of Theorem 1.1 for the case $X = \mathrm{P}^*{}_{n=1}^{\infty} Z(q)$.

THEOREM 3.7. *Fix p with $1 \leq p < 2$. There exists $E \subseteq \mathrm{P}^*{}_{n=1}^{\infty} Z(q)$ and a multiplier ϕ of $\widehat{L_E^p}$ into $\widehat{L^2}$ such that ϕ is not in $M^p|_E$.*

PROOF. The case $p = 1$ follows from the argument given for Proposition 2.1 and the existence of sets $E \subseteq \mathrm{P}^*{}_{n=1}^{\infty} Z(q)$ which are $\Lambda(2)$ but not Sidon. Thus we consider only the case $1 < p < 2$.

Recall that $X_m = \mathrm{P}_{n=1}^{m} Z(q)$ for $m = 1, 2, \ldots$. From Lemmas 3.5 and 2.2 it follows that there exist sequences $\{n_j\}_{j=1}^{\infty}$ and $\{E_j\}_{j=1}^{\infty}$ with $E_j \subseteq X_{n_j}$ and with $\sigma_p(E_j)/\mu_p(E_j) > j$. It follows from the definition of σ_p that for each $j = 1, 2, \ldots$, there exists $\phi_j \in l^\infty(E_j)$ with $\|\phi_j\|_{l^\infty} \leq 1/\mu_p(E_j)$ and with $\|\Phi_j\|_{p,p} > j$ if $\Phi_j \in M^p_{X_{n_j}}$ is such that $\Phi_j|_{E_j} = \phi_j$. Since $\mu_p(E_j) = \|\chi_{E_j}\|_{E_j,p,2}$, we have proved that

(1) there exists a sequence $\{\phi_j\}_{j=1}^{\infty}$ having $\|\phi_j\|_{E_j,p,2} \leq 1$, but such that if $\Phi_j \in M^p_{X_{n_j}}$ satisfies $\Phi_j|_{E_j} = \phi_j$, then $\|\Phi_j\|_{p,p} > j$.

Consider now the map i_j of X_{n_j} into $H_{n_j+1}\backslash H_{n_j}$ defined by $i_j(x_1, \ldots, x_{n_j}) = (x_1, \ldots, x_{n_j}, 1, 0, 0, 0, \ldots)$. Let $\widetilde{E}_j$ be the image $i_j(E_j) \subseteq H_{n_j+1}\backslash H_{n_j}$ and define $E \subseteq \mathrm{P}^*{}_{n=1}^{\infty} Z(q)$ by $E = \bigcup_{j=1}^{\infty} \widetilde{E}_j$. Let $\widetilde{\phi}_j$ be the function defined on $\widetilde{E}_j$ by $\widetilde{\phi}_j[i_j(x)] = \phi_j(x)$ $(x \in E_j)$. Since i_j is affine, it follows easily from (1) that

(a) $\|\widetilde{\phi}_j\|_{\widetilde{E}_j,p,2} \leq 1$,

(b) if $\widetilde{\Phi}_j$ is a function on $\mathrm{P}^*{}_{n=1}^{\infty} Z(q)$ satisfying $\widetilde{\Phi}_j|_{\widetilde{E}_j} = \widetilde{\phi}_j$, then $\|\widetilde{\Phi}_j\|_{M^p} > j$.

Define $\phi \in l^\infty(E)$ by $\phi|_{\widetilde{E}_j} = \widetilde{\phi}_j$. Then it follows from (a), Lemma 3.6, and the fact that $\widetilde{E}_j \subseteq H_{n_j+1}\backslash H_{n_j}$ that ϕ is a multiplier of $\widehat{L_E^p}$ into $\widehat{L^2}$. It follows from (b), though, that ϕ is not in $M^p|_E$.

The idea of using Lemma 3.6 to piece together multipliers was shown to the author by Professor A. Figà-Talamanca. An argument similar to ours in this respect appears in the proof of [5, Theorem B].

We conclude this section with a negative answer to question (ii).

THEOREM 3.8. *There exists a subset E of the group $\mathrm{P}^*{}_{n=1}^{\infty} Z(2)$ such that for some $\alpha \in (0, 1)$, the complex interpolation space $[L_E^1, L_E^2]_\alpha$ is not the space L_E^p, where $1/p = 1 - \alpha/2$.*

PROOF. It suffices to show that there exist E and a sequence $\{\phi_j\}_{j=1}^{\infty}$ of finitely supported functions on E such that

(a) $\|\phi_j\|_{E,1,2} = 2^j$,

(b) $\|\phi_j\|_{E,2,2} = \|\phi_j\|_{l^\infty(E)} = 1$,

(c) $\|\phi_j\|_{E,p,2} \neq O(2^{j(2/p-1)})$.

As before, let $X_j = \mathrm{P}_{n=1}^{j} Z(2)$. Let $E_1 \subseteq X_3$ be the set of nonzero elements in the group X_3, and let G be the character group of X_3. It follows from Lemmas 3.1 and 3.4 and from [1, Chapitre III, Lemme 2] that $\|\chi_{E_1}\|_{E_1,1,2} = 2$ and that for some p with $1 < p < 2$ we have $\|\chi_{E_1}\|_{E_1,p,2} \geq 2^{2/p-1} + \epsilon$ for some $\epsilon > 0$. (These multiplier norms are computed with respect to *normalized* Haar measure on G.) Thus if $E_j = \mathrm{P}_{n=1}^{j} E_1 \subseteq X_{3_j}$, it follows from Lemma 2.3 that

$$\|\chi_{E_j}\|_{E_j,1,2} = 2^j, \quad \|\chi_{E_j}\|_{E_j,p,2} \geq (2^{2/p-1} + \epsilon)^j. \tag{1}$$

We represent $\mathrm{P}^*{}_{n=1}^{\infty} Z(2)$ as $\mathrm{P}^*{}_{j=1}^{\infty} X_{3_j}$ and identify each X_{3_j} with its canonical image in $\mathrm{P}^*{}_{j=1}^{\infty} X_{3_j}$. In this way each $E_j \subseteq X_{3_j}$ becomes a subset of $\mathrm{P}_{j=1}^{\infty} X_{3_j}$, and we let $E = \bigcup_{j=1}^{\infty} E_j$. Since $E \cap X_{3_j} = E_j$, it follows that $\|\chi_{E_j}\|_{E,s,2} = \|\chi_{E_j}\|_{E_j,s,2}$ for $1 \leq s \leq \infty$. In particular, (1) implies that

$$\|\chi_{E_j}\|_{E,1,2} = 2^j, \; \|\chi_{E_j}\|_{E,p,2} \neq O(2^{j(2/p-1)}).$$

Since $\|\chi_{E_j}\|_{l^\infty(E)} = 1$, (a), (b), and (c) are satisfied for the choice $\phi_j = \chi_{E_j}$. The proof is complete.

4. In this section we prove an analogue of Theorem 3.7 for the group Z.

THEOREM 4.1. *Fix p with $1 \leq p < 2$. There exists $E \subseteq Z$ and a multiplier ϕ of $\widehat{L^p_E}$ into $\widehat{L^2}$ such that ϕ is not in $M^p|_E$.*

Once Lemmas 4.2 and 4.3 below are established, the proof for Theorem 4.1 is analogous to the proof of Theorem 3.7.

LEMMA 4.2. *For a given p $(1 < p < 2)$, there exists a sequence $\{E_j\}_{j=1}^{\infty}$ of finite subsets of Z such that $\lambda_p(E_j)/\mu_p(E_j) \to \infty$.*

LEMMA 4.3. *For fixed p with $1 < p < 2$, there exists $K_p < \infty$ such that for any $f \in L^p(T)$,*

$$\left(\sum_{j=1}^{\infty} \left\| \sum_{n=2^j+1}^{2^{j+1}} \hat{f}(n)e^{in\theta} \right\|_{L^p}^2 \right)^{1/2} \leq K_p \|f\|_{L^p}.$$

Lemma 4.3 is a consequence of the classical Littlewood-Paley theorem (see, e.g., [13, Chapter XV, Theorem 2.1]) and Minkowski's inequality. Thus only Lemma 4.2 must be established. To do this, we will use Lemma 2.3, but the proof is slightly complicated by the fact that Z is not a product group. We begin with a series of lemmas.

LEMMA 4.4. *Given a finite subset F of Z^2, $\epsilon > 0$, and p $(1 \leq p < \infty)$, there exists an integer N such that for $m \geq N$ and f any F-polynomial on T^2,* we have

$$(1-\epsilon)\|f\|_{L^p(T^2)} \leqslant \left\| \sum_{(n_1,n_2)\in F} \hat{f}(n_1, n_2) e^{i(n_1+mn_2)\theta} \right\|_{L^p(T)}$$
$$\leqslant (1+\epsilon)\|f\|_{L^p(T^2)}.$$

PROOF. It follows from [3, Lemmas 3.3 and 3.4] that if H_m is the closed subgroup of T^2 given by $H_m = \{(e^{i\theta}, e^{im\theta}): e^{i\theta} \in T\}$ $(m = 1, 2, \ldots)$ and if λ_m is the Haar measure of H_m, then the sequence $\{\lambda_m\}_{m=1}^{\infty}$ converges weak-* to the Haar measure λ of T^2. But then $\lambda_m \longrightarrow \lambda$ uniformly on compact subsets of $C(T^2)$. Since the set $\{|f|^p: f \in L^p_E(T^2), \|f\|_{L^p} = 1\}$ is a compact subset of $C(T^2)$, the lemma follows.

LEMMA 4.5. *Let $E \subseteq Z$ be finite. Suppose that ϕ_1, ϕ_2 are functions on E and that $1 \leqslant p_1 \leqslant p_2 < \infty$. Fix $\epsilon > 0$. Then there exists an integer N such that for $m \geqslant N$, the function ϕ defined on $E + mE$ by $\phi(n_1 + mn_2) = \phi_1(n_1)\phi_2(n_2)$ satisfies*

$$|\|\phi\|_{E+mE,p_1,p_2} - \|\phi_1\|_{E,p_1,p_2} \circ \|\phi_2\|_{E,p_1,p_2}| < \epsilon.$$

PROOF. This follows from Lemma 4.4 (with $F = E \times E$) and Lemma 2.3.

LEMMA 4.6. *Let $E \subseteq Z$ be finite and fix p with $1 < p < 2$. Given $\epsilon > 0$, there exists N such that for $m \geqslant N$ we have*

$$\|\chi_{E+mE}\|_{p,2} > (\|\chi_E\|_{p,2} - \epsilon)^2.$$

PROOF. For $m = 0, 1, 2, \ldots$, let T_m be the translation invariant operator on $L^p(T)$ induced by the multiplier χ_{E+mE}. Let f be an F-polynomial (for some finite $F \supseteq E$) such that $\|f\|_{L^p} = 1$, $\|T_0 f\|_{L^2} > \|\chi_E\|_{p,2} - \epsilon$. Let $g = T_0 f$ and define

$$f_m(e^{i\theta}) = f(e^{im\theta}), \quad g_m(e^{i\theta}) = g(e^{im\theta}) \qquad (m = 1, 2, \ldots).$$

Since $\|ff_m\|_{L^p} \to \|f\|^2_{L^p} = 1$, $\|gg_m\|_{L^2} \to \|g\|^2_{L^2} > (\|\chi_E\|_{p,2} - \epsilon)^2$ (by Lemma 4.4 applied to $F \times F$), it is enough to note that $T_m(ff_m) = gg_m$ for all sufficiently large m.

LEMMA 4.7. *Given p with $1 < p < 2$, there exists a finite set $E \subseteq Z$ satisfying $\|\chi_E\|_{p,2} > \|\chi_E\|_{E,p,2}$.*

PROOF. We sketch the argument. First note that by Lemma 4.4, it suffices to find such an $E \subseteq Z^2$. Let $E = \{(1, 0), (0, 1)\}$ and let $f(e^{i\theta_1}, e^{i\theta_2}) = a_1 e^{i\theta_1} + a_2 e^{i\theta_2}$ be such that $\|f\|_{L^p} = 1$, $\|f\|_{L^2} = \|\chi_E\|_{E,p,2}$. We can assume that $a_1, a_2 \geqslant 0$. If the function of a real variable $t \to \|f + te^{i(n_1\theta_1+n_2\theta_2)}\|_{L^p}$

fails to have a minimum at $t = 0$ for some value of $(n_1, n_2) \in Z^2 \backslash E$, then we are done. But if $t = 0$ is such a minimum for each $(n_1, n_2) \in Z^2 \backslash E$, then $g = |f|^{p-1} \operatorname{sgn} f$ is an E-polynomial, by [8, Lemma 15.10] applied to the linear functional $L\colon h \to \int_{T^2} h|f|^{p-1} \overline{\operatorname{sgn}} f$. And an easy argument shows that if g is an E-polynomial, then g is a positive multiple of f, since $\operatorname{sgn} g = \operatorname{sgn} f$. Hence $|f|$ is constant, and $\|\chi_E\|_{E,p,2} = 1$, a contradiction since L_E^p contains functions of nonconstant modulus.

The proof of Lemma 4.2 is now easy. Lemmas 4.5 and 4.6 show that for any finite $E \subseteq Z$ and $\epsilon > 0$, there exists an integer m such that

$$\frac{\|\chi_{E+mE}\|_{p,2}}{\|\chi_{E+mE}\|_{E+mE,p,2}} > \left(\frac{\|\chi_E\|_{p,2}}{\|\chi_E\|_{E,p,2}}\right)^2 - \epsilon.$$

Thus Lemma 4.7 and an induction argument yield a sequence $\{E_j\}_{j=1}^\infty$ of finite subsets of Z satisfying $\|\chi_{E_j}\|_{p,2}/\|\chi_{E_j}\|_{E_j,p,2} \to \infty$. Since $\|\chi_{E_j}\|_{p,2} = \lambda_p(E_j)$, $\|\chi_{E_j}\|_{E_j,p,2} = \mu_p(E_j)$, this establishes Lemma 4.2.

5. In this section we prove an analogue of Theorem 3.7 for the group $Z(q^\infty)$ and we prove Theorem 1.1 for the group $\mathrm{P}*_{n=1}^\infty Z(q_n)$. We start with a lemma about cyclic groups.

LEMMA 5.1. *Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers. Let G_k be the character group of $Z(n_k)$, considered as a subgroup of T, and fix p $(1 \leq p < \infty)$, a positive integer N, and $\epsilon > 0$. Then there exists a positive integer K such that for $k \geq K$ and any trigonometric polynomial of the form $f(e^{i\theta}) = \sum_{n=1}^N \hat{f}(n)e^{in\theta}$, we have*

$$(1-\epsilon)\|f\|_{L^p(T)} \leq \|f|_{G_k}\|_{L^p(G_k)} \leq (1+\epsilon)\|f\|_{L^p(T)}.$$

PROOF. Let λ_k be the normalized Haar measure on G_k and let λ be Haar measure on T. Since $\lambda_k \to \lambda$ weak-*, the argument in the proof of Lemma 4.4 applies here as well.

Now fix a prime number $q \geq 2$, and identify $Z(q^\infty)$ with a subgroup of T in the usual way. For $k = 1, 2, \ldots$, let H_k be the subgroup of T composed of the q^kth roots of 1. Then $H_1 \subseteq H_2 \subseteq \ldots$, and $\bigcup_{k=1}^\infty H_k = Z(q^\infty)$.

LEMMA 5.2. *Fix p with $1 < p < 2$. There exist sequences $\{E_j\}_{j=1}^\infty$ and $\{n_j\}_{j=1}^\infty$ with $E_j \subseteq H_{n_j+1} \backslash H_{n_j}$ and with $\lambda_p(E_j)/\mu_p(E_j) \to \infty$.*

PROOF. This follows from Lemmas 4.2 and 5.1. The set E_j can be taken to be a subset of a coset of H_{n_j} in H_{n_j+1}.

LEMMA 5.3. *Let G be the character group of $Z(q^\infty)$ and fix p with $1 < p < 2$. There exists K_p such that for any $f \in L^p(G)$,*

$$\left(\sum_{j=2}^{\infty}\left\|\sum_{x\in H_{n_{j+1}}\setminus H_{n_j}}\hat{f}(x)x\right\|_{L^p}^2\right)^{1/2}\leqslant K_p\|f\|_{L^p}.$$

PROOF. Again, this follows from [11, Théorème 3] and Minkowski's inequality.

THEOREM 5.4. *Fix p with $1\leqslant p<2$. There exists $E\subseteq Z(q^\infty)$ and a multiplier ϕ of $\widehat{L^p_E}$ into $\widehat{L^2}$ such that ϕ is not in $M^p|_E$.*

PROOF. Given Lemmas 5.2 and 5.3, the proof is again analogous to that of Theorem 3.7.

Now let $\{q_n\}_{n=1}^\infty$ be an increasing sequence of primes. No theorem of Littlewood-Paley type is available for the group $\mathrm{P}^*{}_{n=1}^\infty Z(q_n)$, and so we are not able to prove an analogue of Theorem 3.7 for this group.(1) Thus we prove only

THEOREM 5.6. *Fix p with $1<p<2$. There exists $E\subseteq \mathrm{P}^*{}_{n=1}^\infty Z(q_n)$ such that $M^p|_E\neq M^p_E$.*

PROOF. In view of the open mapping theorem, it suffices to produce E and a sequence $\{\phi_n\}_{n=1}^\infty$ of finitely supported functions in $l^\infty(E)$ having the following two properties:

(a) $\|\phi_n\|_{E,p,p}\leqslant 1$;

(b) $\|\Phi_n\|_{p,p}>n$ if $\Phi_n\in M^p$ is such that $\Phi_n|_E=\phi_n$.

By passing to a subsequence of $\{q_n\}_{n=1}^\infty$ if necessary, and by using Lemmas 2.2, 4.2, and 5.1, we can assume that the following holds: there exist subsets $E_n\subseteq Z(q_n)$ and functions $\phi_n\in l^\infty(E_n)$ satisfying $\|\phi_n\|_{E_n,p,p}\leqslant 1$ but such that

$$\|\Phi_n\|_{Z(q_n),p,p}>n$$

if $\Phi_n|_{E_n}=\phi_n$. Let $E=\bigcup_{n=1}^\infty E_n$ where each $E_n\subseteq Z(q_n)$ is considered in the usual way as a subset of $\mathrm{P}^*{}_{n=1}^\infty Z(q_n)$, and consider ϕ_n as a function on E with support contained in E_n. Since $E\cap Z(q_n)=E_n$, it follows that $\|\phi_n\|_{E_n,p,p}=\|\phi_n\|_{E,p,p}$. Then it is easy to see that the sequence $\{\phi_n\}_{n=1}^\infty$ satisfies both (a) and (b), and so the proof of the theorem is complete.

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(1) The referee has pointed out to us that a Littlewood-Paley theorem is available for $\mathrm{P}^*{}_{n=1}^\infty Z(q_n)$ (see R. Spector, Bull. Soc. Math. France, Mémoire 24, 1970, Chapter V), and so an analogue of Theorem 3.7 does hold for this group. Thus Theorem 1.1 can be strengthened: THEOREM. *If G is an infinite compact abelian group and if $1\leqslant p<2$, then there exists $E\subseteq X$ and a multiplier ϕ of $\widehat{L^p_E}$ into $\widehat{L^2}$ such that ϕ is not in $M^p|_E$.*

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