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EVERY WEAK PROPER HOMOTOPY EQUIVALENCE IS WEAKLY PROPERLY HOMOTOPIC TO A PROPER HOMOTOPY EQUIVALENCE

BY

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ABSTRACT. We prove that every weak proper homotopy equivalence of σ-compact, locally compact Hausdorff spaces is weakly properly homotopic to a proper homotopy equivalence.

In [6, pp. 489–491], L. Siebenmann obtained a convenient criterion $(\pi_*)_\infty$ for a proper map of finite dimensional polyhedra (polyhedron = locally finite simplicial complex) to be a proper homotopy equivalence. Later, E. M. Brown [1, p. 23], and F. T. Farrell, L. R. Taylor, and J. B. Wagoner [5, Theorem 3.5] claimed to be able to remove the finite dimensional assumption. In [4] we give an example which shows that the finite dimensional assumption *is* necessary. On the positive side, we prove in §2 the following useful (see [2], [8]) theorem.

(1.1) THEOREM. *Let f: $X \to Y$ be a proper map of σ-compact, locally compact Hausdorff spaces. If f is a weak proper homotopy equivalence, then f is weakly properly homotopic to a proper homotopy equivalence.*

(1.2) REMARKS. Chapman and Siebenmann, in developing an obstruction theory for putting boundaries on noncompact Q-manifolds (Q denotes the Hilbert cube) [8], asked two questions.

(1) Is every weak proper homotopy equivalence a proper homotopy equivalence?

(2) Is every weak proper homotopy equivalence weakly properly homotopic to a proper homotopy equivalence?

They obtained the following independent results in [8]. For finite dimensional polyhedra, (1) holds by Siebenmann's $(\pi_*)_\infty$ criterion. For polyhedra with tame

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ends, (2) holds. This uses [7], [8], [9], [10], and [11]. (Note that any Q-manifold can be triangulated as $X \times Q$ for some polyhedron X, and conversely [7], [11].)

In [3] we develop machinery which allows us to prove that a proper map which is a homotopy equivalence and a homotopy equivalence at ∞ (appropriately defined) is a proper homotopy equivalence.

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2. **Proof of Theorem.** We begin by extending Chapman's definition of proper map to a definition of filtered map of filtered spaces. A *filtered space* X consists of a space X, together with a sequence of closed subspaces

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots,$$

with each $X_{n-1} \subset \operatorname{int} X_n$. A map $f\colon X \to Y$ of filtered spaces is called a *filtered map* if for each number $n \geqslant 0$ there exists a number $m \geqslant 0$ with $f(X_m) \subset Y_n$. A filtered space X induces a natural filtration on its cylinder $X \times I$; this yields a natural notion of *filtered homotopy*. There results a filtered category and its filtered homotopy category.

The connection with the proper homotopy theory of σ-compact, locally compact (Hausdorff) spaces (for example, countable locally finite simplicial complexes) is made as follows. Suppose $X = \bigcup_{n=0}^{\infty} K_n$ where $K_0 = \varnothing$, each K_n is compact, and each $K_{n-1} \subset \operatorname{int} K_n$. We associate to X the filtered space

$$\epsilon(X) = \{X = X_0 \supset X_1 \supset X_2 \supset \cdots\}, \quad \text{where}$$
$$X_n \ (=(\epsilon(X))_n) = \operatorname{cl}(X \backslash K_n).$$

$\epsilon(X)$ is called the *end* of X. Ends are unique up to canonical filtered isomorphism. Clearly, a map $f\colon X \to Y$ of σ-compact, locally compact spaces is proper (following Chapman) if and only if f induces a filtered map $\epsilon(f)\colon \epsilon(X) \to \epsilon(Y)$.

We associate to an end the following *mapping telescope*. The telescope of a diagram

$$\bar{X} = \left\{X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \cdots\right\}$$

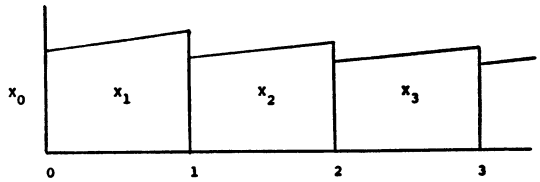
is the space

$$\operatorname{Tel}(\bar{X}) = X_0 \times 0 \cup_{f_1} X_1 \times [0, 1] \cup_{f_2} X_2 \times [1, 2] \cup_{f_3} \cdots,$$

which is filtered by setting

$$\operatorname{Tel}(\bar{X})_n = X_n \times n \cup_{f_{n+1}} X_{n+1} \times [n, n+1] \cup_{f_{n+2}} \cdots,$$

for $n \geqslant 0$.



Tel(X)

Let X be a σ-compact, locally compact space, with $X = \bigcup_{n=0}^{\infty} K_n$, $K_0 = \emptyset$, each K_n compact, and $K_{n-1} \subset \operatorname{int} K_n$. There is a natural projection p: $\operatorname{Tel}(\epsilon(X)) \to X$ ($\epsilon(X)$ is bonded by the inclusions f_i); clearly p is a filtered map. A *proper section* for $\epsilon(X)$ is a filtered map s: $X \to \operatorname{Tel}(\epsilon(X))$ with $ps = \mathrm{id}_x$.

(2.1) *Construction of proper sections.* The Tietze extension theorem yields maps h_n: $\operatorname{cl}(K_n \backslash K_{n-1}) \to [0, 1]$ with $h_n(\operatorname{bd} K_{n-1}) = n - 2$ and $h_n(\operatorname{bd} K_n) = n - 1$ for $n \geqslant 2$. We may glue these maps together to obtain a proper map h: $X \to R^+$ (*notation*: R^+ denotes the set of nonnegative real numbers) such that $h(K_n \backslash K_{n-1}) \subset [n - 2, n - 1]$ for $n \geqslant 1$. Because $(X_{n-1} \backslash X_n) \subset \operatorname{cl}(K_n \backslash K_{n-1})$, there results a map s: $X \to \operatorname{Tel}(\epsilon(X))$, given by the formula

$$s(x) = (x, h(x)). \tag{2.2}$$

Clearly, s is a proper section for $\epsilon(X)$. In fact, each proper section s' for $\epsilon(X)$ comes from a suitable proper map h': $X \to R^+$ and formula (2.2).

Finally, we introduce a relation of *vertical homotopy* into the set of maps $\{X \to \operatorname{Tel}(\epsilon(Y))\}$. Let p: $\operatorname{Tel}(\epsilon(y)) \to Y$ denote the projection. Maps f_0, f_1: $X \to \operatorname{Tel}(\epsilon(Y))$ with $pf_0 = pf_1 = f$: $X \to Y$ are called vertically homotopic if there is a homotopy $H = \{H_t\}$: $X \times I \to \operatorname{Tel}(\epsilon(Y))$ with $H_0 = f_0$, $H_1 = f_1$, and $pH_t = f$ for all t. We call H a *vertical homotopy*. If f_0, f_1, and H are also filtered maps, f_0 and f_1 are called *filtered-vertically-homotopic*. Clearly, any two proper sections for $\epsilon(X)$ are filtered-vertically-homotopic.

We may now begin the proof of Theorem 1.1. Let f: $X \to Y$ be a weak proper homotopy equivalence of σ-compact, locally compact (Hausdorff) spaces. Choose a proper map g: $Y \to X$ which is a weak-proper-homotopy inverse to f. Then choose ends

$$\epsilon(X) = \{X = X_0 \supset X_1 \supset X_2 \supset \cdots\}, \quad \epsilon(Y) = \{Y = Y_0 \supset Y_1 \supset Y_2 \supset \cdots\},$$

such that

(i) $X_n \subset \operatorname{int}(X_{n-1})$ and $Y_n \subset \operatorname{int}(Y_{n-1})$ for $n \geqslant 1$,

(ii) $f(X_n) \subset Y_n$ for $n \geqslant 0$,

(iii) $g(Y_n) \subset X_{n-1}$ for $n \geqslant 1$,

(iv) there exist homotopies H_n: $X \times I \to X$ with $H_n|_0 = \mathrm{id}$ and $H_n|_1 = gf$ for $n \geqslant 0$, and further, $H_n(X_n \times I) \subset X_{n-1}$ for $n \geqslant 1$, and

(v) There exists homotopies $K_n\colon Y \times I \to Y$ with $K_n|_0 = \mathrm{id}$ and $K_n|_1 = fg$ for $n \geqslant 0$, and further, $K_n(Y_n \times I) \subset Y_{n-1}$ for $n \geqslant 1$.

(Such ends are easily obtained by successively passing to cofinal subsystems of any ends of X and Y.)

We shall use the above data to construct a suitable proper homotopy equivalence $f'\colon X \to Y$. Write f_n for $f|_{X_n}$ and g_n for $g|_{Y_n}$. Let $\bar{Z}$ be the inverse system

$$Y_0 \xleftarrow{f_0} X_0 \xleftarrow{g_1} Y_1 \xleftarrow{f_1} X_1 \xleftarrow{g_2} \cdots.$$

Form the homotopy-commutative diagram

$$\begin{tikzcd}
\epsilon(X): & & X_0 \arrow[d, shift right] & X_1 \arrow[l, hook'] \arrow[d, shift right] & X_2 \arrow[l, hook'] \arrow[d, shift right] & \cdots \arrow[l, hook'] \\
\bar{Z}: \arrow[u, shift right] \arrow[d, shift left] & Y_0 \arrow[r, shift left] & X_0 \arrow[l, "f_0"'] \arrow[u, shift left] \arrow[d, shift right] & Y_1 \arrow[l, "g_1"'] \arrow[r, shift left] & X_1 \arrow[l, "f_1"'] \arrow[u, shift left] \arrow[d, shift right] & Y_2 \arrow[l, "g_2"'] \arrow[r, shift left] & X_2 \arrow[l, "f_2"'] \arrow[u, shift left] & \cdots \arrow[l] \\
\epsilon(Y): & Y_0 \arrow[u, shift left] & & Y_1 \arrow[l, hook'] \arrow[u, shift left] & & Y_2 \arrow[l, hook'] \arrow[u, shift left] & & \cdots, \arrow[l, hook']
\end{tikzcd}$$

in which the vertical arrows denote the appropriate identity maps, and the required homotopies are given by conditions (iv) and (v) above.

Diagram (2.3), together with the homotopies (iv) and (v), yields filtered (but *not* proper) maps of mapping telescopes

$$\mathrm{Tel}(\epsilon(X)) \underset{F'}{\overset{F}{\rightleftarrows}} \mathrm{Tel}(\bar{Z}) \underset{G}{\overset{G'}{\rightleftarrows}} \mathrm{Tel}(\epsilon(Y)). \tag{2.4}$$

In order to give explicit formulas for these maps, we regard

$$\mathrm{Tel}\left(W_0 \xleftarrow{h_0} W_1 \xleftarrow{h_1} W_2 \xleftarrow{h_2} \cdots\right)$$

as the union of the "levels" $W_{n-1} \times 0 \cup_{h_{n-1}} W_n \times [0, 1]$. Then F maps $X_{n-1} \times 0 \cup X_n \times [0, 3/4]$ to $X_{n-1} \times 0 \cup_{g_n} Y_n \times [0, 1]$ by the formula

$$F(x, t) = \begin{cases} (H_n(x, 2t), 0) \in X_{n-1} \times 0 & \text{for } 0 \leqslant t \leqslant 1/2, \\ (g_n f_n(x), 0) \in X_n \times 0 & \text{for } t = 1/2, \\ (f_n(x), 4t - 2) \in Y_n \times [0, 1] & \text{for } 1/2 \leqslant t \leqslant 3/4, \end{cases}$$

and F maps $X_n \times [3/4, 1]$ to $Y_n \times 0 \cup_{f_n} X_n \times [0, 1]$ by the formula

$$F(x, t) = \begin{cases} (f_n(x), 0) \in Y_n \times 0 & \text{for } t = 3/4, \\ (x, 4t - 3) \in X_n \times [0, 1] & \text{for } 3/4 \leqslant t \leqslant 1. \end{cases}$$

See Figure I. The map G in diagram (2.4) has an analogous description.

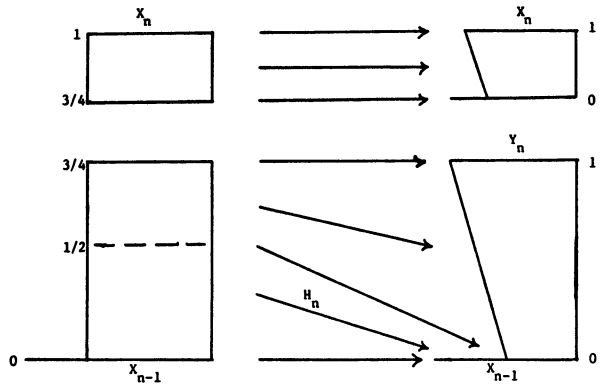


FIGURE I

The map F': $\mathrm{Tel}(\bar{Z}) \to \mathrm{Tel}(\epsilon(X))$ maps $X_{n-1} \times 0 \cup_{g_n} Y_n \times [0, 1]$ into $X_{n-1} \times 0 \cup X_n \times [0, 1]$ according to the formula

$$\begin{cases} F'(x, 0) = (x, 0), \\ F'(y, t) = (g_n(y), 0), \end{cases}$$

and maps $Y_n \times 0 \cup_{f_n} X_n \times [0, 1]$ into $X_{n-1} \times 0 \cup X_n \times [0, 1]$ according to the formula

$$\begin{cases} F'(y, 0) = (g_n(y), 0), \\ F'(x, t) = \begin{cases} (g_n f_n(x), 0) & \text{for } t = 0, \\ (H_n(x, 1 - 2t), 0) & \text{for } 0 \leqslant t \leqslant 1/2, \\ (x, 2t - 1) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases} \end{cases}$$

See Figure II. The function $(\ , t) \to (H_n(\ , 1 - 2t), 0)$ used in the definition of F' is indicated by "$-H$" in Figure II to indicate that $1 - 2t$ decreases as t increases. The map G' in diagram (2.4) has an analogous description.

Our construction also yields *filtered homotopies*

$$\begin{aligned} &F'F \sim \mathrm{id}_{\mathrm{Tel}(\epsilon(X))}, FF' \sim \mathrm{id}_{\mathrm{Tel}(\bar{Z})}, \quad \text{and} \\ &G'G \sim \mathrm{id}_{\mathrm{Tel}(\epsilon(Y))}, GG' \sim \mathrm{id}_{\mathrm{Tel}(\bar{Z})}. \end{aligned} \tag{2.5}$$

These homotopies arise from first deforming maps H: $X_n \times I \to X_{n-1}$ of the form H_n + "$-H_n$":

$$H(x, t) = \begin{cases} H_n(x, 2t) & \text{for } 0 \leqslant t \leqslant 1/2, \\ H_n(x, 2 - 2t) & \text{for } 1/2 \leqslant t \leqslant 1 \end{cases}$$

to the constant homotopy while keeping the top and bottom of the cylinder fixed, and then moving at most two levels within the telescopes. The first two homotopies are shown in Figures IIIa and IIIb. The formulas are complicated and not enlightening, and therefore omitted.

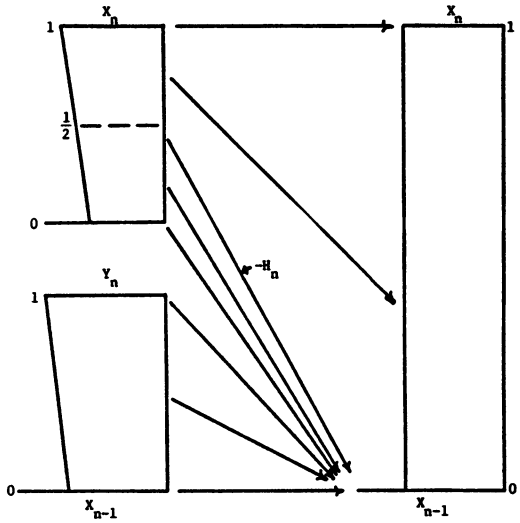
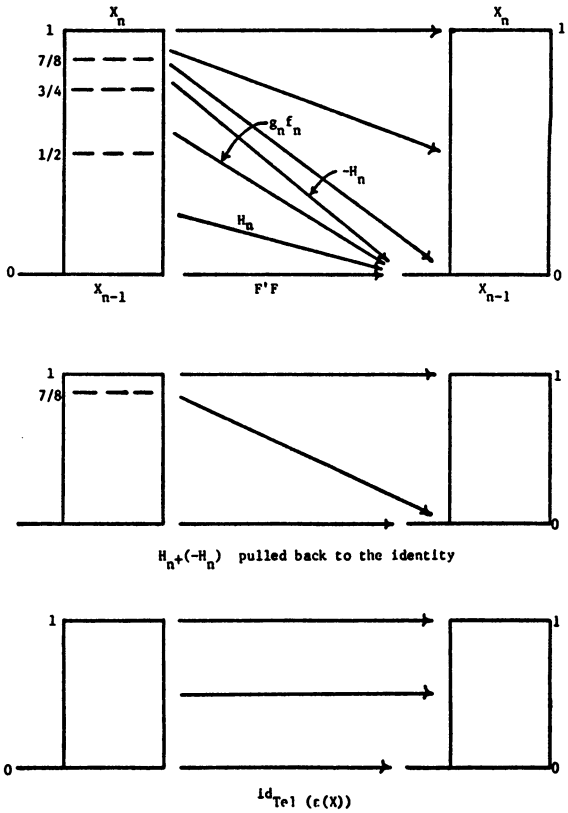
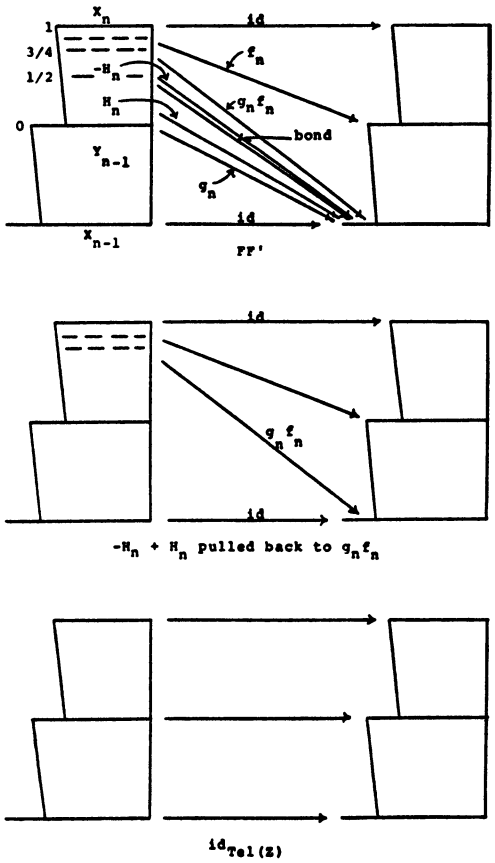


FIGURE II



Deforming F^1F to $id_{Tel(c(X))}$

FIGURE IIIa



Deforming FF' to $\mathrm{id}_{\mathrm{Tel}(Z)}$

FIGURE IIIb

Now write p: $\mathrm{Tel}(\epsilon(X)) \to X$ and p': $\mathrm{Tel}(\epsilon(Y)) \to Y$ for the projections. Choose proper maps h: $X \to R^+$ and h': $Y \to R^+$ with the property that $h(X_n \backslash X_{n-1}) \subset [n-2, n-1]$ and $h'(Y_n \backslash Y_{n-1}) \subset [n-2, n-1]$ as in (2.1). (This is possible by property (i) of $\epsilon(X)$ and $\epsilon(Y)$.) As in formula (2.2), there result proper sections s for $\epsilon(X)$ and s' for $\epsilon(Y)$, respectively. Define maps f': $X \to Y$ and g': $Y \to X$ to be the composites

$$(2.6)\qquad \begin{cases} X \xrightarrow{s} \mathrm{Tel}(\epsilon(X)) \xrightarrow{F} \mathrm{Tel}(\bar{Z}) \xrightarrow{G'} \mathrm{Tel}(\epsilon(Y)) \xrightarrow{p'} Y, \\ Y \xrightarrow{s'} \mathrm{Tel}(\epsilon(Y)) \xrightarrow{G} \mathrm{Tel}(\bar{Z}) \xrightarrow{F'} \mathrm{Tel}(\epsilon(X)) \xrightarrow{p} X, \end{cases}$$

respectively. Because f' and g' are composites of filtered maps, they are filtered maps, and thus proper maps.

To complete the proof, we shall verify the following claims.

Claim 1. The maps f' and g' are proper homotopy inverses, hence the map f': $X \to Y$ is a proper homotopy equivalence.

Claim 2. The maps $f', f\colon X \to Y$ are weakly-properly-homotopic.

VERIFICATION OF CLAIM 1. Consider the commutative solid-arrow diagram

$$\begin{tikzcd}
\mathrm{Tel}(\epsilon(X)) \arrow[r, "G'F"] & \mathrm{Tel}(\epsilon(Y)) \arrow[r, dashed, "\mathrm{id}"] \arrow[dr, "p'"] & \mathrm{Tel}(\epsilon(Y)) \arrow[r, "F'G"] & \mathrm{Tel}(\epsilon(X)) \arrow[d, "p"] \\
X \arrow[u, "s"] \arrow[r, "f'"] & Y \arrow[ur, "s'"] \arrow[r, "g'"] & X. &
\end{tikzcd} \tag{2.7}$$

There exists a vertical homotopy H between the maps

$$H_0 \equiv s'p' \circ G'F \circ s,\ H_1 \equiv G'F \circ s\colon X \rightrightarrows \mathrm{Tel}(\epsilon(Y));$$

given by the formula

$$H(x, \tau) = (f'(x), (1 - \tau) \cdot \pi(H_0(x)) + \tau \cdot \pi(H_1(x)))$$

where π denotes the projection $\mathrm{Tel}(\epsilon(Y)) \to R^+$. Because

$$\pi H_0(x) = h'f'(x) \quad \text{and} \quad |\pi H_1(x) - h(x)| \leqslant 2$$

by construction, πH_1 is a proper map and H is a *filtered* vertical homotopy. Hence the maps

$$g'f', p \circ F'G \circ G'F \circ s\colon X \rightrightarrows X$$

are properly homotopic. But the composite $F' \circ GG' \circ F$ is *filtered-homotopic* to $\mathrm{id}_{\mathrm{Tel}(\epsilon(X))}$ by composites of the filtered homotopies of formula (2.5). Hence the map $g'f'$ is properly homotopic to id_X. Similarly, $f'g'$ is properly homotopic to id_Y, and Claim 1 follows.

VERIFICATION OF CLAIM 2. The construction of the telescopes $\mathrm{Tel}(\epsilon(X))$, $\mathrm{Tel}(\epsilon(Y))$ and $\mathrm{Tel}(\bar{Z})$ is readily extended to give telescopes

$$\mathrm{Tel}(\tilde{X}) \equiv \mathrm{Tel}\left(X \xleftarrow{\mathrm{id}} X \xleftarrow{\mathrm{id}} \cdots\right) \cong X \times R^+,$$

$$\mathrm{Tel}(\tilde{Y}) \equiv \mathrm{Tel}\left(Y \xleftarrow{\mathrm{id}} Y \xleftarrow{\mathrm{id}} \cdots\right) \cong Y \times R^+,$$

$$\mathrm{Tel}(\tilde{Z}) \equiv \mathrm{Tel}\left(Y \xleftarrow{f} X \xleftarrow{g} Y \xleftarrow{f} X \xleftarrow{g} \cdots\right).$$

Also, because the homotopies H_n and K_n (used to define the maps $F\colon \mathrm{Tel}(\epsilon(X)) \to \mathrm{Tel}(\tilde{Z})$ and $G'\colon \mathrm{Tel}(\tilde{Z}) \to \mathrm{Tel}(\epsilon(Y))$ (see formula (2.4) and the following discussion)) are defined on all of $X \times I$ and $Y \times I$ respectively (see conditions (iv) and (v), above) we obtain maps $\tilde{F}\colon \mathrm{Tel}(\tilde{X}) \to \mathrm{Tel}(\tilde{Z})$ and $\tilde{G}'\colon \mathrm{Tel}(\tilde{Z}) \to \mathrm{Tel}(\tilde{Y})$ and a commutative diagram

$$\begin{tikzcd} \mathrm{Tel}(\epsilon(X)) \arrow[r, "F"] \arrow[d, hook] & \mathrm{Tel}(\bar{Z}) \arrow[r, "G'"] \arrow[d, hook] & \mathrm{Tel}(\epsilon(Y)) \arrow[d, hook] \\ \mathrm{Tel}(\widetilde{X}) \arrow[r, "\widetilde{F}"] & \mathrm{Tel}(\widetilde{Z}) \arrow[r, "\widetilde{G}'"] & \mathrm{Tel}(\widetilde{Y}). \end{tikzcd} \tag{2.8}$$

Now let B be a compactum in Y. We shall first define an auxiliary map f'': $X \to Y$ depending upon B, a homotopy H: $X \times I \to Y$ with $H_0 = f'$ and $H_1 = f''$, and a compactum A in X such that $H((X \backslash A) \times I) \subset (Y \backslash B)$. Both H and A depend upon B. We shall then define a similar homotopy from f'' to f.

To define f'', choose an integer n so that $Y_n \subset (Y \backslash B)$. Let h_n: $X \to R^+$ be the map

$$h_n(x) = \min(h(x), n)$$

where h is the map used to construct a proper section for $\epsilon(X)$. Let f'': $X \to Y$ be the composite $p'\widetilde{G}'\widetilde{F}(x, h_n(x))$. (Recall that p' is the projection $\mathrm{Tel}(\epsilon(Y)) \to Y$. See Figure IV.

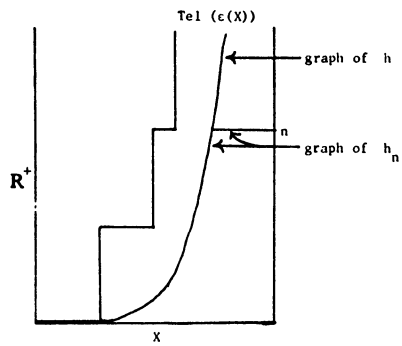


FIGURE IV

The required homotopy is given by

$$H(x, \tau) = p' \circ \widetilde{G}'\widetilde{F}(x, (1 - \tau) \cdot h(x) + \tau \cdot h_n(x)).$$

Then $H(X_{n+1} \times I) \subset Y_n)$. If A is the compactum $\mathrm{cl}(X \backslash X_{n+1})$, then

$$H((X \backslash A) \times I) \subset (Y \backslash B),$$

as required.

It remains to find a compactum A' in X and a homotopy H': $X \times I \to Y$ such that $H'_0 = f''$, $H''_1 = f$, and $H'((X \backslash A') \times I) \subset (Y \backslash B)$. To do this, observe that

$$p' \circ \widetilde{G}'\widetilde{F}(x, n) = f(x),$$

so that H' may be given by the formula

$$H'(x, \tau) = p' \circ \widetilde{G}'\widetilde{F}(x, (1 - \tau) \cdot h_n(x) + \tau \cdot n).$$

See Figure V. Finally, $H'^{-1}(Y \backslash B) \supset H'^{-1}(Y_n) \supset X_{n+1} \times I$ so that we may take $A' = \mathrm{cl}(X \backslash X_{n+1})$. □

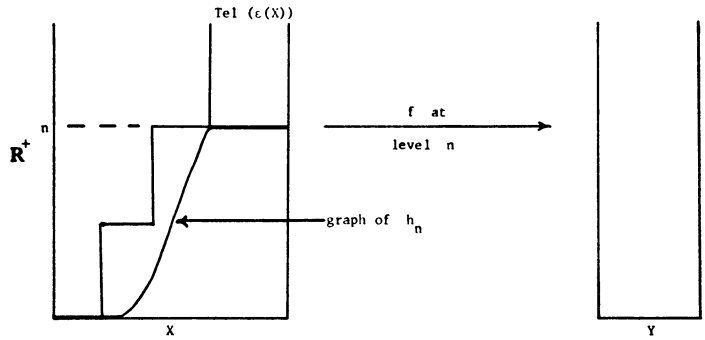


FIGURE V

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