

THE INFINITESIMAL STABILITY OF SEMIGROUPS
 OF EXPANDING MAPS

BY

RICHARD SACKSTEDER

ABSTRACT. The concept of C^∞ infinitesimal stability for representations of a semigroup by C^∞ maps is defined. In the case of expanding linear maps of the torus T^d it is shown that certain algebraic conditions assure such stability.

1. Introduction. Let S be an abstract semigroup and let $\phi: S \rightarrow C^\infty(X, X)$ be a representation of S by C^∞ maps of a manifold X so that $\phi(st) = \phi(s) \circ \phi(t)$. Let $\pi: T(X) \rightarrow X$ be the tangent bundle of X . A map $\Psi: S \rightarrow C^\infty(X, T(X))$ is called a C^∞ infinitesimal perturbation of ϕ if:

- (a) $\pi \circ \Psi(s) = \phi(s)$, that is $\Psi(s)$ is a section of the pullback of $T(X)$ by $\phi(s)$, and
- (b) for every s and t in S ,

$$(1.1)_{st} \quad \Psi(st) = \Psi(s) \circ \phi(t) + D\phi(s) \circ \Psi(t).$$

All of the terms are sections of the pullback of $T(X)$ by $\phi(st)$, so the addition makes sense. The motivation for the relation (1.1)_{st} is as follows: If $\phi_\epsilon(s)$ is a family of representations depending smoothly on the parameter ϵ , $\phi_0(s) = \phi(s)$ and $d\phi(s)/d\epsilon|_{\epsilon=0} = \Psi(s)$, then (1.1)_{st} is the result of differentiating $\phi_\epsilon(st) = \phi_\epsilon(s)\phi_\epsilon(t)$ at $\epsilon = 0$.

An infinitesimal deformation Ψ of ϕ is called a C^∞ infinitesimal conjugate if there is a vector field f such that for every s in S ,

$$(1.2)_s \quad f \circ \phi(s) - D\phi(s) \circ f = \Psi(s).$$

Here every term is a section of the pullback of $T(X)$ by $\phi(s)$. The motivation for (1.2)_s is analogous to that for (1.1)_{st}. The representation ϕ is said to be infinitesimally C^∞ stable if every C^∞ infinitesimal perturbation of ϕ is a C^∞ infinitesimal conjugate of ϕ .

If S is a free semigroup $\Psi(s)$ can be specified arbitrarily, except for the condition (a) above, on a free set of generators, and then (1.1)_{st} determines Ψ on

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all of S . When there are relations in the semigroup, $(1.1)_{st}$ yields constraints on $\Psi(s)$ even if s is a generator. Obviously, the more relations in S , the easier it is for a representation to be C^∞ stable.

As the motivation described above suggests, the concept of infinitesimal stability is of interest because of its relevance to smooth one parameter families of deformation of representations. However, the concept is of further potential interest because it may be of use in proving the stability of representations via the technique of the Moser implicit functions theorem (cf. [3] or [6]). There are difficulties in applying the technique to stability problems for semigroups, stemming from the fact that the smoothing operators tend to destroy relations in the semigroup. I have not yet been able to overcome these difficulties for the representations considered below. A special case of the problems treated here is discussed in [4]. An alternate approach to the problem of relating infinitesimal stability to stability might be by means of an analogue of the Mather theorem (cf. [1] or [2]) applicable to representations of semigroups.

2. Special representations. It will always be assumed below that the semigroup S is finitely generated by elements s_0, \dots, s_k , where s_0 is central, that is $s_0 t = t s_0$ for every t in S . Of course, $s_0 t = t s_0$ for all t follows from the special cases $t = s_i, i = 1, \dots, k$. Thus, $\Psi(s_0 s_i) = \Psi(s_i s_0)$ and the corresponding relations $(1.1)_{st}$ provide the constraints on the infinitesimal deformations of representation ϕ of S .

Another specialization employed below is that X will be the d -torus, $T^d = R^d/Z^d$, and the representation ϕ will be by linear maps, that is by maps whose lifts to R^d are linear. The following notation will be used:

$$N_i = \text{lift of } \phi(s_i) \text{ to } R^d,$$

$$G_i = \text{lift of } \Psi(s_i) \text{ to } R^d.$$

Then N_i is a nonsingular $d \times d$ matrix with integer entries, and $G_i(x) \equiv G_i(x + n)$ for all x in R^d and n in Z^d .

Now the constraints imposed on G_i are

$$(2.1)_i \quad G_0 \circ N_i - N_i \circ G_0 = G_i \circ N_0 - N_0 \circ G_i,$$

and G_i is an infinitesimal conjugate if there is a solution F of

$$(2.2)_i \quad F \circ N_i - N_i \circ F = G_i,$$

where F is the lift of the f in $(1.2)_s$.

Our main theorem is as follows:

THEOREM 2.1. *Let S be a semigroup generated by s_0, \dots, s_k , where s_0 is central, and let ϕ be a representation of s by linear maps of T^d . Then ϕ is C^∞ infinitesimally stable provided*

(2.4) N_i is an expanding map ($i = 0, \dots, k$), and for any n in Z^d ,

(2.5) $N_i^* n \equiv 0 \pmod{N_0^*}$ ($i = 1, \dots, k$) implies $n \equiv 0 \pmod{N_0^*}$.

The condition (2.4) means that for some $\lambda > 1$, $|N_i x| \geq \lambda |x|$ holds for every x in R^d (cf. [5]), and in (2.5) $m \equiv 0 \pmod{N_0^*}$ means that $m = N_0^* z$ for some z in Z^d . Here the asterisk denotes transpose with respect to the usual inner product on R^d .

Actually, a somewhat stronger result than Theorem 2.1 will be proved. This result (Theorem 4.1) below states roughly that if the relations (2.1)_{*i*} are satisfied approximately, (2.2)_{*i*} will be satisfied approximately. This stronger version of Theorem 2.1 is proved here because it seems likely that applications to stability questions of the sort discussed at the end of the preceding section will require it.

3. **Lemmas.** *Let Y be a real finite dimensional vector space with inner product denoted by $u \cdot v$ and norm $|u|^2 = u \cdot u$. The norm $\|F\|_\alpha$ ($\alpha \geq 0$) for functions $F: T^d \rightarrow Y$ is defined by*

$$\|F\|_\alpha = \sum (1 + |n|)^\alpha |a_n|,$$

where the summation is over all n in Z^d , $F \sim \sum a_n e(nx)$ is the Fourier series for F , and $e(nx) = \exp(2\pi i n \cdot x)$. Note that each a_n is an element of Y . The space of F 's such that $\|F\|_\alpha < \infty$ is a Banach space which we denote by B_α .

If $L^2 = L^2(T^d, Y)$ is the space of square integrable functions on T^d with values in Y , and t_i is a nonsingular linear map of T^d which lifts to N_i on R^d , $V_i F(x) = F(t_i(x))$ defines an isometry of L^2 and bounded linear map of the function spaces B_α ($\alpha \geq 0$). If V_i^* is the adjoint of V_i on L^2 , V_i^* is a left inverse of V_i . Since $B_\alpha \subseteq B_0 \subseteq L^2$ for $\alpha \geq 0$, V_i^* acts on B_α and is easily seen to be a contraction on any B_α . It is easily checked that V_i^* can be written explicitly as follows:

$$V_i^* F(x) = |\det N_i|^{-1} \sum_z F(N_i^{-1}(x + z)),$$

where the summation is over elements z of Z^d such that $N_i^{-1}z$ is in some fixed fundamental domain for T^d . (Here and below we make use of the fact that functions F on T^d can be viewed as functions on R^d satisfying $F(x) = F(x + z)$ for all z in Z^d .)

If M^p is a p -linear map from R^d to R^d and N is a linear map from R^d to R^d , $M^p N_p$ will denote the p -linear map defined by:

$$M^p N_p(\xi^1, \dots, \xi^p) = M^p(N\xi^1, \dots, N\xi^p),$$

where each ξ^j is in R^d . The norm for such a map M^p is defined in terms of a fixed norm on R^d by:

$$|M^p| = \text{Sup} \left\{ |M^p(\xi^1, \dots, \xi^p)| : \prod_{j=1}^p |\xi^j| \leq 1 \right\}.$$

LEMMA 3.1. *Let N be an expanding $d \times d$ matrix. Then for any $\theta > 0$, there is a $P = P(\theta, N)$ such that if $p \geq P$,*

$$|N \circ M^p N^{-1}| \leq \theta |M^p|$$

holds for every p -linear map M^p .

The proof is an immediate consequence of the definition of the norm of a p -linear map and the fact for any expanding linear map N , $|N^{-1}| < 1$.

If G_i is as above, G_i^p will denote the p th derivative of G_i ($0 \leq p < \infty$). If $p \geq 1$, $G_i^p(x)$ is at each point x in R^d a p -linear map from R^d to R^d . Moreover, $G_i^p(x) = G_i^p(x + z)$ for z in Z^d , so $G_i^p(x)$ can be regarded as defined for x in T^d . Since N_j is linear, the p th derivative of $G_i \circ N_j$ at x in T^d is given by $V_j G_i^p(x) N_{jp}$. Defining F^p analogously to G_i^p and differentiating (2.2) _{i} p times, gives

$$(3.1)_i^p \quad V_i F^p N_{ip} - N_i \circ F^p = G_i^p.$$

(The case $p = 0$ is understood as (2.2) _{i} itself.)

Applying V_i^* to this equation and composing each component of the multilinear part with N_i^{-1} on the right gives

$$(3.2)_i^p \quad F^p - N_i \circ V_i^* F^p N_{ip}^{-1} = V_i^* G_i^p N_{ip}^{-1}.$$

The next lemma shows the existence of solutions of equations of the same form as (3.2) _{i} ^{p} .

LEMMA 3.2. *Let t be an expanding map of T^d , suppose that the linear map N is the lift of t to R^d and let V be the corresponding isometry of $L^2(T^d)$. Let B be a C^∞ map from T^d to R^d and let B^q denote its q th derivative. Then there is a unique solution F of*

$$(3.3)^0 \quad F - N \circ V^* F = V^* B,$$

whose q th derivative F^q satisfies

$$(3.3)^q \quad F^q - N \circ V^* F^q N_q^{-1} = V^* B^q N_q^{-1}.$$

For sufficiently large α , $\|F\|_\alpha \leq \text{const } \|B\|_\alpha$.

(Here V^* on the right side of (3.3) ^{q} and (3.3) ^{0} has no effect since V^* is onto. The relations are written as they are to emphasize their similarity to (3.2) _{i} ^{p} .)

PROOF. Define

$$(3.4)^p \quad F^p = \sum_{s=1}^{\infty} N^s \circ V^{-s} B^p N_p^{-s},$$

where V^{-s} means $(V^*)^s$. If p is large enough the series converges uniformly on T^d by Lemma 3.1. For such large p , the formal computation showing that F^p satisfies (3.3)^p is justified.

The difference Δ^p of two solutions of (3.3)^p would have to satisfy $\Delta^p = N \circ V^{-1} \Delta^p N_p^{-1}$ hence by iteration $\Delta^p = N^s \circ V^{-s} \Delta^p N_p^{-s}$. Applying Lemma 3.1 shows that for large p the right side goes uniformly to 0 as $s \rightarrow \infty$, hence the solution F^p of (3.3)^p is unique for large p at least.

Formally it is clear from the explicit formula for V^* given above that F^{q+1} is the derivative of F^q and there is no difficulty in justifying the formalities for $q \geq p$.

The proposition in the appendix shows how F and F^q for $0 \leq q < p$ are constructed. Note that here $F(0)$ cannot be chosen arbitrarily. To see this let $F_0(x)$ be the antiderivative of $F^1(x)$ satisfying $F_0(0) = 0$. Setting $F(x) = F(0) + F_0(x)$ and substituting in (3.3)⁰ with $x = 0$ gives

$$(I - N) \circ F(0) = N \circ (V^* F_0)(0) + (V^* B)(0).$$

Since $I - N$ is invertible, $F(0)$ is uniquely determined by this relation.

The estimates for $\|F\|_\alpha$ for $\alpha \geq p$ follow from those in the proposition in the appendix and simple estimates for $F(0)$.

The next lemma depends on the relation

$$(3.5)_i^0 \quad G_0 \circ N_i - N_i \circ G_0 - G_i \circ N_0 + N_0 \circ G_i = R_i,$$

where R_i is a map from R^d to R^d satisfying $R_i(x) \equiv R_i(x + z)$ for x in R^d and z in Z^d . The relation (3.5)⁰_i generalizes (2.1)_i. Differentiating it p times gives

$$(3.5)_i^p \quad V_i G_0^p N_{ip} - N_i \circ G_0^p - V_0 G_i^p N_{0p} + N_0 \circ G_i^p = R_i^p.$$

LEMMA 3.3. Assume (3.2)⁰ and (3.5)^p_i for $p \geq 0$ and some $i = 1, \dots, k$. Let $H_i^p = F^p N_{ip} - N_i \circ V_i^* F^p - V_i^* G_i^p$. Then

$$(3.6) \quad H_i^p N_{0p} - N_0 \circ V_0^* H_i^p = -V_0^* V_i^* R_i^p,$$

hence, by Lemma 3.2, $\|H_i\|_\alpha \leq \text{const} \|R_i\|_\alpha$. In particular, F^p satisfies (3.2)^p_i if $R_i = 0$.

PROOF. Let $h_0 = V_0 F^p N_{0p} - N_0 \circ F^p$ and $h_i = V_i F^p N_{ip} - N_i \circ F^p$. It is easy to check that

$$(3.7) \quad V_0 h_i N_{0p} - N_0 \circ h_i - V_i h_0 N_{ip} + N_i \circ h_0 = 0.$$

Letting $Q_i = h_i - G_i^p$ and $Q_0 = h_0 - G_0^p$, (3.5) $_i^p$ and (3.7) give

$$(3.8) \quad V_0 Q_i N_{0p} - N_0 \circ Q_i - V_i Q_0 N_{ip} + N_i \circ Q_0 = -R_i^p.$$

Applying $V_0^* V_i^*$ to both sides of (3.8) gives

$$(3.9) \quad V_i^* Q_i N_{0p} - N_0 \circ V_0^* V_i^* Q_i = -V_0^* V_i^* R_i^p,$$

since (3.2) $_0^p$ implies that $V_0^* Q_0 = 0$. But since $H_i^p = V_i^* Q_i$, (3.9) is the same as (3.6).

4. **Proof of Theorem 2.1.** The special case in which $R_i = 0$ in (3.5) $_i^0$ in the following theorem implies Theorem 2.1.

THEOREM 4.1. *Assume (2.4), (2.5), and (3.5) $_i^0$ for $i = 1, \dots, k$. Then the solution F of (3.2) $_0^0$ satisfies*

$$(4.1) \quad F \circ N_i - N_i \circ F = G_i + \epsilon_i \quad (i = 0, \dots, k),$$

where for large α , $\|\epsilon_i\|_\alpha \leq \text{const} (\sum_{j=1}^k \|R_j\|_\alpha)$, for $i = 0, \dots, k$.

PROOF. Let the Fourier series of Q_0 , Q_i , and R_i^p be

$$Q_0 \sim \sum a(n)e(nx), \quad Q_i \sim \sum b_i(n)e(nx), \quad R_i^p \sim \sum c_i(n)e(nx),$$

where Q_0 and Q_i are as in the last section. Then

$$H_i = V_i^* Q_i \sim \sum b_i(n/N_i^*)e(nx),$$

where conventionally $b_i(n/N_i^*) = 0$ if $n \not\equiv 0 \pmod{N_i^*}$ and $b_i(n/N_i^*) = b_i(m)$ if $n = N_i^* m$ for m in Z^d . Defining $b_i(n/N_0^*)$ and $a(n/N_0^*)$ analogously, one sees that (3.8) gives

$$(4.2) \quad -c(n) = N_i \circ a(n) - a(n/N_i^*)N_{ip} + b_i(n/N_0^*)N_{0p} - N_0 \circ b_i(n).$$

Note that $V_0^* Q_0 = 0$ implies that

$$(4.3) \quad a(n) = 0 \quad \text{if } n \equiv 0 \pmod{N_0^*}.$$

Now suppose that m is such that $a(m) \neq 0$, hence $m \not\equiv 0 \pmod{N_0^*}$ by (4.3). The assumption (2.5) then implies that for some i , $N_i^* m \not\equiv 0 \pmod{N_0^*}$ and (4.2) gives

$$(4.4) \quad a(m) = N_i \circ a(N_i^* m)N_{ip}^{-1} - N_0 \circ b_i(N_i^* m)N_{ip}^{-1} + c_i(N_i^* m)N_{ip}^{-1}$$

since $b_i(N_i^* m/N_0^*) = 0$.

Assume from now on that p is large enough so that Lemma 3.1 holds for $N = N_1, \dots, N_k$ and θ such that $\theta k < \beta < 1$.

Then (4.4) implies that

$$(4.5) \quad |a(m)| \leq \theta \{ |a(N_i^* m)| + K(|b_i(N_i^* m)| + |c_i(N_i^* m)|) \},$$

where K is a constant. If $B(m)$ is defined by

$$B(m) = \sum_{j=1}^k \{ |b_j(N_j^* m)| + |c_j(N_j^* m)| \},$$

then (4.5) implies

$$(4.6) \quad |a(m)| \leq \theta \{ |a(N_i^* m)| + KB(m) \}.$$

Continuing in this manner, one obtains a sequence M_1, M_2, \dots such that each M_j is N_1^* or N_2^*, \dots , or N_k^* and if $S_v = M_v \circ M_{v-1} \circ \dots \circ M_1$,

$$|a(m)| \leq \theta^v |a(S_v m)| + K \sum_{j=1}^v \theta^j B(S_{j-1} m)$$

and letting $v \rightarrow \infty$ gives

$$(4.7) \quad |a(m)| \leq K \sum_{j=1}^{\infty} \theta^j B(S_{j-1} m).$$

For a fixed integer j , any n in Z^d can appear as an $S_j m$ for some m and S_j in at most k^j ways, hence (4.7) implies that

$$(4.8) \quad \begin{aligned} \|Q_0\|_{\alpha} &= \sum_m (1 + |m|)^{\alpha} |a(m)| \leq K \sum_n \sum_{j=0}^{\infty} (1 + |n|)^{\alpha} (k\theta)^j B(n) \\ &\leq K(1 - \beta)^{-1} \sum_n (1 + |n|)^{\alpha} B(n). \end{aligned}$$

The definition of $B(n)$ and Lemma 3.3 imply that

$$\sum_n (1 + |n|)^{\alpha} B(n) \leq \text{const} \sum_{i=1}^k \|R_i^p\|_{\alpha},$$

hence (4.8) gives

$$(4.9) \quad \|Q_0\|_{\alpha} \leq \text{const} \sum_{i=1}^k \|R_i^p\|_{\alpha}.$$

Applying V_0^* to (3.8) and using (4.9) shows that

$$(4.10) \quad Q_i - N_0 \circ V_0^* Q_i = \lambda_i^p,$$

where $\|\lambda_i^p\|_{\alpha} \leq \text{const} \|R_i^p\|_{\alpha}$, by $V_0^* Q_0 = 0$ and the boundedness of V_0^* in $\|\cdot\|_{\alpha}$. Applying Lemma 3.2 to (4.10) shows that for $\alpha \geq p$,

$$\|Q_i\|_{\alpha} \leq \text{const} \|R_i^p\|_{\alpha},$$

which together with (4.9) is the assertion of the theorem.

APPENDIX. *Integration on T^d .* Let Y be a finite dimensional vector space. Here Y will almost always be one of the spaces $L_s^q(R^d; R^d)$, the space of symmetric q -linear maps from R^d to R^d . Let the continuous map $G: T^d \times R^d \rightarrow Y$ be linear in the R^d variable. Such a map is said to be *integrable* (or *locally integrable*) if for every smooth closed curve γ (or every smooth closed curve

$\gamma \sim 0$) on T^d defined by the function $x(t)$,

$$\int_{\gamma} G(x(t), x'(t)) dt = 0.$$

If $H(0) \in Y$ is specified and G is integrable, there is a unique $H: T^d \rightarrow Y$ whose derivative is G .

There is a simple analogue of classical calculus term by term integration theorems for functions and their derivatives defined on T^d .

PROPOSITION. *Let A_1, A_2, \dots be a sequence of functions mapping T^d to R^d and let A_s^p denote the p th derivative of A_s . If for some p the series $\sum_{s=1}^{\infty} |A_s^p(x)|$ converges uniformly, there are uniquely defined functions $F^q: T^d \rightarrow L_s^q(R^d; R^d)$, ($q = 1, \dots, p$) such that F^q is the derivative of F^{q-1} and $F^p(x) = \sum_{s=1}^{\infty} A_s^p(x)$. Also, if $F(0)$ is given, there is a uniquely determined $F: T^d \rightarrow R^d$ such that F^q is the q th derivative of F . Moreover, for $\alpha \geq p$,*

$$\|F - F(0)\|_{\alpha} \leq \text{const } \|F^p\|_{\alpha-p}.$$

The proof only needs to be sketched, since it is classical. The series for F^p converges uniformly and absolutely. Viewing F^p as a map from $T^d \times R^d$ to $L_s^{p-1}(R^d; R^d)$ one sees that F^p is integrable in the above sense, since each term is. Therefore, F^p has an antiderivative which will be locally integrable by the symmetry of A^p . One and only one choice of the constant of integration makes the antiderivative into an integrable function in the above sense.

Calling the integrated series with the choice of constant that makes it integrable F^{p-1} , one is in a position to continue the argument in this way until F^1 has been obtained. At the final step, integrating F^1 to obtain F , there is no preferred choice of the constant of integration, so $F(0)$ can be specified arbitrarily. The estimates for $\|F - F(0)\|_{\alpha}$ are easy to check.

REFERENCES

1. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Math., vol. 14, Springer-Verlag, New York and Heidelberg, 1973. MR 49 #6269.
2. J. N. Mather, *Stability of C^{∞} mappings. II. Infinitesimal stability implies stability*, Ann. of Math. (2) 89 (1969), 254–291. MR 41 #4582.
3. J. Moser, *A rapidly convergent iteration method and non-linear differential equations. II*, Ann. Scuola Norm. Sup. Pisa (3) 20 (1966), 499–535. MR 34 #6280.
4. R. Sacksteder, *A belian semi-groups of expanding maps*, Recent Advances in Topological Dynamics, Lecture Notes in Math., vol. 318, Springer-Verlag, New York and Berlin, 1973, pp. 235–248.
5. M. Shub, *Endomorphisms of compact differentiable manifolds*, Amer. J. Math. 91 (1969), 175–199. MR 39 #2169.
6. S. Sternberg, *Celestial mechanics. Part II*, Benjamin, New York, 1969.

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, NEW YORK, NEW YORK 10036