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ASYMPTOTICALLY AUTONOMOUS MULTIVALUED DIFFERENTIAL EQUATIONS(1)

BY

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ABSTRACT. The asymptotic behavior of solutions of the perturbed autonomous multivalued differential equation $x' \in F(x) + G(t, x)$ is examined in relation to the behavior of solutions of the autonomous equation $x' \in F(x)$ assuming that all solutions of the latter approach zero as t approaches ∞.

For multivalued functions F and G whose values are nonempty subsets of d-dimensional Euclidean space, R^d, the generalized differential equation

$$x' \in F(x) + G(t, x) \tag{1}$$

is said to be asymptotically autonomous if $G(t, x)$ becomes small in some sense as $t \longrightarrow \infty$. The main result of this investigation establishes the relationship of the asymptotic behavior of solutions of (1) to that of solutions of the autonomous equation

$$x' \in F(x). \tag{2}$$

THEOREM 1. *Let F be a positive-homogeneous upper semicontinuous mapping from R^d (d-dimensional Euclidean space) to the nonempty, compact, convex subsets of R^d such that all solutions of* (2) *approach zero as $t \longrightarrow \infty$. Let G be a mapping from R^{1+d} to the nonempty subsets of R^d such that $G(t, \cdot) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on nonempty compact subsets of R^d. If ϕ is a bounded solution of* (1) *on* $[0, \infty)$ *then $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$.*

If F and G are single-valued functions, denoted by f and g, respectively, the equations (1) and (2) are ordinary differential equations and the asymptotic behavior of the solutions is discussed, for example, by Strauss and Yorke. One of their results [7, p. 180] guarantees that all (classical) solutions of

$$x' = f(x) + g(t, x) \tag{3}$$

which are bounded on $[t_0, \infty)$ tend to zero as $t \longrightarrow \infty$ provided that f and g are

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continuous vector-valued functions, that all solutions of the unperturbed autonomous equation approach zero as $t \longrightarrow \infty$, and that $g(t, x)$ "mostly approaches zero". The last condition, which is defined in [7, p. 176] is satisfied, if for example, $g(t, \cdot)$ approaches zero as $t \longrightarrow \infty$ uniformly on compact subsets of R^d. Other treatments of asymptotically autonomous ordinary differential equations may be found in [1]–[4] and [6]–[10].

A perturbation-type result for generalized differential equations was developed by Lasota and Strauss [5, p. 169] as an aid in their investigation of autonomous ordinary differential equations. This result, tailored to suit the present context, is presented below.

LEMMA 2. *Let F be a positive-homogeneous upper semicontinuous mapping from R^d to the nonempty, compact convex subsets of R^d such that every solution of* (2) *approaches zero as $t \longrightarrow \infty$. Then there exist $\epsilon > 0$ and $K > 1$ such that for $t_0 > 0$ and $x_0 \in R^d$ each solution of*

$$x' \in F(x) + \epsilon B(|x|), \qquad x(t_0) = x_0 \tag{4}$$

can be continued to $+\infty$ and satisfies

$$|x(t)| \leqslant K|x_0| \exp(-\epsilon(t - t_0)) \tag{5}$$

for all $t \geqslant t_0$.

A *solution* of (1) is an absolutely continuous d-vector valued function which satisfies (1) almost everywhere on some nondegenerate interval. For $\epsilon > 0$, $x \in R^n$, and $A \subset R^n$ denote the Euclidean norm of x by $|x|$ and the norm of A by $\|A\| = \sup\{|x|: x \in A\}$. The distance from x to A is defined by $d(x, A) = \inf\{|x - y|: y \in A\}$ and the ϵ-neighborhood of A is the set $N(A, \epsilon) = \{y \in R^n: d(y, A) < \epsilon\}$. The closed-origin-centered ball of radius ϵ is denoted by $B(\epsilon)$.

The multivalued mapping H from R^n to the nonempty compact subsets of R^d is said to be *upper semicontinuous* if to each $\epsilon > 0$ and $x \in R^n$ there corresponds $\delta > 0$ such that $H(y) \subset N(H(x), \epsilon)$ provided $|x - y| < \delta$. The set-valued mapping H defined on R^n is said to be *positive-homogeneous* if $H(rx) = rH(x) = \{rz: z \in H(x)\}$ for all $x \in R^n$ and $r > 0$. The statement $H(t) \longrightarrow \infty$ means that to each $\epsilon > 0$ there corresponds $T > 0$ such that $H(t) \subset B(\epsilon)$ for all $t \geqslant T$; that is, $\|H(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$.

A variation of Theorem 1, in which the perturbation term depends only on t, provides an approach to the proof of the main result.

THEOREM 3. *Let F and G satisfy the hypotheses of Theorem* 1 *and in addition assume that G is independent of x. Then all solutions* (*not just the bounded solutions*) *of*

$$x' \in F(x) + G(t) \tag{6}$$

on $[0, \infty)$ *approach zero as* $t \longrightarrow \infty$.

The proof of this theorem is based on the observation that if ϕ is a solution of (6) for which $G(t) \subset \epsilon B(|\phi(t)|)$ for all $t \geqslant 0$ then $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$ according to Lemma 2; whereas, if $G(t) \not\subset \epsilon B(|\phi(t)|)$ for all $t \geqslant 0$ then $\epsilon B(|\phi(t)|) \subset B(\|G(t)\|)$, and $\phi(t) \longrightarrow 0$ since $\|G(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$.

PROOF OF THEOREM 3. Let ϵ and K be as in Lemma 2 and let ϕ be a solution of (6) at least on $[0, \infty)$. Define the sets I and J by

$$I = \{t \geqslant 0\colon\ G(t) \subset \epsilon B(|\phi(t)|)\} \tag{7}$$

and

$$J = \{t \geqslant 0\colon\ G(t) \not\subset \epsilon B(|\phi(t)|)\}; \tag{8}$$

clearly $I \cup J = \{t \geqslant 0\}$ and $I \cap J = 0$. In the light of the previous remarks, it remains to be shown that $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$ when both I and J are unbounded sets. Since the solution approaches zero on unbounded increasing sequences from J, it suffices to show that ϕ approaches zero along an arbitrary unbounded increasing sequence from I; let $\{t_k\}$ be such a sequence. For $k = 1, 2, 3, \ldots$, let I_k denote the component (maximal connected subset) of I which contains t_k and let d_k denote the length of this component. Let $s_k = \inf\{t \in I_k\}$ and assume, without loss of generality, that $s_1 > 1$; clearly, $s_k \nearrow \infty$ as $k \longrightarrow \infty$. The continuity of ϕ provides for each positive integer k a corresponding $\delta_k < 1$ such that

$$|\phi(s_k) - \phi(t)| < 1/(2k) \quad \text{for } |t - s_k| < \delta_k; \tag{9}$$

in addition, if $d_k > 0$, choose $\delta_k < d_k$. Choose auxiliary sequences $\{\tau_k\} \subset J$ and $\{\tau_k^*\} \subset I$ such that $s_k - \delta_k \leqslant \tau_k \leqslant s_k$ and $s_k \leqslant \tau_k^* \leqslant s_k + \delta_k$ for each positive integer k; these selections guarantee that

$$|\phi(\tau_k) - \phi(s_k)| < 1/(2k) \tag{10}$$

and

$$|\phi(\tau_k^*) - \phi(s_k)| < 1/(2k). \tag{11}$$

Consequently, for $t \in I_k$, ϕ satisfies

$$|\phi(t)| \leqslant \begin{cases} |\phi(s_k)| + 1/(2k) & \text{for } t \leqslant \tau_k^*, \quad (12) \\ K|\phi(\tau_k^*)| \exp(-\epsilon(t - \tau_k^*)) & \text{for } t \geqslant \tau_k^*. \quad (13) \end{cases}$$

The estimate in (12) follows from (9) and the choice of τ_k^*; whereas, the estimate in (13) follows from Lemma 2. These estimates can be modified by

(10) and (11) to obtain

$$|\phi(t)| \leq K(|\phi(\tau_k)| + 1/k) \quad \text{for } t \in I_k. \tag{14}$$

In particular, since $t_k \in I_k$ and $\tau_k \in J$, it follows that $\phi(\tau_k) \longrightarrow 0$ as $k \longrightarrow \infty$ which forces $\phi(t_k) \longrightarrow 0$ as $k \longrightarrow \infty$; thus $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$, which concludes the proof.

The proof of Theorem 1 follows almost as an immediate consequence of Theorem 3.

PROOF OF THEOREM 1. Let ϕ be a bounded solution of (1) which is defined at least on $[0, \infty)$, and let C denote a compact subset of R^d which contains $\phi(t)$ for all $t \geq 0$. Define the multivalued function H by

$$H(t) = \{y \in G(t, x): x \in C\}.$$

Clearly, $H(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and ϕ is a solution of

$$x' \in F(x) + G(t, x) \subset F(x) + H(t). \tag{15}$$

An application of Theorem 3 yields the desired results.

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