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CENTRALISERS OF C^∞ DIFFEOMORPHISMS

BY

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ABSTRACT. It is shown that if F is a hyperbolic contraction of R^n, coordinates may be chosen so that not only is F a polynomial mapping, but so is any diffeomorphism which commutes with F. This implies an identity principle for diffeomorphisms G_1 and G_2 commuting with an arbitrary Morse-Smale diffeomorphism F of a compact manifold M: *if* G_1, $G_2 \in Z(F)$, then $G_1 = G_2$ on an open subset of $M \Rightarrow G_1 \equiv G_2$ on M.

Finally it is shown that under a certain linearisability condition at the saddles of F, $Z(F)$ is in fact a Lie group in its induced topology.

Introduction. Let f be a C^∞ diffeomorphism of R^n onto itself which fixes the origin, and let $Df_0\colon R^n \longrightarrow R^n$ be its first derivative at 0. We shall describe f as a *sink* on R^n if it is hyperbolic and a topological contraction: i.e., (i) every eigenvalue λ of Df_0 satisfies $|\lambda| < 1$ and (ii) $\bigcap_{n=0}^{\infty} f^n(U) = \{0\}$ for any bounded set U containing the origin. The k-jet of f, denoted f_k, is an element of $L^k(n)$, the group of k-jets of local diffeomorphisms of R^n which preserve the origin. We denote by $L^\infty(n)$ the group of formal power series; $L^\infty(n) = \operatorname{inv}\lim L^k(n)$. The theorem of K.-T. Chen and S. Sternberg [2], [7] applied to sinks, implies that there is k (computed from the eigenvalues of Df_0) such that the germ of f is conjugate by a C^∞ diffeomorphism germ g, to some L^k-conjugate of f_k, $\bar{f}_k$: thus

$$gfg^{-1} = \bar{f}_k. \tag{I}$$

We describe the normal form $\bar{f}_k$ in §2. Two sinks f and h are conjugate if and only if $\bar{f}_k$ and $\bar{h}_k$ are conjugate in $L^\infty(n)$.

It is our purpose to show that the space of diffeomorphisms commuting with f admits a finite dimensional parametrisation. The first main result is

THEOREM 1. *Let h be any local diffeomorphism of R^n such that $\bar{f}_k h = h\bar{f}_k$. Then $h = h_k$.*

An obvious corollary is that conjugating functions g in equation (I) are unique up to elements of the centraliser in $L^k(n)$ of $\bar{f}_k$.

The theorem is a generalisation of a corresponding theorem of N. Kopell [3] in which $\bar{f}_k = \bar{f}_1$, i.e., f is linearisable. Thus by an argument in her paper,

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there follows an identity principle for a sink, or for that matter, for a C^∞ Morse-Smale diffeomorphism of a compact C^∞-manifold M without boundary. (For definitions, see [6].)

COROLLARY 1. *If g_1 and g_2 both commute with f, and if U is an open set of $R^n(M)$, then $g_1 = g_2$ on U implies $g_1 \equiv g_2$.*

A second corollary may be adduced which describes the space of solutions to the equation $[X, Y] = 0$, where X and Y are C^∞ vector fields on R^n, and the flow ϕ_t generated by Y consists of hyperbolic sinks.

COROLLARY 2. *Let Y be a vector field germ such that Y is an elementary contracting critical point: i.e., the eigenvalues μ of DZ_0 satisfy* $\mathrm{Re}(\mu) < 0$. *Let* $\mathfrak{Z}(Y) = \{X \mid [X, Y] = 0\}$. *Then $\mathfrak{Z}(Y)$ is the Lie algebra of $Z(\overline{\phi}_1)$ the centraliser in $L^k(n)$ of $\overline{\phi}_1$ (k chosen as above).*

In order to prove the theorem, we give in §2 a normal form $\overline{f}_k$ for f; it is related to the real Jordan form which the matrix f_k has in a certain faithful linear representation of $L^k(n)$. The jet $\overline{f}_k$ is an invariant of C^∞-conjugacy of sinks. Indeed, using the representation one can give an alternative proof of the formal content of Sternberg's theorem.

The second main result is an extension of Theorem 1 to Morse-Smale diffeomorphisms. To a Morse-Smale diffeomorphism f, and to an orbit $\{x, f(x), \ldots, f^m(x)\}$ in $\Omega(f)$, there is associated the spectrum of $D(f^m)_x$. At least one such orbit is a *sink*, for which all the eigenvalues of $D(f^m)_x$ have absolute value less than 1. A *source* for f is a sink for f^{-1} and any other point of $\Omega(f)$ is called a *saddle*.

Now let $\mathrm{Diff}^\infty(M)$ denote the topological group of C^∞ diffeomorphisms of a compact manifold M, topologised by the C^∞ topology (see, for example [5]). $\mathrm{Diff}^\infty(M)$ contains $Z(f) = \{g \in \mathrm{Diff}^\infty(M) \mid gf = fg\}$ as a closed subgroup.

THEOREM 2. *Suppose f is a Morse-Smale diffeomorphism such that at each saddle y,*

$$\text{(II)} \qquad \lambda_i \neq \prod_{j=1}^{n} \lambda_j^{m_j},$$

where if m is the period of f on y, λ_i are the eigenvalues of $D(f^m)_y$, m_j are nonnegative integers, and there is $j \neq i$ with $m_j \neq 0$. Then as a topological group $Z(f)$ is equivalent to a Lie group.

In a subsequent paper we prove that generically $Z(f)$ is in fact discrete; such f cannot, for example, be a time-one map for a C^∞ vector field.

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1. **Preliminaries.** We shall adopt some notations of [4], which we briefly recall. $C^{\infty}(R^n)$ denotes the local ring of C^{∞} mapping germs at the origin and M its maximal ideal. Analogously, let N be the ideal in $R[x_1, \ldots, x_n]$ of polynomials without constant term. N is generated as a real vector space by the monomials–we denote the monomial $x_1^{i_1} \ldots x_n^{i_n}$ by x^I, where $I = (i_1, \ldots, i_n)$ is an ordered n-tuple of nonnegative integers. For such I, $I! = i_1! \ldots i_n!$ and $\deg I = i_1 + \ldots + i_n$. We shall need the fact that there is a canonical identification of N with $\Sigma_{i=1}^{\infty} O^i V$ (where $O^i V$ denotes the ith symmetric power of $V \approx R^n$) under which monomials in x are identified with monomials in the standard basis of V.

The Taylor homomorphism j_k: $M \longrightarrow N$ is defined by

$$j_k(f) = \sum \frac{1}{I!} \frac{\partial^{\deg I} f}{\partial x^I}(0) x^I,$$

the sum being over all I with $1 \leqslant \deg I \leqslant k$. As is well known, j_k induces an isomorphism $M/M^{k+1} \approx N/N^{k+1}$, this quotient being denoted $J^k(n, 1)$, the ring of k-jets of vanishing real C^{∞} functions. Having introduced N, we may speak of "complex k-jets": these are elements of N/N^{k+1} when N is the ideal of complex polynomials in n-variables without constant term. $J^k(n, n) = J^k(n, 1) \otimes R^n$ is the space of k-jets of functions R^n to R^n and $L^k(n) \subset J^k(n, n)$ the group of jets invertible for the operation of composition.

There is a right linear action of $L^k(n)$ on $J^k(n, 1)$ defined by $x^I \cdot g = j_k(g^I)$ where $g = (g_1, \ldots, g_n)$ is in $L^k(n)$. In the basis of monomials, g has the matrix G_{IJ} = coefficient of g^J on x^I. Hence by sending g to the adjoint of the linear map G_{IJ}, there is obtained a representation ρ_k: $L^k(n) \longrightarrow GL(J^k(n, 1))$. The matrix of $\rho_k(g)$, in the basis of monomials, is G_{IJ} = coefficient of g^I on x^J. The indices I, J range over all monomials in n variables of degree not greater than k.

The following facts are easy to see.

(i) ρ_k is faithful (and presents $L^k(n)$ as an algebraic group).

(ii) If $\deg I = 1 = \deg J$, $(\rho_k(g))_{IJ}$ is the Jacobian matrix of g, $j_1(g) = Dg_0$. Moreover, if $1 < r \leqslant k$ and $\deg I = r = \deg J$, $(\rho_k(g))_{IJ}$ is the matrix of $Dg \circ \ldots \circ Dg$: $O^r V \longrightarrow O^r V$ (in the basis of monomials). This implies that the eigenvalues of $\rho_k(g)$ are all monomials λ^I of degree $\leqslant k$ in the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of Dg_0.

(iii) If f is a diffeomorphism of R^n, and if $\deg I = 1$, $\deg J = r \leqslant k$, then $\rho_k(f_k)_{IJ}$ is the matrix of $D^r f$: $O^r V \longrightarrow V$ taken in the basis $\{J! x^J \mid \deg J = r\}$.

2. **Normal forms of diffeomorphisms.** *1-jets.* Let A be a real linear operator. Then there is a basis $\{z_1, \bar{z}_1, \ldots, z_r, \bar{z}_r, w_{2r+1}, \ldots, w_n\}$ of C^n in which

the matrix of A is in Jordan form: if z_i is a (generalised) eigenvector for λ, so is $\bar{z}_i$ for $\bar{\lambda}$, and w_i are the real eigenvectors. By the real Jordan form of A, we mean the matrix which A takes in the basis $\{\mathrm{Re}(z_1), \mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_r), w_{2r+1}, \ldots, w_n\}$.

EXAMPLE.

$$\begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda & 1 & 0 \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda & 0 & 1 \\ 0 & 0 & \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ 0 & 0 & -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \bar{\lambda} & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \bar{\lambda} \end{pmatrix}.$$

Notice that there is an involution $z_i \longleftrightarrow \bar{z}_i$ of the basis of C^n which induces a corresponding involution σ of the monomials, which we denote by $z^I \longleftrightarrow z^{\sigma(I)}$. If

$$z^I = z_1^{i_1}\bar{z}_1^{j_1} \ldots z_r^{i_r}\bar{z}_r^{j_r} w, \qquad z^{\sigma(I)} = z_1^{j_1}\bar{z}_1^{i_1} \ldots \bar{z}_r^{i_r} w.$$

Complex jets. A complex k-jet or formal power series (∞-jet) F is in normal form providing

(i) $\rho_1(F_1)$ is in Jordan normal form,

(ii) $\rho_r(F_r)$ is upper triangular for $k \geqslant r \geqslant 1$,

(iii) $(\rho_r(F_r))_{IJ} \neq 0 \Rightarrow \lambda^I = \lambda^J$.

Real jets. A real k-jet or formal power series F is in normal form providing:

(i) $\rho_1(F_1)$ is in real Jordan normal form, so there is a matrix A_1, being a sum of blocks $\binom{1\ \ 1}{i\ \ -i}$ so that $A_1\rho_1(F_1)A_1^{-1}$ = Jordan form of $\rho_1(F_1)$.

(ii) The complex power series $A_1 F_r A_1^{-1}$ is in normal form for each $k \geqslant r \geqslant 1$.

It follows from (ii) that $(A_k)_{IJ} = \overline{(A_k)_{\sigma(I)\sigma(J)}}$, and conversely, if this condition is satisfied for a complex formal power series or jet A_k, the eigenvalues of whose linear part occur in conjugate pairs, then A_k is derived from a real f.p.s. via conjugation by A_1 (see [1]).

A diffeomorphism germ is in normal form if its associated formal power series is in real normal form.

REMARKS. (1) The normal form is a conjugacy invariant of formal power series or jets, or diffeomorphisms, but it need not be unique. For instance, if $\lambda \neq 1$ is real, the 3-jets

$$F(x, y, z) = (\lambda x, \lambda^2 y + x^2, \lambda^3 z + xy + x^3),$$
$$G(x, y, z) = (\lambda x, \lambda^2 y + x^2, \lambda^3 z + xy)$$

are distinct normal forms which are conjugate in $L^3(3)$, by $I + (0, x^2, 0)$.

(2) The matrix $\rho_k(F_\infty)$ will in general not be in Jordan form for $k > 1$.

The following lemma is a generalisation of a lemma of Sternberg [7].

LEMMA. *For any (real) invertible FPS F_∞ there is a (real) invertible FPS G with $GF_\infty G^{-1}$ in (real) normal form.*

PROOF. We construct $G = \varprojlim G_k$ by induction on k. If $k = 1$, this is the familiar real Jordan normal form theorem for the Jacobian F_1. For the inductive step, we observe that if $\rho_k(F_k)$ is in (real) normal form, so is $\rho_{k+1}(F_k)$. (Any entry $\langle f^I, x^J\rangle$ of $\rho_{k+1}(F_k)$ satisfies $\langle f^I, x^J\rangle = \Sigma\langle f^{I_1}, x^{J_1}\rangle\langle f^{I_2}, x^{J_2}\rangle$ where the sum is over all monomials $f^{I_1}f^{I_2} = f^I$ and $x^{J_1}x^{J_2} = x^J$. By inductive hypothesis, some summand can be nonzero only when $\lambda^{I_1} = \lambda^{J_1}$ and $\lambda^{I_2} = \lambda^{J_2}$, in which event $\lambda^I = \lambda^J$. As before, λ^I denotes a monomial in the eigenvalues of F_1.)

It follows that if G_k is chosen so that $\rho_k(G_kF_kG_k^{-1})$ is in (real) normal form, then $\rho_{k+1}(G_kF_{k+1}G_k^{-1})$ will be in (real) normal form except possibly for entries f_{iJ} with $\deg i = 1$, $\deg J = k + 1$. (These entries represent the contribution of D^kF: $O^kV \longrightarrow V$.) Dividing up

$$J^{k+1}(n, 1) = \frac{M}{M^2} \oplus \sum_{1<r<k+1} \frac{M^r}{M^{r+1}} \oplus \frac{M^{k+1}}{M^{k+2}}$$

we see that $\rho_{k+1}(F_\infty)$ has, in this decomposition, the form

$$X = \begin{bmatrix} A & B & D \\ 0 & C & E \\ 0 & 0 & F \end{bmatrix},$$

where

$$\begin{pmatrix} A & B & 0 \\ 0 & C & E \\ 0 & 0 & F \end{pmatrix}$$

is in normal form. A is the Jacobian and F is its symmetric power of degree $(k + 1)$. Conjugating X by a matrix

$$Y = \begin{bmatrix} I & 0 & G \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we obtain $D - AG + GF$ in the top right-hand corner.

To prove the lemma is to solve $(D - AG + GF)_{iJ} = 0$ when $\lambda^i \neq \lambda^J$; i.e., $D_{iJ} = \Sigma A_{ij}G_{jK} - \Sigma G_{iK}F_{KJ}$ with $1 \leqslant j \leqslant n$ and $\deg K = k + 1$.

Using the assumption that $\rho_{k+1}(F_k)$ is upper triangular (as a complex matrix), this equation becomes

$$(*) \qquad D_{iJ} = G_{iJ}(A_{ii} - F_{JJ}) + \sum_{i<j} A_{ij}G_{jJ} - \sum_{K<J} G_{iK}F_{KJ}.$$

Supposing G_{jJ} is known for $j < i$, G_{iK} for $K < J$, we may solve inductively for G_{iJ} when $A_{ii} \neq F_{JJ}$ (i.e. $\lambda^i \neq \lambda^J$).

To prove the statement for real jets, we must know that the basis change taking F_1 into its real Jordan form takes some real jet into Y. According to Birkhoff [1], the condition that this be so is that $\langle y^I, x^J\rangle = \overline{\langle y^{\sigma(I)}, x^{\sigma(J)}\rangle}$, where the involution σ on monomials is defined as above. It is clear in equation $(*)$ that if D, A, F satisfy the condition, G will satisfy it by the same induction when $\lambda^i \neq \lambda^J$; and otherwise the choice of $G_{iJ} = 0$ also satisfies the condition.

REMARKS. (1) The lemma above constitutes the formal content of the theorem of Chen and Sternberg as in [2], [7] and [8], in the sense that by choosing a diffeomorphism with G_∞ as above, we may conjugate F by G to a diffeomorphism whose jet is in normal form. The analytic content is then to prove that two C^∞ hyperbolic diffeomorphisms with the same normal form are C^∞-conjugate.

(2) If $\lambda^i \neq \lambda^I$ for any monomial λ^I of degree > 1, the normal form is linear, whereupon F is C^∞ conjugate to a linear map.

(3) If F is a sink (source), there is k such that $\lambda^i = \lambda^I \Rightarrow \deg I \leqslant k$, so that the normal form of F is a k-jet, to which F is C^∞-conjugate.

3. **Centralisers.** The crucial observation of this section is that if $L\colon V \longrightarrow V$ is a linear map, and if $V = \Sigma V_i$ is its decomposition into generalised eigenspaces, then if $LM = ML$, each V_i is M-invariant. Applying this fact to $\rho_k(F_k)$ in normal form, if G_k is a commuting jet, then $\rho_k(G_k)$ also has the property $\rho_k(G_k)_{IJ} \neq 0 \Rightarrow \lambda^I = \lambda^J$; although of course it need not be upper triangular. Furthermore, if F_k is a contraction and k is not less than the maximal degree on the right-hand side of relations $\lambda^i = \lambda^J$, then if G_∞ commutes, $G_\infty = G_k = G_{k+r}$ for all r. In particular, F_k^{-1} (or G_k^{-1}) coincides with the inverse of F_k in $L^\infty(n)$, and so the polynomial mappings F_k, F_k^{-1}, G_k, G_k^{-1} are globally defined diffeomorphisms of R^n.

The theorem is an application of the following

LEMMA. *Let F_k be a contracting jet in normal form with k as above. Considered as a polynomial diffeomorphism of R^n, for x, y in a compact $K \subset R^n$, we have*

$$(1) \qquad |F_k^{-m}(x) - F_k^{-m}(y)| \leqslant p(m)\lambda^{km}|x - y|,$$

$$(2) \qquad |F_k^{m}(x) - F_k^{m}(y)| \leqslant q(m)\mu^{km}|x - y|,$$

where p, q are polynomials; λ^{-1} = *smallest eigenvalue of* F_1, μ = *largest eigenvalue of* F_1.

PROOF. We prove (1). Let $\alpha: V \longrightarrow S^k V$ be defined by $x \longrightarrow x^I$; i.e., if $\{e_j\}$ is a basis of V and the symmetric products of these vectors are denoted e^I, then the coefficient of $\alpha(x)$ on e^I is x^I. Clearly α has a uniform Lipschitz constant on K.

If π is the projection $S^k V \longrightarrow V$, then $F_k^m(x) = \pi \circ \rho(F_k)^m \circ \alpha(x)$ for any $m \in Z$, because F_k is in normal form. Hence

$$|\pi \circ \rho(F_k^{-m}) \circ \alpha(x) - \pi \circ \rho(F_k^{-m}) \circ \alpha(y)| \leqslant |\pi|\,|\rho(F_k^{-m})|\,|\alpha(x) - \alpha(y)|$$
$$\leqslant \text{const}|\rho(F_k)^{-m}|\,|x - y|.$$

We may write $\rho(F_k) = SU$ where these two linear maps commute, S is semisimple—we may suppose diagonal—and U is unipotent. Then $\rho(F_k)^{-m} = S^{-m}U^{-m}$ and

$$|\rho(F_k)^{-m}| \leqslant |S^{-m}|\,|U^{-m}| \leqslant \lambda^{km}p(m).$$

REMARK. $(\deg p) + 1$ = nilpotence degree of $(U - I)$.

PROOF OF THEOREM 1. According to the observation at the head of this section, with k as above, if g commutes with F_k, then for all $r \leqslant \infty$, $\rho_{k+r}(g) = \rho_k(g)$. Let g_k be the polynomial diffeomorphism of degree k such that $\rho_k(g_k) = \rho_k(g)$, then $g_kF_k = F_kg_k$, so that we may write $gg_k^{-1} = I + h$, commuting with F_k, and such that $h(x) = O(x^r)$ on a small enough neighbourhood U_r of 0 (for any r). In fact, $h \equiv 0$. Since $h(x) = (I + h - I)(x)$, we have

$$|h(x)| \leqslant |F_k^{-m}(I + h)F_k^m - F_k^{-m}F_k^m(x)| \leqslant p(m)\lambda^{km}|(I + h)F_k^m(x) - F_k^m(x)|$$
$$\leqslant p(m)\lambda^{km}|h(F_k^m(x))| \leqslant p(m)\lambda^{km}|F_k^m(x)|^r$$
$$\leqslant \text{const}\; p(m)\lambda^{km}q(m)^r\mu^{kmr}|x|^r.$$

Choosing r and U_r such that $\lambda\mu^r < 1$, and taking the limit of the right-hand side as $m \longrightarrow \infty$, we see $h(x) \equiv 0$, because the exponential convergence of $(\lambda\mu^r)^{km}$ dominates the polynomial convergence of $p(m)q(m)^r$.

4. Centralisers of Morse-Smale diffeomorphisms. In this section we prove Theorem 2, stated in the Introduction. If p is the number of points in $\Omega(f)$, since $Z(f)$ acts on $\Omega(f)$, there is a homomorphism from $Z(f)$ to S_p whose kernel contains the identity component $Z(F)_0$: we show $Z(F)_0$ is a Lie group. Moreover, we may suppose $\Omega(f)$ consists of fixed points because $Z(f) \subset Z(f^r)$ for any r.

If $\{S_i\}$ are the sources in $\Omega(f)$ and $\{N_j\}$ the sinks, we may choose coordinates for their unstable and stable manifolds, $w^u(S_i)$ and $w^s(N_j)$, so that f is in normal form, and there are Lie groups $G_i = Z(f|_{w^u(S_i)})$, $G_j = Z(f|_{w^s(N_j)})$. Then if $g \in Z(f)$ acts trivially on $\Omega(f)$, g leaves invariant the $w^u(S_i)$ and $w^s(N_j)$, and so $g|_{w^u(S_i)} \in G_i$, $g|_{w^s(N_j)} \in G_j$. Hence there is defined a homomorphism $R\colon Z(f)_0 \longrightarrow \Pi_{i,j} G_i \times G_j$.

By the identity principle this is injective, and it is continuous for the C^∞ topology on $Z(f)_0$. The conclusion will follow from the fact that R is a closed map. In other words, if g_m are C^∞ diffeomorphisms in $Z(f)_0$, such that $g_m \longrightarrow g$ C^∞-uniformly on compacta in $\bigcup w^u(S_i) \cup w^s(N_j) = M \sim \{\text{saddles}\}$, then the convergence is in fact uniform on M; in particular, g is C^∞ and hence in the image of R.

We restrict attention to a saddle, about which, by assumption (see [8]), coordinates may be chosen making f linear. In this coordinate system, there are unique f-invariant linear subspaces E^u and E^s which span R^n; the expanding and contracting eigenspaces of the (now linear) map f. The diffeomorphisms g_m, hence the map g, leave these subspaces invariant, whereupon from Theorem 1, $g_m^u = g_m|_{E^u}$ and $g_m^s = g_m|_{E^s}$ are already uniformly convergent on compacta in $E^u \cup E^s$; it follows that $g|_{E^u \cup E^s}$ is C^∞.

Since f is linear, $f = f^u \times f^s\colon E^u \times E^s \longrightarrow E^u \times E^s$. This means f commutes with each $g_m^u \times g_m^s$, and therefore with $g_m \circ (g_m^u \times g_m^s)^{-1}$. This sequence is the identity along $E^u \cup E^s$, and converges uniformly on all compacta to g if and only if $g_m \longrightarrow g$ uniformly on all compacta. Hence, we may make the simplifying assumption that $g_m = g = I$ on $E^u \cup E^s$.

The first step is to show uniform C^0-convergence. Let $g_m = S_m + U_m$ be the coordinate expansion of g_m; thus, $S_m(U_m)\colon R^n \longrightarrow E^s(E^u)$. Then $f^{-r}U_m f^r = U_m$ and $f^t S_m f^{-t} = S_m$ for every $r, t \in Z$ and every m, by the linear and hyperbolic properties of f. Consequently, in any norm

$$|g_m(x) - g(x)| \leqslant |f^{-r}(U_m f^r(x) - Uf^r(x))| + |f^t(S_m f^{-t}(x) - Sf^{-t}(x))|.$$

If $W \subset R^n - \{0\}$ is a compact set containing fundamental regions for f^u and f^s, then by uniform convergence on W, for high m, $|U_m - U| + |S_m - S| < \epsilon$ on W. But for x in any compact neighborhood of zero, there are numbers r, t so that $f^r(x)$ and $f^{-t}(x)$ are in W. The result follows from the fact that f^{-r} acts contractively on E^u and f^s contractively on E^s.

The argument for convergence of the higher derivatives is more delicate: we observe that $D^k f = 0$ for $k \geqslant 2$ (f is linear). Then by applying the chain rule ($f = Df$ is a constant linear map)

$$(**) \qquad (Df)^r D^k g_m(x)(\underbrace{Df \times \ldots \times Df}_{k})^{-r} = D^k g_m(f^r x)$$

for all k, r, m, x. This equation is in fact the rth iteration of the linear operator $F(A) = Df \circ A \circ (Df \times \ldots \times Df)^{-1}$ operating on L^k_{sym} the space of symmetric

k-multilinear maps. If $L^k_{\text{sym}} = L^u \oplus L^s \oplus L^c$ is the decomposition into eigenspaces for F for the eigenvalues of absolute value > 1, < 1, and $= 1$ respectively, then the argument used in the C^0 case above may be applied to show that the L^u and L^s components of $D^kG_m - D^kG$ tend to zero as $x \longrightarrow 0$. It remains to settle the component on L^c. Note that for $F|_{L^c}$, with a norm on L^k_{sym}, $|F^r| \leqslant p(r)$ for some polynomial p.

We claim that G_m has infinite contact with the identity along $E^u \cup E^s$. If $x \in E^s$, we see that $F^r(D^kg^c_m(x)) \longrightarrow D^kg^c_m(0)$ as $r \longrightarrow \infty$. For a linear map, this can only happen for a fixed point, and so $D^kg^c_m$ is constant along E^s (similarly, E^u, using F^{-1}). In fact, for every $k > 1$, $D^kg^c_m(0) = 0$, since, any jet J_k commuting with the linear map $f = Df_0$, satisfies $J_k = J_1$ by the assumption $\lambda_i \neq \Pi\lambda_j^{n_j}$. But $g_m = I$ along E^u and E^s (by the simplifying assumption above), and this implies that $J_1 = I$: this proves the claim.

We can now show that (g_m) is uniformly convergent on compacta. By (**), for the L^c component, we have

$$\begin{aligned}|D^kg_m(x) - D^kg(x)| &= |F^{-r}(D^kg_m(f^r(x))) - F^{-r}(D^kg(f^r(x)))| \\ &\leqslant p(r)|D^kg_m(f^r(x)) - D^kg(f^r(x))| \\ &\leqslant p(r)(|D^k(g_m - g)(f^r(x)) - D^k(g_m - g)(z)| + |D^k(g_m - g)(z)|).\end{aligned}$$

For any x in a small compact neighborhood of 0, there is r such that $f^r(x) \in W$, then $z \in E^u$ may be chosen to minimise the distance between $f^r(x)$ and E^u. By the preceding, $D^k(g_m - g)(z) = 0$. Now applying the Mean Value Theorem, where $|A|_W$ denotes supremum on W,

$$|D^k(g_m - g)(f^r(x)) - D^k(g_m - g)(z)| \leqslant |D^{k+1}(g_m - g)|_W |f^r(x) - z| \text{ const } \lambda^r$$

$$\text{for some } 0 < \lambda < 1.$$

Since exponential convergence still dominates polynomial convergence, this completes the proof. g_m is for each k, C^k-uniformly convergent on compacta, R is a closed map, and g is a C^∞ diffeomorphism.

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